PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #6

(1) The Blume-Capel model is a spin-1 version of the Ising model, with Hamiltonian

$$H = -J \sum_{\langle ij \rangle} S_i \, S_j - \Delta \sum_i S_i^2 \ ,$$

where $S_i \in \{-1\,,\,0\,,\,+1\}$ and where the first sum is over all links of a lattice and the second sum is over all sites. It has been used to describe magnetic solids containing vacancies ($S = 0$ for a vacancy) as well as phase separation in $^4\mathrm{He} - {}^3\mathrm{He}$ mixtures ($S = 0$ for a ^{4}He atom). This problem will give you an opportunity to study and learn the material in §§5.2,3 of the notes. For parts (b), (c), and (d) you should work in the thermodynamic limit. The eigenvalues and eigenvectors are such that it would shorten your effort considerably to use a program like `Mathematica` to obtain them.

(a) Find the transfer matrix for the $d = 1$ Blume-Capel model.

(b) Find the free energy $F(T, \Delta, N)$.

(c) Find the density of $S = 0$ sites as a function of T and Δ.

(d) *Exciting!* Find the correlation function $\langle \, S_j \, S_{j+n} \, \rangle$.

(2) DC Comics superhero Clusterman and his naughty dog Henry are shown in fig. 1. Clusterman, as his name connotes, is a connected diagram, but the diagram for Henry contains some disconnected pieces.

(a) Interpreting the diagrams as arising from the Mayer cluster expansion, compute the symmetry factor s_γ for Clusterman.

(b) What is the *total* symmetry factor for Henry and his disconnected pieces? What would the answer be if, unfortunately, another disconnected piece of the same composition were to be found?

(c) What is the lowest order virial coefficient to which Clusterman contributes?

(3) The Tonks gas is a one-dimensional generalization of the hard sphere gas. Consider a one-dimensional gas of indistinguishable particles of mass m interacting via the potential

$$u(x - x') = \begin{cases} \infty & \text{if } |x - x'| < a \\ 0 & \text{if } |x - x'| \geq a \ . \end{cases}$$

Let the gas be placed in a finite volume L. The hard sphere nature of the particles means that no particle can get within a distance $\frac{1}{2}a$ of the ends at $x = 0$ and $x = L$. That is, there

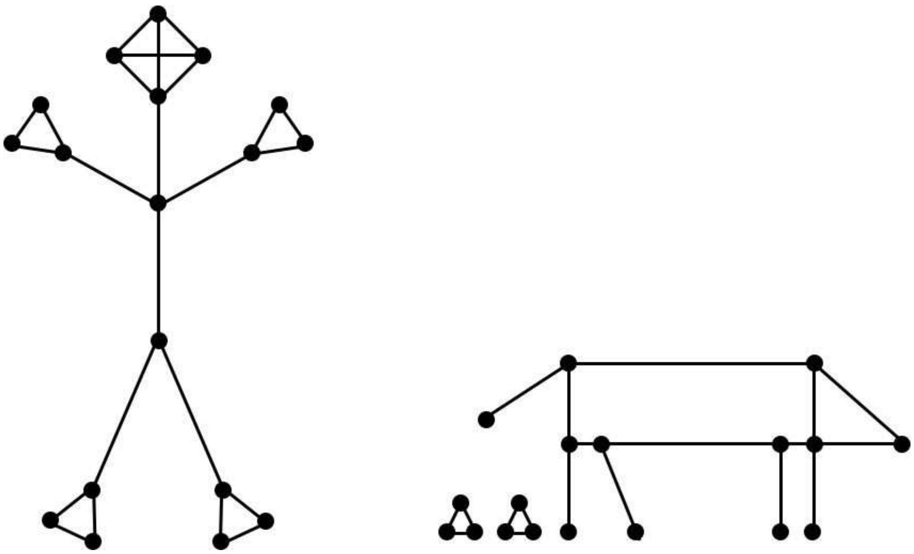

Figure 1: Mayer expansion diagrams for Clusterman and his dog.

is a one-body potential $v(x)$ acting as well, where

$$v(x) = \begin{cases} \infty & \text{if } x < \frac{1}{2}a \\ 0 & \text{if } \frac{1}{2}a \le x \le L - \frac{1}{2}a \\ \infty & \text{if } x > L - \frac{1}{2}a \ . \end{cases}$$

(a) Compute the N particle partition function $Z(T, L, N)$ for the Tonks gas. Present a clear derivation of your result. Please try your best to solve this, either by yourself or in collaboration with classmates. If you get stuck there is an $Sk\log$*Wiki* page on the web on "1-dimensional hard rods" where you can look up the derivation.

(b) Find the equation of state $p = p(T, L, N)$.

(c) Find the grand potential $\Omega(T, L, \mu)$. *Hint : There is a small subtlety you must appreciate in order to obtain the correct answer.*

(4) In §5.5.3 of the notes, the virial equation of state is derived for a single species of particle.

(a) Generalize eqn. 5.160 to the case of two species interacting by $u_{\sigma\sigma'}(r)$, where σ and σ' are the species labels.

(b) For a plasma, show from Debye-Hückel theory that the pair correlation function is $g_{\sigma\sigma'} \propto \exp\big(-\sigma\sigma' q^2 \phi(r)/k_{\rm B}T\big)$, where σ and σ' are the signs of the charges (magnitude q), and $\phi(r)$ is the screened potential due to a unit positive test charge.

(c) Find the equation of state for a three-dimensional two-component plasma, in the limit where T is large.