PHYSICS 110A : CLASSICAL MECHANICS
HW 3 SOLUTIONS

(1) Taylor 6.6

(a)
Here we are working with $ds = \sqrt{dx^2 + dy^2}$. For a function $y = y(x)$ we will pull out a dx to have:

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = dx\sqrt{1 + (y')^2}.$$

(b)
Similarly for a function $x = x(y)$ we have:

$$ds = \sqrt{dx^2 + dy^2} = dy\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = dy\sqrt{1 + (x')^2}.$$

(c)
Now for cylindrical coordinates we should remember the line element looks like:

$$d\vec{l} = dr\hat{r} + rd\phi\hat{\phi} + dz\hat{z}.$$

So for a function $r = r(\phi)$ we have:

$$ds = \sqrt{dr^2 + r^2 d\phi^2} = d\phi\sqrt{r^2 + \left(\frac{dr}{d\phi}\right)^2} = d\phi\sqrt{r^2 + (r')^2}.$$

An alternate and equivalent way to do this is to begin from the Euclidean distance in (a) and write x and y (and z, if needed) in the coordinate system you are transforming to. In this case

$$x = r\cos\phi \;\text{ and }\; y = r\sin\phi$$

so

$$dx = \tfrac{\partial x}{\partial r}dr + \tfrac{\partial x}{\partial \phi}d\phi = \cos\phi dx - r\sin\phi d\phi$$

$$dy = \tfrac{\partial y}{\partial r}dr + \tfrac{\partial y}{\partial \phi}d\phi = \sin\phi dx + r\cos\phi d\phi$$

in which case, after some algebra and use of basic trig identities (which you should go through), we have

$$ds = \sqrt{(\cos^2\phi + \sin^2\phi)dr^2 + r^2(\cos^2\phi + \sin^2\phi)d\phi^2} = \sqrt{dr^2 + r^2 d\phi^2}$$

which recapitulates what we have above in a completely equivalent manner. This method is often useful when you don't know the specific measure or line element of a specific coordinate system.

(d)
And for a function $\phi = \phi(r)$ we have:

$$ds = \sqrt{dr^2 + r^2 d\phi^2} = dr\sqrt{1 + r^2\left(\frac{d\phi}{dr}\right)^2} = dr\sqrt{1 + r^2\left(\phi'\right)^2}.$$

(e)
For a function $\phi = \phi(z)$ we have:

$$ds = \sqrt{dz^2 + R^2 d\phi^2} = dz\sqrt{1 + R^2\left(\frac{d\phi}{dz}\right)^2} = dz\sqrt{1 + R^2\left(\phi'\right)^2}.$$

(f)
For a function $z = z(\phi)$ we have:

$$ds = \sqrt{dz^2 + R^2 d\phi^2} = d\phi\sqrt{R^2 + \left(\frac{dz}{d\phi}\right)^2} = dz\sqrt{R^2 + \left(z'\right)^2}.$$

Finally for spherical coordinates we have:

$$d\vec{l} = dr\hat{r} + r d\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}.$$

(g)
So for a function $\theta = \theta(\phi)$ we have:

$$ds = \sqrt{R^2 d\theta^2 + R^2\sin^2\theta d\phi^2} = Rd\phi\sqrt{\sin^2\theta + \left(\frac{d\theta}{d\phi}\right)^2} = Rd\phi\sqrt{\sin^2\theta + \left(\theta'\right)^2}.$$

(h)
And for a function $\phi = \phi(\theta)$ we have:

$$ds = \sqrt{R^2 d\theta^2 + R^2\sin^2\theta d\phi^2} = Rd\theta\sqrt{1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2} = Rd\theta\sqrt{1 + \sin^2\theta\left(\phi'\right)^2}.$$

(2) Taylor 6.11

We want to find the path $y = y(x)$ for which the integral:

$$\int_{x_1}^{x_2} \sqrt{x}\sqrt{1 + y'^2}dx,$$

is stationary.

For this we turn to the Euler-Lagrange equation:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0, \tag{1}$$

Where $f = \sqrt{x}\sqrt{1+y'^2}$.

As f is not explicitly dependent on y we have $\frac{\partial f}{\partial y'} = constant$ or:

$$\frac{\sqrt{x}y'}{\sqrt{1+y'^2}} = k.$$

Solving for y' we have:

$$y' = \frac{k}{\sqrt{x-k^2}}.$$

Which is a separable differential equation which can be solved like:

$$dy = \int_{x_0}^{x} dx \frac{k}{\sqrt{x-k^2}}.$$

Which has the solution:

$$y = 2k\sqrt{x-k^2} - C.$$

Where $C = 2k\sqrt{x_0 - k^2} + y_0$. So this leads us to an equation for a parabola as such:

$$x = \frac{(y+C)^2}{4k^2} + k^2.$$

(3) Taylor 6.22

The equation to find the area between the string and the x-axis is as so:

$$Area = \int_0^{x_f} y dx.$$

A hint is given to change this into the form:

$$Area = \int_0^{l} f ds, \tag{2}$$

so that we can deal with something we know, l, the length of the string.

Our normal ds element is as such:

$$ds = \sqrt{dx^2 + dy^2}.$$

Which can be rearranged to get:

$$dx = \sqrt{ds^2 - dy^2} = ds\sqrt{1 - \frac{dy}{ds}^2} = ds\sqrt{1-y'^2}.$$

This will give us an f in equation 2 above of:

$$f = y\sqrt{1 - y'^2}.$$

Now since there is no explicit dependence on s in f we can use the 'first integral' as in equation 6.43 of the text. So we will have:

$$f - y'\frac{\partial f}{\partial y'} = constant.$$

Which for us will be:

$$\frac{y}{\sqrt{1 - y'^2}} = k.$$

Where k is some constant.

This in turn leads to:

$$y' = \sqrt{1 - (y/k)^2}.$$

Integrating this we have:

$$\arcsin(y/k) = s/k;$$

or:

$$y = k\sin(s/k); \tag{3}$$

Now we can go back to out definition of dx from above:

$$dx = ds\sqrt{1 - y'^2}.$$

Integrating this we get:

$$x = k - k\cos(s/k); \tag{4}$$

Putting equation 3 and 4 together we have:

$$(x - k)^2 + y^2 = k^2.$$

Which is the equation of a circle with radius k. If we use the boundary conditions $y(s = 0) = y(s = \ell) = 0$ and $x(s = 0) = 0$ we obtain $k = \frac{\ell}{\pi}$.

(4) Taylor 6.23

The integral for time takes the form:

$$t = \int_{s_i}^{s_f} \frac{ds}{v}.$$

For us ds will be:

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + y'^2}.$$

And the velocity v is:

$$v = \sqrt{(v_0 \cos\phi + Vy)^2 + v_0^2 \sin^2\phi}.$$

So we have:

$$t = \int_{x_i}^{x_f} \frac{dx\sqrt{1+y'^2}}{\sqrt{(v_0\cos\phi + Vy)^2 + v_0^2\sin^2\phi}}.$$

Now when ϕ and y' are small we can approximate:

$$ds = dx\sqrt{1+y'^2} \approx dx(1+\frac{1}{2}y'^2),$$

and,

$$v \approx v_0 + Vy.$$

So we have:

$$t = \int_{x_i}^{x_f} \frac{dx(1+\frac{1}{2}y'^2)}{v_0(1+ky)}.$$

Here our functional is:

$$f = \frac{1+\frac{1}{2}y'^2}{1+ky}.$$

Now this is not explicitly dependent on the variable x so we may use the 'first integral' as we discussed in discussion section Thursday night but did not finish. There is a subtlety, however. The first integral is as follows:

$$f - y'\frac{\partial f}{\partial y'} = C.$$

For us this looks like so:

$$\frac{1+\frac{1}{2}y'^2}{1+ky} - \frac{y'^2}{1+ky} = \frac{1-\frac{1}{2}y'^2}{1+ky} = C.$$

This is a first order differential equation. Now we are told the solution looks as follows:

$$y = \lambda x(D-x).$$

Plugging in to our differential equation we get:

$$1 - \frac{1}{2}y'^2 = C(1+ky).$$

Or:

$$1 - \frac{1}{2}\lambda^2 D^2 - 2\lambda^2 x^2 + 2\lambda^2 Dx = C + Ck[\lambda x D - \lambda x^2]. \quad (5)$$

We get an equation for λ by matching coefficients of each power of x on either side of the equations. Matching the x^2 coefficients requires $C = \frac{2\lambda}{k}$. Then matching the constant (x^0) term requires

$$1 - \frac{1}{2}\lambda^2 D^2 = \frac{2\lambda}{k}.$$

Which has solution:

$$\lambda = \frac{\sqrt{4 + 2k^2D^2} - 2}{kD^2};$$

as advertised.

You may also do this using the Euler-Lagrange equation (equation 1) to get a second order differential equation:

$$y''[1 - ky] + k - \frac{k}{2}y'^2 = 0.$$

This will lead to the same equation for λ.

(5) Taylor 6.25

Let's start by stating the parameterized equation for x and y:

$$x = a(\theta - \sin\theta),$$

and

$$y = a(1 - \cos\theta).$$

Now the differential is $ds = \sqrt{dx^2 + dy^2}$ or:

$$ds = \sqrt{dx^2 + dy^2} = d\theta\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}.$$

So:

$$ds = \sqrt{(a(1 - \cos\theta))^2 + (a\sin\theta)^2} = a\sqrt{2 - 2\cos\theta}.$$

And we can find the velocity using conservation of energy:

$$v = \sqrt{2g(y - y_0)} = \sqrt{2ga(\cos\theta_0 - \cos\theta)}.$$

Where we inserted the above definition for x and where $y_0 = a(1 - \cos\theta)$(remember in this picture gravity is in the positive y-direction).

Putting this together with our definition of time integral we have:

$$t = \int_{s_i}^{s_f} \frac{ds}{v} = \int_{\theta_0}^{\pi} d\theta \frac{a\sqrt{2(1 - \cos\theta)}}{\sqrt{2ga(\cos\theta_0 - \cos\theta)}} = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} d\theta \frac{\sqrt{(1 - \cos\theta)}}{\sqrt{(\cos\theta_0 - \cos\theta)}}.$$

So now we have this integral to complete. The book suggests a change of variables $\theta = \pi - 2\alpha$. With this change we can find:

$$t = 2\sqrt{\frac{a}{g}} \int\limits_0^{\alpha_0} d\alpha \frac{\cos\alpha}{\sqrt{(\sin^2\alpha_0 - \sin^2\alpha)}}.$$

Let's do a substitution $u = \frac{sin\alpha}{\sin\alpha_0}$:

$$t = 2\sqrt{\frac{a}{g}} \int\limits_0^{1} \frac{du}{\sqrt{1-u^2}} = 2\sqrt{\frac{a}{g}}\frac{\pi}{2} = \pi\sqrt{\frac{a}{g}}.$$

So this means no matter where you let go of the car, the time to get to the bottom is the same. Qualitatively if you were to move the car's initial position further up the track, the extra distance the car needs to travel is exactly balanced by the increased slope of the track, which gives the car greater velocity more quickly.