PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #1

(1) Prove that for $x \geq 0$ and $y \geq 0$ that

$$(x - y)\,(\ln x - \ln y) \geq 0 \ .$$

(2) A *Markov chain* is a process which describes transitions of a discrete stochastic variable occurring at discrete times. Let $P_i(t)$ be the probability that the system is in state i at time t. The evolution equation is

$$P_i(t+1) = \sum_j Q_{ij}\, P_j(t) \ .$$

The *transition matrix* Q_{ij} satisfies $\sum_i Q_{ij} = 1$ so that the total probability $\sum_i P_i(t)$ is conserved. The element Q_{ij} is the *conditional probability* that for the system to evolve to state i given that it is in state j. Now consider a group of Physics graduate students consisting of three theorists and four experimentalists. Within each group, the students are to be regarded as indistinguishable. Together, the students rent two apartments, A and B. Initially the three theorists live in A and the four experimentalists live in B. Each month, a random occupant of A and a random occupant of B exchange domiciles. Compute the transition matrix Q_{ij} for this Markov chain, and compute the average fraction of the time that B contains two theorists and two experimentalists, averaged over the effectively infinite time it takes the students to get their degrees. *Hint:* Q is a 4×4 matrix.

(3) Consider a q-state generalization of the Kac ring model in which $\mathbb{Z}_q$ spins rotate around an N-site ring which contains a fraction $x = N_{\rm F}/N$ of flippers on its links. Each flipper cyclically rotates the spin values: $1 \to 2 \to 3 \to \cdots \to q \to 1$ (hence the clock model symmetry $\mathbb{Z}_q$).

(a) What is the Poincare recurrence time?

(b) Make the *Stosszahlansatz*, *i.e.* assume the spin flips are stochastic random processes. Then one has

$$P_\sigma(t+1) = (1-x)\, P_\sigma(t) + x\, P_{\sigma-1}(t) \ ,$$

where $P_0 \equiv P_q$. This defines a Markov chain

$$P_\sigma(t+1) = Q_{\sigma\sigma'}\, P_{\sigma'}(t) \ .$$

Decompose the transition matrix Q into its eigenvectors. *Hint:* The matrix may be diagonalized by a simple Fourier transform.

(c) The eigenvalues of Q may be written as $\lambda_\alpha = e^{-1/\tau_\alpha}\, e^{-i\delta_\alpha}$, where τ_α is a relaxation time and δ_α is a phase. Find the spectrum of relaxation times. What is the longest finite relaxation time?

(d) Suppose all the spins are initially in the state $\sigma = q$. Write down an expression for $P_\sigma(t)$ for all subsequent times $t \in \mathbb{Z}^+$. Plot your results for different values of x.

(e) Simulate the system and show how it exhibits both equilibration and Poincaré recurrence.

Hint: It may be helpful to study carefully the solution to problem 5.1 (*i.e.* problem 1 of assignment 5) from F08 Physics 140A. You can access this through the link to the 140B website on the 210A course web page.

(4) A ball of mass m executes perfect one-dimensional motion along the symmetry axis of a piston. Above the ball lies a mobile piston head of mass M which slides frictionlessly inside the piston. Both the ball and piston head execute ballistic motion, with two types of collision possible: (i) the ball may bounce off the floor, which is assumed to be infinitely massive and fixed in space, and (ii) the ball and piston head may engage in a one-dimensional elastic collision. The Hamiltonian is

$$H = \frac{P^2}{2M} + \frac{p^2}{2m} + MgX + mgx \ ,$$

where X is the height of the piston head and x the height of the ball. Another quantity is conserved by the dynamics: $\Theta(X - x)$. *I.e.*, the ball always is below the piston head.

(a) Choose an arbitrary length scale L, and then energy scale $E_0 = MgL$, momentum scale $P_0 = M\sqrt{gL}$, and time scale $\tau_0 = \sqrt{L/g}$. Show that the dimensionless Hamiltonian becomes

$$\bar{H} = \tfrac{1}{2}\bar{P}^2 + X + \frac{\bar{p}^2}{2r} + rx \ ,$$

with $r = m/M$, and with equations of motion $d\bar{X}/d\bar{t} = \partial\bar{H}/\partial\bar{P}$, *etc.* (Here the bar indicates dimensionless variables: $\bar{P} = P/P_0$, $\bar{t} = t/\tau_0$, *etc.*) What special dynamical consequences hold for $r = 1$?

(b) Compute the microcanonical average piston height $\langle\bar{X}\rangle$. The analogous dynamical average is

$$\langle\bar{X}\rangle_T = \lim_{T\to\infty} \frac{1}{T} \int\limits_0^T dt\, \bar{X}(t) \ .$$

You should use a *restricted* microcanonical ensemble where $\bar{X} \geq \bar{x}$ is enforced. When computing microcanonical averages, it is helpful to use the Laplace transform, discussed toward the end of §3.3 of the notes. (It is possible to compute the microcanonical average by more brute force methods as well.)

(c) Compute the microcanonical average of the rate of collisions between the ball and the floor. Show that this is given by

$$\Big\langle \sum_i \delta(t - t_i) \Big\rangle = \big\langle \Theta(v)\, v\, \delta(x - 0^+) \big\rangle \ .$$

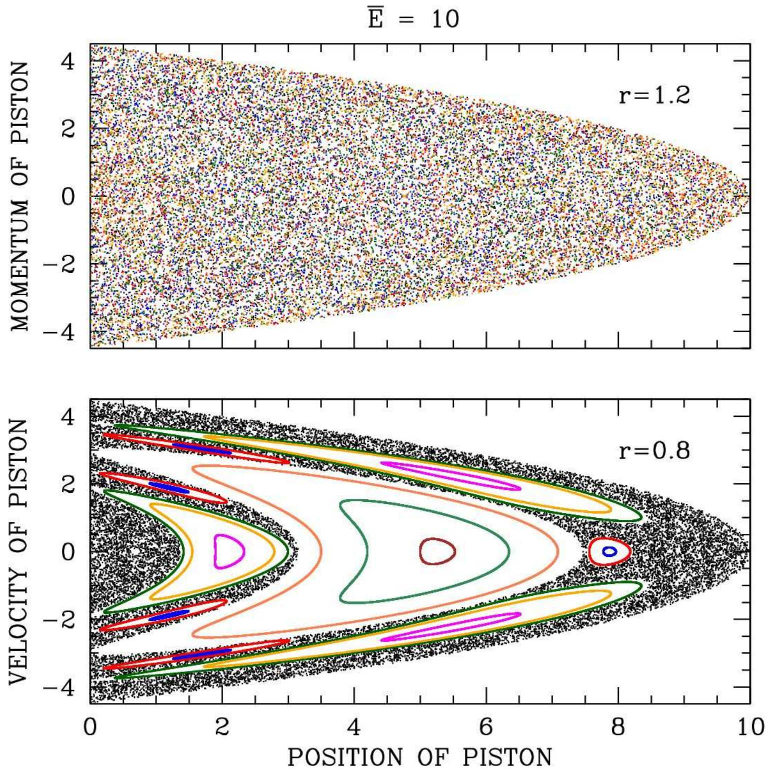

Figure 1: Poincaré sections for the ball and piston head problem. Each color corresponds to a different initial condition. When the mass ratio $r = m/M$ exceeds unity, the system apparently becomes ergodic.

The analogous dynamical average is

$$\langle \gamma \rangle_T = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \sum_i \delta(t - t_i) \ ,$$

where $\{t_i\}$ is the set of times at which the ball hits the floor.

(d) How do your results change if you do not enforce the dynamical constraint $\bar{X} \geq \bar{x}$?

(e) Write a computer program to simulate this system. The only input should be the mass ratio r (set $\bar{E} = 10$ to fix the energy). You also may wish to input the initial conditions, or perhaps to choose the initial conditions randomly (all satisfying energy conservation, of course!). Have your program compute the microcanonical as well as dynamical averages in parts (b) and (c). Plot out the Poincaré section of $\bar{P}$ *vs.* $\bar{X}$ for those times when the ball hits the floor. Investigate this for several values of r. Just to show you that this is interesting, I've plotted some of my own numerical results in fig. 1.