Experiment 3: Resonance in LRC Circuits Driven by Alternating Current

Introduction

In last week's laboratory you examined the LRC circuit when constant voltage was applied to it. During this laboratory you will examine the same circuit but now you will apply alternating current (AC). Our functions of interest from last week contained time as the independent variable. These functions can generally be referred to as "time-domain" functions. In this laboratory the functions of interest will be in the "frequency-domain". That is, the independent variable is frequency.

1 Physics

1.1 Complex Impedance Revisited

This topic was previously introduced in Experiment 1, but a few of the impedances were simply stated. The form of the impedance for a capacitor and for an inductor will be proven below.

The relationships of current to voltage for capacitors and inductors were explained in Experiment 2 section 1.2. These relationships are rewritten here.

$$V_C = \frac{Q}{C} = \frac{1}{C}\int I dt \tag{1}$$

$$V_L = L\frac{dI}{dt} \tag{2}$$

Again the voltage/current relationship for the capacitor is best written in terms of derivatives. Differentiating each side with respect to time equation 1 becomes:

$$\frac{dV_C}{dt} = \frac{I}{C} \tag{3}$$

Now assuming that alternating current is flowing through these types of elements, we will prove the impedances which were only stated in Experiment 1. Recall complex impedance is only applicable when one uses AC.

Complex impedance is a measure of the ratio of voltage to current and it also contains the phase offset between the two. This relationship is summarized in the following equation.

$$Z = \frac{\tilde{V}}{\tilde{I}} \tag{4}$$

Now using the assumption that voltage is alternating:

$$\tilde{V} = V_0 e^{i\omega t} = Z\tilde{I} \tag{5}$$

Replacing the current in the inductor equation we have:

$$\tilde{V}_L = L\frac{d}{dt}\left(\frac{\tilde{V}_L}{Z_L}\right)$$

$$= \frac{i\omega L}{Z_L}V_{L0}e^{i\omega t} \tag{6}$$

$$= \frac{i\omega L}{Z_L}\tilde{V}_L$$

This implies:

$$Z_L = i\omega L \tag{7}$$

In the same way, the voltage/current relationship for capacitors becomes:

$$\tilde{I} = C\frac{d}{dt}\left(V_{C0}e^{iwt}\right)$$

$$= i\omega C\left(V_{C0}e^{iwt}\right) \tag{8}$$

$$= i\omega C\left(Z_C\,\tilde{I}\right)$$

This implies:

$$Z_C = \frac{1}{i\omega C} \tag{9}$$

2 Mathematical Analysis

2.1 Properties of Complex Numbers

Complex numbers are of the form $c = a + ib$ where a and b are real numbers and $i = \sqrt{-1}$. A complex number can be visualized in a two-dimensional plane as it is in Figure 1. Figure 1a shows a complex number in Cartesian coordinates. Figure 1b shows the same complex number in polar coordinates which is of the form: $c = re^{i\theta}$. From well-known trigonometric relations we have the follow relationships between Cartesian and polar coordinates.

$$r = \sqrt{a^2 + b^2} \qquad a = r\cos(\theta)$$

$$\theta = \arctan\left(\frac{b}{a}\right) \qquad b = r\sin(\theta) \tag{10}$$
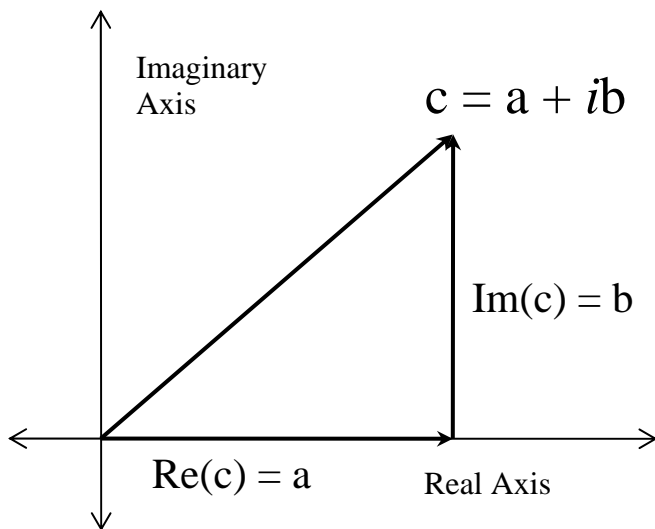
**Figure 1a** *A complex vector, c, is shown in Cartesian coordinates.*
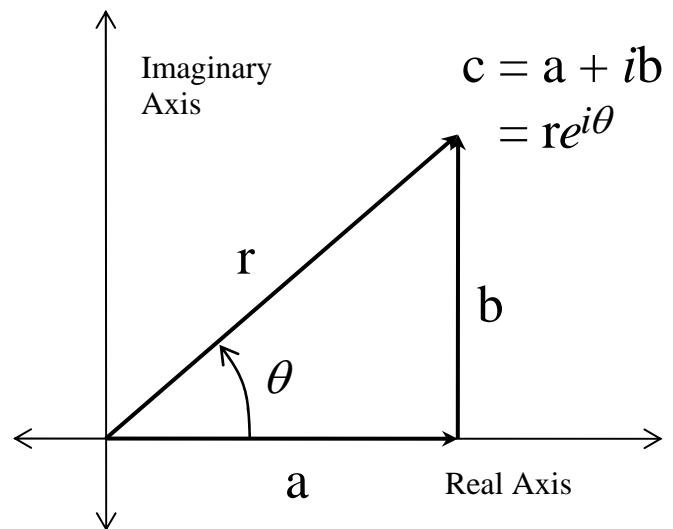
Figure 1b *A complex vector, c, is shown in polar coordinates*

2.2 The LRC Series Circuit Driven by AC

Figure 2 is a schematic of the circuit you will construct during this laboratory. Using Kirchhoff's Law on the single loop of this circuit we have:

$$V_S + V_L + V_R + V_C = 0 \qquad \textbf{(11)}$$

Notice that in Figure 2 there are three sources of resistance. The voltage drops of these three are all included in V_R. Also, in subsequent equations R represents the total resistance in the loop and R_R represents the resistance of only the resistor.
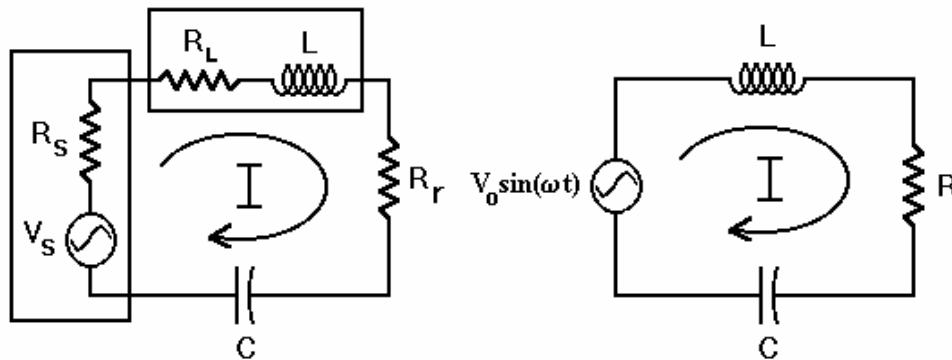
Figure 2 *The complete LRC series circuit is drawn in (a) with all three resistors, including V_S for the Signal Generator, R_L for the Inductor and R_R for the resistor which you put into the circuit. An equivalent circuit is drawn in (b) with the single resistor $R = (R_S + R_L + R_R)$ representing the total of all of the circuit resistance. The internal voltage source, $V_S(t)$, is a sine wave, and the current through all of the components is I.*

Replacing the voltage of the circuit elements in equation 11 we have:

$$V_S - I(Z_L + Z_R + Z_C) = 0 \rightarrow V_S = I\left(i\omega L + R + \frac{1}{(i\omega C)}\right) \tag{12}$$

The polar expression for the total impedance will turn out to be the most useful. Using the equations in block 10 we have:

$$\left[R + i\left(\omega L - \frac{1}{\omega C}\right)\right] = \sqrt{R^2 + L^2\left(\omega - \frac{1}{\omega LC}\right)^2}\, \exp\left[i\arctan\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)\right] \tag{13}$$

We now wish to substitute ω_0 and Q, which are the same parameters from last week, into the expression for the magnitude of the total impedance and then simplify. First recall:

$$\omega_0 = \sqrt{\frac{1}{LC}}, \text{ and } Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{\omega_0 RC} \tag{14}$$

$$|Z_{Total}| = \sqrt{R^2 + L^2\left(\omega - \frac{1}{\omega LC}\right)^2}$$

$$= R\sqrt{1 + \frac{L^2}{R^2}\left(\omega - \frac{\omega_0^2}{\omega}\right)^2}$$

$$= R\sqrt{1 + \frac{L^2\omega_0^2}{R^2}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2} \tag{15}$$

$$= R\sqrt{1 + \frac{L}{R^2 C}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}$$

$$= R\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}$$

We can also simplify the angle as well. The result is stated below.

$$\arctan\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right) = \arctan\left[\frac{1}{R}\sqrt{\frac{L}{C}}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]$$

$$= \arctan\left[Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right] \tag{16}$$

Now let us find the current based on the impedance and the assumption that the voltage is of the form, $\tilde{V} = V_0 e^{i\omega t}$.

$$\tilde{V} = \tilde{I}Z \rightarrow V_0 e^{iwt} = |Z_{Total}|\exp\left[i\arctan\left(Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right)\right]\tilde{I}$$

$$\rightarrow \tilde{I} = \frac{V_0}{|Z_{Total}|}\exp\left[i\left(\omega t - \arctan\left(Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right)\right)\right]$$

(17)

2.3 Voltage Over the Resistor

The voltage over the resistor is the current times the impedance of the resistor (simply its own resistance). Let us evaluate its magnitude:

$$|V_R| = |I||Z_R| = V_0\frac{|Z_R|}{|Z_{Total}|}$$

$$= \frac{V_0 R_R}{R\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}}$$

(19)

Notice the impedance of the resistor, R_R, is only the resistance of the resistor and not the total resistance, R. Also, notice that the voltage drop over the resistor is greatest when $\omega = \omega_0$. This frequency is called the resonance frequency.

Below in figure 3 there is a graph of this function. The "half power points", ω_1 and ω_2, are also shown in figure 3. Since $P \propto V^2$, the half power points occur when the voltage decreases to $1/\sqrt{2}$ of its peak value, or

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}} \rightarrow Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2 = 1$$

(20)

which happens twice, $\omega = \omega_1$, below resonance, and $\omega = \omega_2$, above resonance. If you solve the two quadratic equations, you find that:

$$Q = \frac{\omega_0}{\omega_2 - \omega_1}$$

(21)

This is the quick way to measure **Q**, and it is exact, having no approximations in its derivation.

During your experiment you will determine the resonance frequency visually by varying the frequency until the amplitude is at its highest value. You will "eye-ball" the frequency and for that

reason the uncertainty of your measurement is half of the range over which you are unsure if your maximum amplitude is changing.

Question 3.1

What is the propagated uncertainty for Q in Equation 21?

Question 3.2

When determining the linear resonance frequency, you determine that the amplitude is greatest between 21.03 Hz and 22.73 Hz. What do you record for the linear resonance frequency (with its uncertainty)? What is the angular resonance frequency (with its uncertainty)?
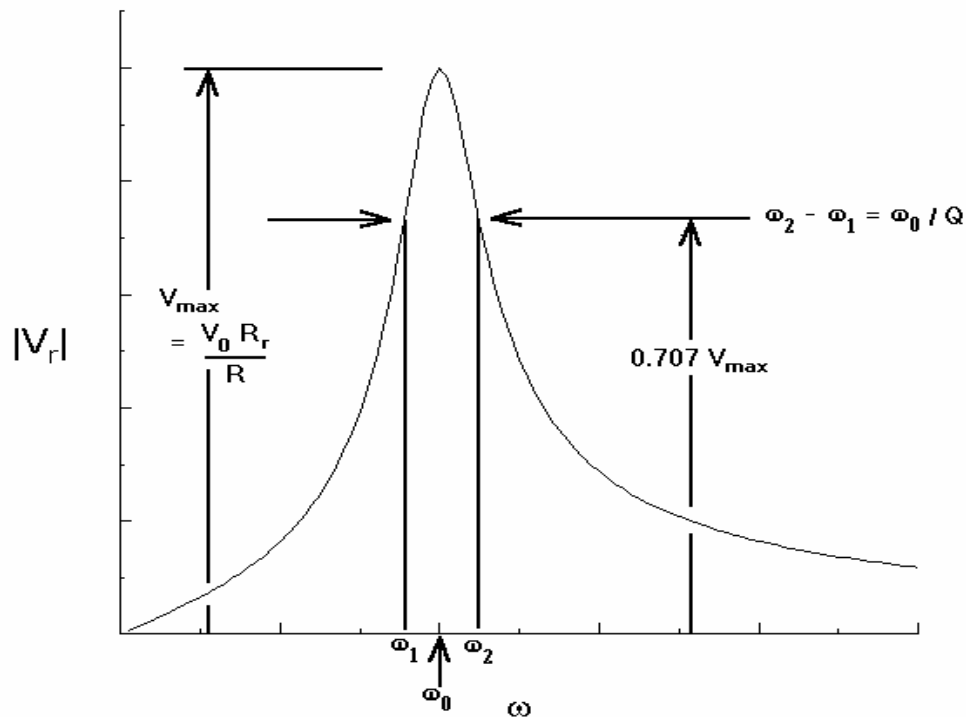
Figure 3 *The Response Function of the LRC series-resonant circuit. A constant amplitude signal (V_0) drives the circuit, and changes in frequency lead to changes in the current that flows, which you measure as the voltage drop across the resistor. The half-power points, ω_1 and ω_2, are shown along with their relation to the quality factor Q and the resonant frequency, ω_0, which corresponds to the peak of the response function. As the Q increases, the width of the function decreases.*

2.4 The Q Multiplier

At the resonance frequency there is a very easy way to determine Q. The ratio of the amplitude of voltage of the capacitor to the amplitude of the driving voltage is simply Q. The easiest way to derive this result is to find the magnitude of the voltage drop across the capacitor and divide by the driving voltage amplitude.

$$
\begin{aligned}
V_{CO} &\equiv \left| V_C(\omega_0) \right| \\
&= \frac{V_0}{\left| Z_{Total} \right| \omega_0 C} = \frac{V_0}{\omega_0 RC} \\
&= \frac{V_0}{R}\sqrt{\frac{L}{C}} = V_0 Q \\
&\rightarrow Q = \frac{V_{C0}}{V_0}
\end{aligned}
\tag{22}
$$

In this derivation we recall equation 15 and use the fact that when $\omega = \omega_0$ $\left| Z_{Total} \right| = R$.

Question 3.3

What is the propagated error for Q in equation 22?

2.5 Voltage Phase Offsets of Circuit Elements

In this lab you will be asked to measure the phase offset from the driving voltage of the circuit elements at various important frequencies. Below the phase offsets as a function of angular frequency for the resistor, capacitor and inductor are derived. Having already found the current of the circuit this process is fairly simple although the simplification looks messy. The voltage drop over each element is the product of the current, which is found in equation 17, and the impedance of that element. Thus, for a resistor

$$
\begin{aligned}
\tilde{V}_R &= \tilde{I}\,Z_R = \tilde{I}\,R_R \\
&= \frac{V_0 R_R}{\left| Z_{Total} \right|} \exp\left[i\left(\omega t - \arctan\left(Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) \right) \right) \right]
\end{aligned}
\tag{23}
$$

Define:

$$
\phi_R(\omega) \equiv -\arctan\left(Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) \right)
\tag{24}
$$

Notice that at the resonance frequency the phase offset of the resistor is zero.

Below in Figure 4 there is a graph of this phase offset. Recall that ω_1 and ω_2 are two solutions to the following equation:
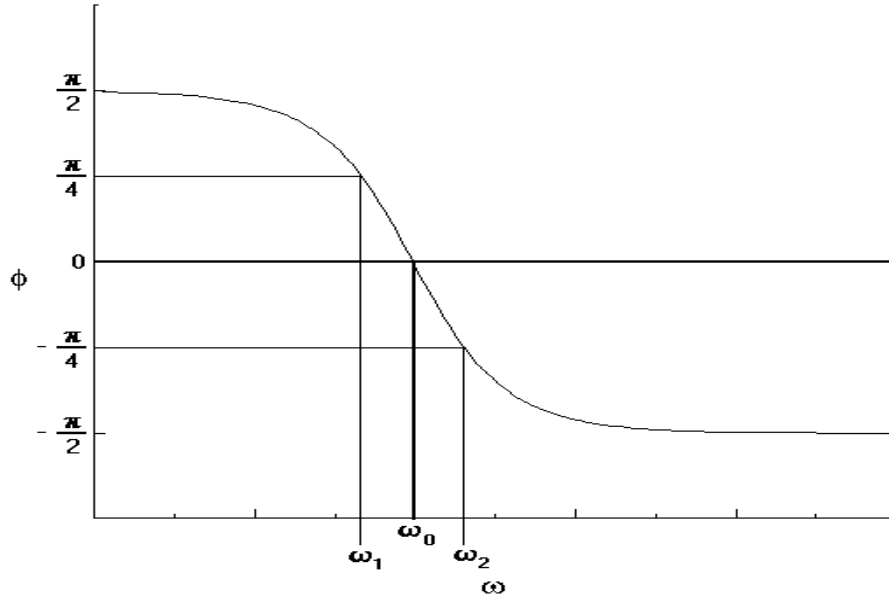
$$Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2 = 1 \rightarrow Q\left|\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right| = 1 \tag{25}$$

$$Q\left(\frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1}\right) = -1 \tag{26}$$

$$Q\left(\frac{\omega_2}{\omega_0} - \frac{\omega_0}{\omega_2}\right) = 1 \tag{27}$$

Placing equations 26 and 27 into equation 24 we determine

$$\phi_R(\omega_1) \equiv -\arctan(-1) = \frac{\pi}{4} \qquad\qquad \phi_R(\omega_2) \equiv -\arctan(+1) = -\frac{\pi}{4} \tag{28}$$

Figure 4 *Phase (ϕ_R) as a function of angular frequency. Note that the phase shift is positive at low frequencies and negative for high frequencies. Resonance is at zero phase shift, and the half-lower points are at plus and minus $\pi/4$.*

Following the same procedure the phase offsets for the inductor and capacitor are:

$$\tilde{V}_L = \tilde{I}\,Z_L = \tilde{I}\,i\omega L$$
$$= \frac{V_0\omega L}{|Z_{Total}|}\exp\left[i\left(\omega t - \arctan\left(Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right) + \frac{\pi}{2}\right)\right] \tag{29}$$

Define:

$$\phi_L(\omega) \equiv -\arctan\left(Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right) + \frac{\pi}{2} \tag{30}$$

$$\tilde{V}_C = \tilde{I}\,Z_C = \tilde{I}\,\frac{-i}{\omega C}$$
$$= \frac{V_0}{|Z_{Total}|\omega C}\exp\left[i\left(\omega t - \arctan\left(Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right) - \frac{\pi}{2}\right)\right] \tag{31}$$

Define:

$$\phi_C(\omega) \equiv -\arctan\left(Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right) - \frac{\pi}{2} \tag{32}$$

Notice that we have used the identity $i = e^{i\frac{\pi}{2}}$. Also notice that the resonance frequency $\phi_C(\omega_0) = -\pi/2$ and $\phi_L(\omega_0) = \pi/2$.

3   The Experiment

3.1   Target Values for R, L and C

In the same way you selected your circuit elements in experiment 2 you must do so for this laboratory as well. For this laboratory Q should be between 4 and 10. ω_0 should be between 10^4 and $5\cdot10^5$ rad/s (or the linear frequency should be between 1.6 and 80 kHz). Below is a table with the listed options for R, L and C.

Circuit Element	Options
Resistor (R)	10, 51, 100, 200, 510, and 1000 Ω
	There is also a variable resistor.[*]
Inductor (L)	0.55, 2.5, 5 and 10mH (milli Henry)
Capacitor (C)	0.001, 0.01, 0.1 and 1.0μF (micro Farad)

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[*] When using the variable resistor, measure your final resistance with the multimeter.

When you arrive in the lab measure the resistance and capacitance of the elements you have selected and determine your theoretical values of Q and ω_0 with your measured values. Also, before you begin using the oscilloscope, CALIBRATE it. You must calibrate both voltage channel 1 and channel 2 because you are going to be using both.

Question 3.4 – Mandatory

What are your target values for R, L and C? What values of Q and ω_0 do you expect?

Hint: if your Q from experiment 2 was between 4 and 10, then you can re-use those values.

3.2 Determining Q and ω_0 Experimentally

Figure 5 shows generally how the LRC series current is to be assembled. However, you will also be measuring the voltage of individual circuit elements (i.e. the capacitor, resistor and inductor) at different times. Recall that one lead from the signal generator is ground. There will also be a ground from the oscilloscope, which displays the voltage over that circuit element. If you do not change the order of circuit elements you may short-circuit your loop. In Experiment 1, Section 1.4 "Common Grounding", this concept is explained. If you do not remember, please re-read this section.

Set up the circuit in Figure 5 and connect the oscilloscope across R_R (i.e., measure V_R). This is proportional to the current flowing in the circuit and should show a large amplitude when you change the frequency of the signal generator (sinusoidal wave) so as to pass through resonance, $\omega_0 (= 2\pi f_0)$.

Now find ω_1 and ω_2 by changing frequency until the sinusoidal wave amplitude V_R decreases to 0.707 of the voltage at resonance. Verify that equation 21 gives you a value of Q close to what you calculated.
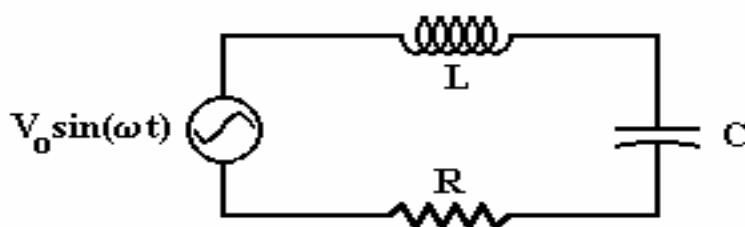
Figure 5 *Setup for LRC circuit*

3.3 Graphing Frequency Response

Now measure several values of the amplitude of the resistor's voltage, V_R, at different frequencies, recording them in a table. Make sure that at least 3 points are below ω_1, 10 points between ω_1 and ω_2 and 3 points above ω_2 (include the ω_0, ω_1, and ω_2 points as well). Plot V_R as a function of ω for your data (with error bars). Fit your data to the functional form of equation 19:

$$y = \frac{A}{\sqrt{1 + B^2 \left(\dfrac{x}{C} - \dfrac{C}{x} \right)^2}}$$

and from the coefficients of the fit, determine values (with error) for ω_0 and Q.

Question 3.5

What are the coefficients A, B, and C in the equation above? What is the independent variable x?

3.4 Q Dependence on R

Restore your circuit to the original form (cf. Figure 5), but increase R to some new value, for the total resistance of the circuit. Since you expect that $Q = \dfrac{1}{R}\sqrt{\dfrac{L}{C}}$, the new value of Q should be $\dfrac{R_{Old}}{R_{New}}$ of your original Q. Verify by finding the new ω_1 and ω_2, and check that ω_0 does not change.

Question 3.6

What is the relationship between Q_{New} and Q_{Old} if only the resistance is changed from R_{New} to R_{Old} ?
Hint: *L and C are not changed. Write out the equation for Q_{New} and Q_{Old} and substitute.*

3.5 The Q Multiplier

Common ground leads on our electronic equipment (the oscilloscope and signal generator) are typically the same or very close in voltage.   Thus you cannot connect the scope across the capacitor in Figure 5. If the oscilloscope ground is connected to the node between R_R and C, then you will effective remove the resistor from the circuit. Rearrange the circuit so that the capacitor is the last element. This allows you to measure the voltage across the capacitor without "shorting" the resistor.   For diagrams and a more thorough explanation review Experiment 1, Section 1.4 (Common Grounding).

When viewing more than one signal, the oscilloscope may picture the two or at least one of the signals as fuzzy or "jumpy." This is certainly true if each signal has a slightly different ground voltage. The oscilloscope can only make one steady at a time when the two grounds are different. You can make the grounds exactly the same by connecting the TTY output on the signal generator (the middle BNC port) to the lower right external trigger on the oscilloscope. Make sure to turn the two switches above the external trigger to external. Keep this connected for section 3.6 as well.

Now measure V_{CO}, the amplitude of the voltage across the capacitor *at resonance*. (Make sure that you stay at the resonant frequency for this section of the experiment). Next, disconnect the entire circuit from the generator and measure a proper value for V_S (the internal voltage of the generator). Determine the Q of the circuit by taking the ratio of the capacitor voltage amplitude at resonance to the drive

voltage (cf. Equation 22). Compare this value of Q to the expected value found earlier. Comment on why the circuit was removed from the signal generator before measuring V_S.

Hint: *If the amplitude of the source signal is too great for you to read on the oscilloscope screen, then you must turn the <u>signal voltage</u> knob at the lower right on the face of the signal generator. Afterward you can take the proper measurement.*

Analysis
Compute the following:
- Theoretical value of Q;
- Theoretical value of ω_0 ;
- Theoretical bandwidth $\omega_1 - \omega_2$;
- Experimental bandwidth $\omega_1 - \omega_2$;
- Experimental Q using measured $\omega_1 - \omega_2$ and ω_0 ;
- Compare the ratio of Q_{New} to Q_{Old} to the inverse ratio of R_{New} to R_{Old}.

Compute t–values for the following:
- Experimental ω_0 with theoretical ω_0 ;
- Q from bandwidth with theoretical Q;
- Q from fit with theoretical Q.

Conclusions
    Highlight the themes of the lab and the physics the experiment verifies. You should discuss the errors you encounter in the lab and how you could improve the lab if you had to repeat it. If your results are unexpected or your t–values are high, you should identify possible explanations.

    Which method is 'best' at determining Q? What defines 'best' in your analysis?

Hints on reports
- Print (preferably type) neatly—if your TA cannot read it, you could lose points.
- Be organized—if your TA cannot find it, you could lose points.
- Report your data, including plots—if your data is not in your report, your TA does know you did it.
- Record uncertainty.
- Propagate uncertainty.
- Write your final answers with proper significant figures.