Experiment 4: Refraction and Interference with Microwaves

Introduction

Many phenomena whose study comes under the heading of "physical optics" arise from certain integral relationships between the wavelength of the radiation and dimensions of the experimental apparatus. This experiment involve such ratios, these studies will be done with microwaves, i.e., very short radio waves having wavelengths of a few centimeters, rather than with visible light, where changes of system dimensions over a few wavelengths are very difficult to produce and control. This experiment will take advantage of interference patterns (constructive interference to be precise) in order to determine fundamental values of both the microwave and the mediums through which it will travel.

1   Physics

1.1 Electromagnetic Radiation

The term electromagnetic radiation encompasses a wide variety of radiative phenomena that seem at first to be quite different from one another. For example, radio waves, visible light, and the "gamma rays" which emanate from radioactive substances all fall into this category. In fact, they are manifestations of the same process, i.e., they all are propagating waves of electric and magnetic field energy. The only way in which they differ from each other is in their wavelengths (and frequencies). All electromagnetic radiation propagates at a velocity $c = 3$ x 10^8 m/sec in vacuum. Thus, while radio waves may have wavelengths of several thousand meters, light waves 5 x 10^{-5} cm and γ-rays 10^{-10} cm or shorter, they are the same kind of wave traveling at the same velocity (in vacuum).

The phenomena of reflection and refraction of electromagnetic waves are particularly familiar in the instance of visible light, where our means of detection is so very good. Reflection, of course, is what makes most objects visible to us, and refraction is responsible for the functioning of eyeglass and camera lenses. Refraction is also responsible for the distortion of the image of an object when viewed through a glass of water.

1.2 Index of Refraction and Snell's Law

To discuss these processes in more detail, we must establish the notion of refractive index. This quantity, usually labeled n, is simply the factor by which an electromagnetic wave is slowed upon entering a material medium from a vacuum region. In other words, the velocity of the waves in this refractive medium is:

$$v = \frac{c}{n} \tag{1}$$

Now, it is evident that the number of waves per second incident from a vacuum region onto the surface of a refracting medium is the same as the number per second entering that medium, i.e., the frequency (f) of the waves must be the same in both regions. Otherwise, there would be a pile-up of

waves formed at the interface. Therefore, because we know that $f\lambda = c/n$, (λ is wavelength) we can conclude that in the refractive region the wavelength is decreased such that

$$\lambda_0 = n\lambda \tag{2}$$

Here, λ_0 is the vacuum wavelength.

Snell's Law is a direct result of boundary conditions, which will be explained below. The result is

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2) \tag{3}$$

2 Mathematical Analysis

2.1 Boundary Conditions

The way in which refraction is most noticeable is in the bending of light rays. You are probably familiar with the way in which prisms and lenses bend beams of light, and nearly everyone has seen the apparent bending of a stick thrust into water at some oblique angle with the surface. We may relate the bending of rays quite easily to the refractive index by a geometrical construction, as follows.

We imagine that a plane wave is traveling through a medium of index n_1. By plane wave, we mean a disturbance uniform in a plane, which propagates along a ray direction perpendicular to the plane. The electric field and magnetic field vectors associated with the wave lie in the plane of the wave, and at right angles to each other; the space and time dependence of the electric field is given in vector notation by

$$\mathbf{E} = E_0\hat{x}\sin(kz - \omega t) \tag{4}$$

and the magnetic field by

$$\mathbf{H} = H_0\hat{y}\sin(kz - \omega t) \tag{5}$$

when the ray, or propagation vector, is along the z-axis.

Consider now a plane wave, such as the one depicted in figure 1, incident upon a plane boundary separating regions 1 and 2, which have indices n_1 and n_2, respectively. We will assume for now that $n_2 > n_1$, and assign an angle θ_1 between the wave and the interface. At a time t_1, the wave enters a portion of the surface of width x, and at time t_2 it has just crossed completely through it.
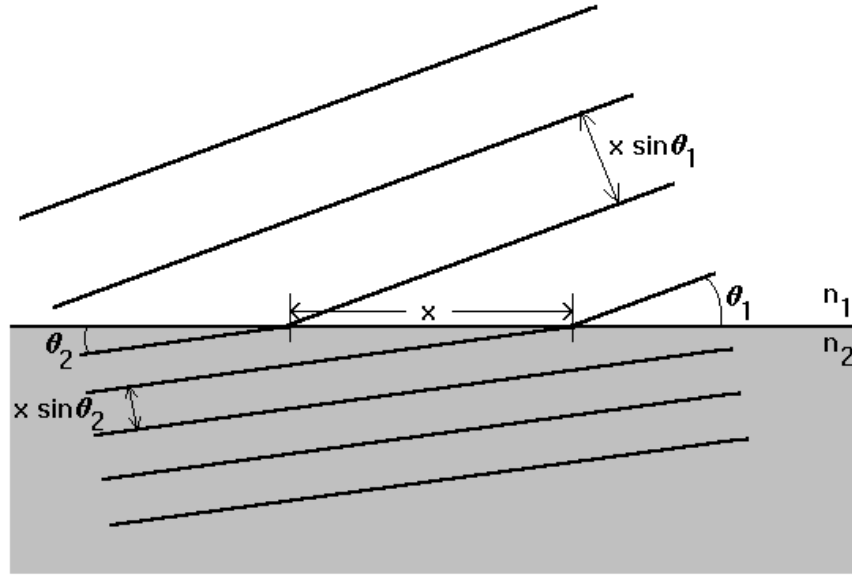
Figure 1 *Waves incident on an interface between index of refraction n_1 and n_2.*

We can see from figure 1 that:

$$\frac{c}{n_1}(t_2 - t_1) = x\sin(\theta_1) \tag{6}$$

i.e., that in the time interval $t_2 - t_1$, the incident wave has moved a distance $x\sin(\theta_1)$. However, now one also sees that the part of the wave which was just entering region 2 at time t_1 has moved a distance $x\sin(\theta_2)$ at time t_2. Therefore,

$$n_1 x\sin\theta_1 = c(t_2 - t_1) = n_2 x\sin\theta_2 \tag{7}$$

or

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2) \tag{8}$$

This is Snell's law, which relates ray bending to a step in refractive index at a surface. Ordinarily, one draws the rays associated with the wave, as shown in figure 2. Our description of events at the interface is still incomplete, however, because in addition to the wave transmitted through the surface, some portion of the incident energy is reflected. The complete ray diagram is, therefore, modified as shown in figure 2 to include both transmitted and reflected rays, where the angle of reflection is the same as the angle of incidence. (You may prove this result to yourself in the same manner as we proved Snell's law.)

So far, the fraction of the wave intensity transmitted, and the fraction reflected, has not been specified. We will not develop this subject here, since it involves somewhat advanced concepts in

electromagnetic theory. However, stated without proof, the result which applies for normal incidence $(\theta_1 = \theta_2 = 0)$ is:
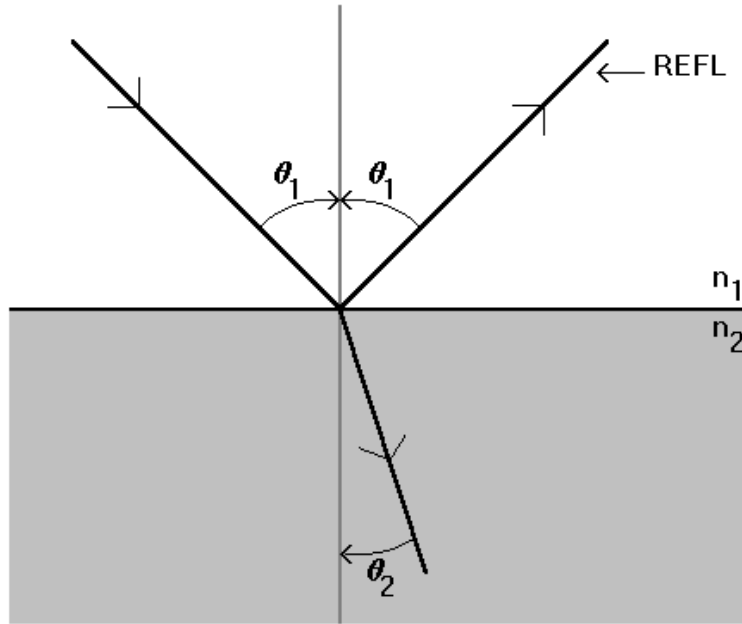

Figure 2 *Rays incident on an interface between index of refraction $n_1 < n_2$. Angles are measured with respect to the normal to the interface. The refracted ray is at the new angle θ_1, while the reflected ray is shown as REFL.*

$$E_{trans} = E_0 \frac{2n_1}{n_1 + n_2} \qquad (9)$$

and

$$E_{refl} = E_0 \frac{n_1 - n_2}{n_1 + n_2} \qquad (10)$$

where E_0 is the incident electric field strength. Now, the intensity of an electromagnetic wave, or the energy transported by the wave across unit area each second, is proportional to E^2; therefore, the intensity reflected back along a light beam incident on a refractive interface is:

$$I_{refl} = I_0 \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2 \qquad (11)$$

Question 4.1

Consider a light ray in air, whose index is essentially 1.00 (that of vacuum), normally incident on a sheet of glass, for which, typically, $n_2 = 1.5$. What fraction of the initial intensity is reflected?

The result of question 4.1 is familiar. Nearly all the light incident on a window passes through it while only enough is reflected to allow you to see dim images. This image is often too dim to discern during the day because the light from the other side is much more intense. When there is little or no light coming from the opposite side you can clearly see the dim reflection.

2.2 The Critical Angle

An interesting special situation arises for waves passing from a refracting medium into another region of smaller refractive index. We recall that $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$, and so $\theta_1 < \theta_2$ for $n_2 < n_1$. Suppose we increase θ_1, the angle of incidence, until $\sin(\theta_2) = 1$; then,

$$\sin \theta_1 = \frac{n_2}{n_1} \qquad (12)$$

and the transmitted ray is moving just parallel to the interface. If we increase θ_1 still further, a rather remarkable thing happens, and it is called, appropriately enough, **total internal reflection**. The interface now acts like a perfect mirror, and simply reflects all of the incident energy back into region 1, at an angle equal to θ_1. This is the general principle behind fiber optic cables and light guides.

The angle for which $\sin \theta_1 = \frac{n_2}{n_1}$ is called the critical angle, $\theta_C = \theta_1$.
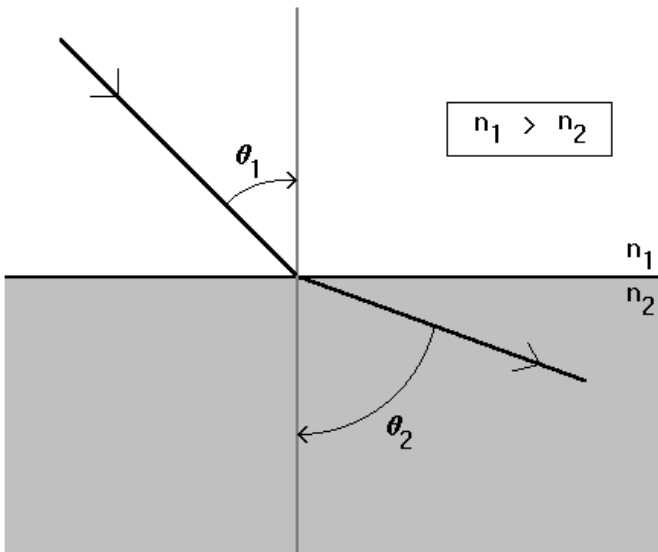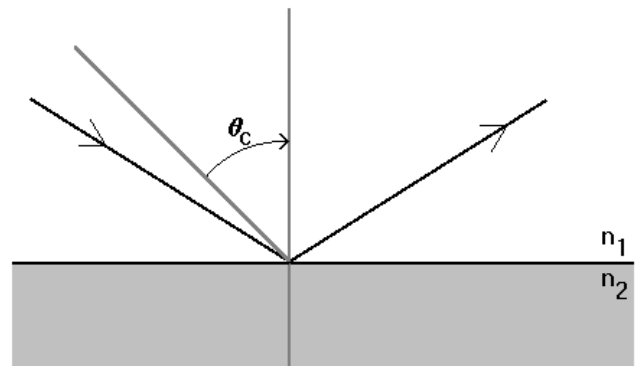
Figure 3 (left) *Rays incident on an interface between index of refraction $n_1 > n_2$, showing increase in the angle θ_2 to the normal for the transmitted ray*

Figure 4 (above right) *Rays incident on an interface between indices of refraction $n_1 > n_2$. The critical angle, θ_C, is shown for the case of total internal reflection.*

2.3 Standing Waves

When an electromagnetic wave strikes a conductor, it bounces off. However, the situation is slightly more complicated than that because there are magnetic and electric boundary conditions which dictate the direction of each field after it comes in contact with the new medium. The most important boundary condition needed to understand the condition of the reflected wave is that of the parallel* electric field. The parallel electric field must be continuous on each side of the interface for all time.

A primary property of conductors is that they all have zero electric fields inside them. This is why metal can be used to shield electromagnetic waves. However, the electric and magnetic fields do not simply disappear; the wave is completely reflected. You can proof this to yourself, by taking the limit as n_2 goes to infinity in equations 9 and 10. A conductor's index of refraction is infinite.

In the case of normal incidence, the reflected wave must have an electric field which is exactly opposite of the direction of the incoming wave. This is because all of the electric field is parallel to the surface of the conductor. Recall that the parallel component of the electric field is continuous and that there is no electric field inside the conductor. In order to satisfy both of these conditions the reflected wave must have equal and opposite electric field just after reflection. Thus, the net electric field right next to the conductor is zero, but it is not zero everywhere thereafter.

Figure 5 shows still images of two waves (green and blue) traveling past each other. The red wave is the sum of the two waves at various times. The red wave is the standing wave.
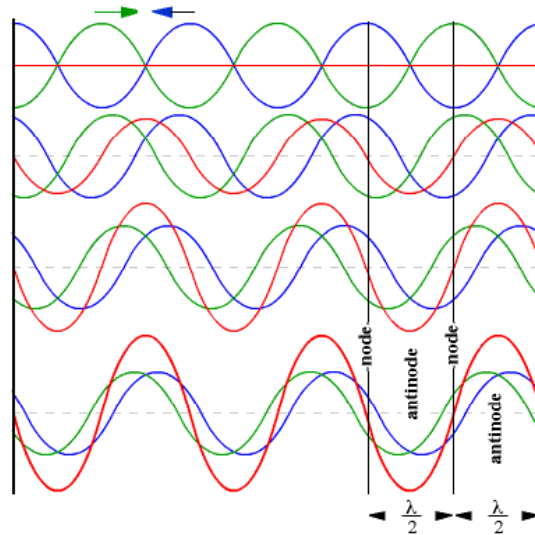

Figure 5 *This picture depicts a standing wave (in red) created by two waves each having the same wavelength traveling past each other at the same velocity.*

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* Parallel to the surface of the conductor.

2.4 Inference Patterns

Consider a wave incident normally on a flat slab of paraffin[†] or at least a slab of homogenous material. In addition to the wave reflected from the front surface there will be a refracted (transmitted) wave propagating into the paraffin slab. This transmitted wave will similarly give rise to reflected (back into the paraffin) and transmitted (out of the paraffin) waves at the rear surface of the slab. This second reflected wave will again be partially transmitted back through the front surface. If the original reflected wave and the second wave, which was transmitted through the paraffin, are in phase with each other they will add (constructive interference). If they are out of phase they will partially cancel (destructive interference).

Total constructive interference is when two waves are perfectly in phase. When this occurs, they combine to produce a wave with an amplitude, which is the sum of the first two. Total destructive interference is when two waves are exactly out of phase. The new amplitude of this wave will be the difference between the first two. The totally destructive amplitude will also be the lowest amplitude possible when combining these two waves. Likewise, the totally constructive amplitude will be the largest. See figure 6 below to see what total constructive and total destructive interference look like when both waves have the same amplitude.
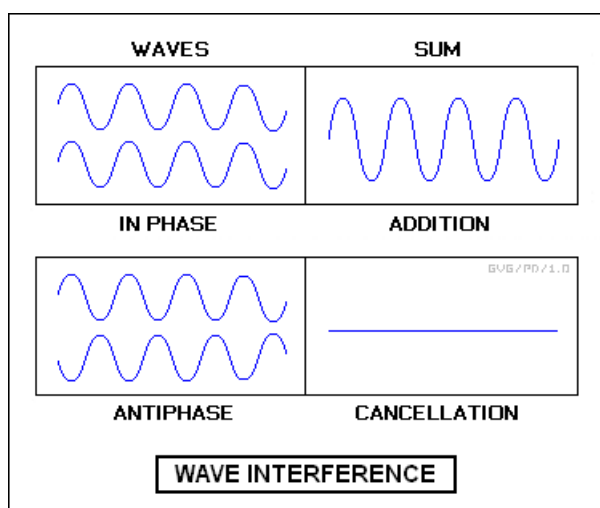

Figure 6 *The top left shows two waves in phase and the top right shows the addition of the two. Likewise, the bottom left shows two waves out of phase and the bottom right shows their cancellation.*

Let us examine the conditions necessary for constructive interference of the two reflected waves to occur. We have two waves, one (1) reflected from the front surface and the other (2) from the rear surface of the slab. We will suppose for now that upon reflection no change of phase occurs for either wave; the direction of travel is simply reversed instantaneously. Recall that the frequency f of the wave is unchanged as it enters or leaves a medium of different refractive index. Therefore, if the wave (2) transmitted into the slab and reflected off its rear surface has just undergone an integral number of oscillations when it reaches the front surface again, it will be exactly in phase with the wave (1) which has just reflected off the front surface. These two waves add together to give a strong reflection and we have that constructive interference must depend upon the total time spent by wave (2) within the slab

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[†] Paraffin is a particular type of hydrocarbon. Wax, including candles and crayons, are typically made primarily from paraffin wax.

since this determines the number of oscillations which have occurred while the wave is within the slab. The time Δt is given in terms of the wave velocity $v = \frac{c}{n}$ within the material by

$$\Delta t = \frac{2d}{v} = \frac{2dn}{c} \tag{13}$$

The number m of oscillations in the time Δt is just the frequency of oscillation times the amount of time Δt.

$$m = f\Delta t = \frac{2dnf}{c} = \frac{2dn}{\lambda_0} = \frac{2d}{\lambda} \tag{14}$$

where λ is the wavelength within the slab of refractive index n. Note that an alternative way of viewing this relation is that m is the number of wavelengths contained in the total path length $2d$. Thus, in the case considered above where there is no phase change for any of the reflections, the condition for constructive interference of the reflected waves is that m be an integer.

The above conclusions still apply if upon reflection the waves (1) and (2) do undergo a phase change, so long as the phase change is the same for both waves. It is possible, however, that a difference in phase shift can result from the two reflections. This should not be surprising since one reflection (at the front face) occurs for a beam moving from a medium of low refractive index (air) to one of high refractive index (paraffin), whereas the other reflection is for a beam in a high index medium passing to a low index medium. This is similar to the reflection of oscillations at the end of a rope. The reflections are different depending on whether or not the end of the rope is tied to a fixed support or is flapping free.

Suppose that one of the waves has its phase shifted by 180° and the other is not shifted at all. In this case, the relation derived above, that $2d = m\lambda$ (m an integer), will give a minimum in the reflected intensity rather than a maximum. Thus, it is possible to distinguish the case of only one wave being reflected with a 180° phase shift from that of either both or neither of the waves being shifted in phase by 180°.

A paraffin slab of variable thickness can be constructed in the lab by placing the two prisms together as shown in figure 7.
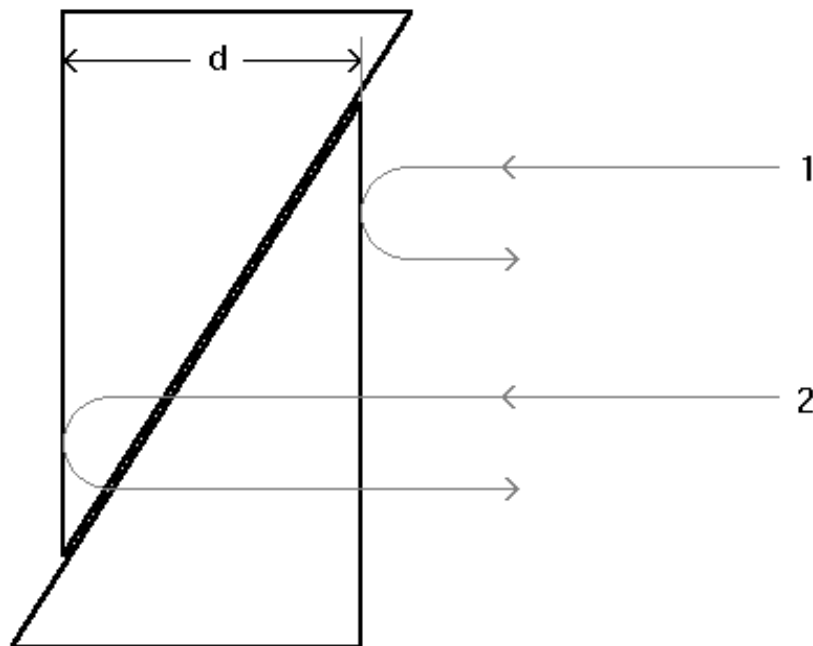
Figure 7 *Interference between two amplitudes: waves scattering from the front and the back of the block. Slide one prism relative to the second to change d.*

3 The Experiment

3.1 The Equipment
Required Equipment
- Microwave transmitter and receiver units
- Two 30°-60-°90° triangular solid (paraffin or lucite) prisms
- Goniometer (unit with angular scale and arm that moves about central pivot) – See figure 8 below.

Your basic equipment consists of a microwave generator (or transmitter), and a receiver. The transmitter consists of a Gunn-effect oscillator, a short piece of rectangular channel called a waveguide, which conducts the wave energy from one point to another, and a horn antenna, which directs the radiation into a fairly narrow beam in a selected direction.

The receiver gathers wave energy from a certain area over the wave fronts by means of its horn antenna and directs that wave energy into a short waveguide section containing a crystal rectifier. The rectifier is aligned along the electric vector **E** in the waveguide, and has the property of allowing current to flow along **E** in one polarity, but not in the other; it thus generates a pulsating direct current that is rectified and then displayed with the meter on the receiver unit.

The transmitter and receiver are mounted at the ends of two arms that rotate about a center stage. Since much of what you will study and measure involves the change in ray direction caused by some

physical process, the stage is provided with an angular scale. The process of interest will occur on the central stage.
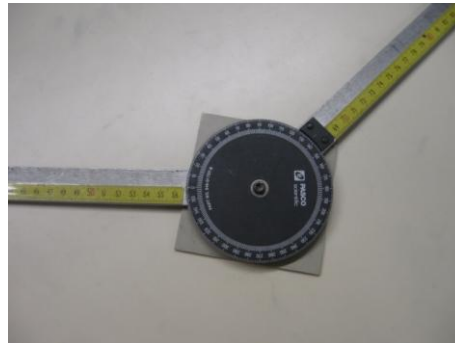

Figure 8 – *Close view of goniometer's angular scale.*

3.2 Measuring λ_0 in Free Space with Interference

The manufacturer specifies the frequency of the microwaves to be 10.525 GHz (1 GHz = 10^9 Hz.).

Question 4.2

What is the wavelength of the microwave with the frequency 10.525 GHz?

In order to determine the wavelength in free space of the microwave you must create a standing wave and find the distance between consecutive maximum values. Set up a reflector an integral number of 1/2 wavelengths away from the transmitter so as to produce a standing wave. The optimal distance from source to reflector will produce the largest ratio between measurements at node and anti-nodes. Use the detector probe to measure the distance between nodes (minimal field) and anti-node (maximal field). The receiver should not be used to measure the standing wave pattern because the horn would seriously alter the pattern. The small detector probe cable should be plugged into the receiver and the receiver horn pointed away from the transmitter. Note that the small detector probe has a very weak output. It will not output more than 0.1mA. In order to make the signal as strong as possible, make the distance between the transmitter and the reflector between 10 and 25 half-wavelengths.

Place the reflector, detector, and transmitter (in that order) on one arm of the goniometer as shown in figure 9. Move the detector in order to measure the distance between several (at least 4) consecutive peaks in the standing wave.

When you compute the wavelength you will find that is relatively short. Give some thought to what procedure will provide the best accuracy. For example, would it be better to measure the distances between successive minima and average them or to measure the distance between pairs of minima separated by several wavelengths? How well does the specified frequency (wavelength) agree with your measurements?

Hint: *Microwaves can be reflected from surfaces that are too rough to reflect images at optical wavelengths. "Specular" reflection (all reflected rays are at an angle equal to the angle of incidence) can be obtained so long as the wavelength of the microwaves is large relative to the scale of the roughness on the surface. This means the tabletop or objects upon it can reflect waves and confuse your results unless precautions are taken.*
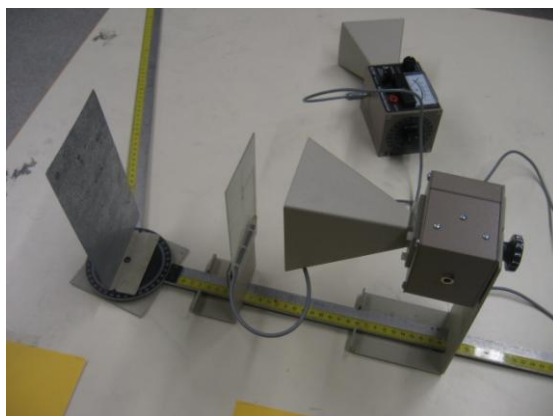

Figure 9 *In the picture above, the experimental setup for the standing wave is shown. The distance from the base of the horn of the transmitter to the reflector should be an integer number of half-wavelengths.*

3.3 Determining the Index of Refraction

Disconnect the detector probe, and place the transmitter and the receiver on the two arms of the goniometer. Place a single solid prism so that its face closest to the transmitter is perpendicular to the beam direction, so that the beam will not bend upon entering the prism. The opposite face of the prism is now at a known angle with respect to the beam, and therefore the beam will bend when exiting the prism. Make sure that the axis of the goniometer coincides with the location at which the beam exits and refraction occurs.

Change the angle of the receiver until you get a maximum signal. This angle will be θ_2 of the refracted beam (θ_1 is known from the geometry of the prism). With the wavelength (measured in section 3.2) and the two angles, you can use Snell's law to calculate the index of refraction of the prism.

In order to correct for systematic errors in determining the angles, you should repeat the experiment with the prism mirror inverted with respect to the beam axis (see figure 10), and average your results.
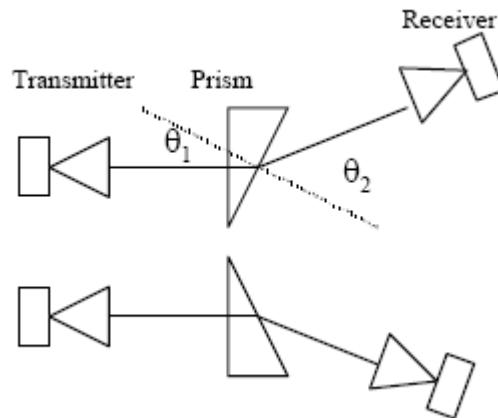
Figure 10 *The experimental setup for 3.3.*

Question 4.3

To which medium do n_1 and n_2 correspond? Which one do you know already?

3.4 Measuring λ in Material with Interference

Construct a slab of variable thickness by using two solid prisms as shown in figure 7. By observing the variations in reflected intensity with slab thickness, find the wavelength, λ, of the microwaves in the material. Construct a graph of *m* vs. *d* for maxima (or *m* + 1/2 vs. *d* for minima) using ORIGIN to obtain the wavelength.

Hint: *If your signal is weak, you may place a reflector on the backside of the two prisms.*

Question 4.4

Why does the detector find alternating maximum and minimum values as you vary the thickness of the slabs?

Question 4.5

Let us assume that you cannot find $m = 1$ as a reference point to begin your measurements. The best you can do is find a single maximum value and start consecutive measurements there. Starting at an arbitrary point and counting will change the functional form you will plot. How will $d = \dfrac{1}{2}m\lambda$ change?

Analysis

Compute the following:
- Wavelength with manufacturer specified frequency;
- Measured Wavelength in air;
- Determination of index of refraction of slabs;
- Wavelength in free space based on index of refraction and wavelength in slabs;
  Using equation 2 and the results from section 3.3 and 3.4 find λ_0.

Compute t–values for the following:
- Comparison of expected wavelength to that measured in air;
- Comparison of expected wavelength to that inferred from index of refraction of slabs and wavelength in slabs.

Conclusions

Highlight the themes of the lab and the physics the experiment verifies. You should discuss the errors you encounter in the lab and how you could improve the lab if you had to repeat it. If your results are unexpected or your t–values are high, you should identify possible explanations.

Hints on reports
- Print (preferably type) neatly—if your TA cannot read it, you could lose points.
- Be organized—if your TA cannot find it, you could lose points.
- Report your data, including plots—if your data is not in your report, your TA does know you did it.
- Record uncertainty.
- Propagate uncertainty.
- Write your final answers with proper significant figures.