Applications of Contour Integration

Here are some examples of the techniques used to evaluate several different types of integrals. Everything is based on the Cauchy integral theorem (really the Cauchy-Goursat theorem to avoid questions about the continuity of the derivative)

$$\oint_C f(z)\,dz = 0 \tag{1}$$

the Cauchy integral formula

$$2\pi i f(z_0) = \oint_C \frac{f(z)}{z - z_0}\,dz \tag{2}$$

and the derivative formula

$$\frac{2\pi i}{n!} f^{(n)}(z_0) = \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz \tag{3}$$

where in all of these equations $f(z)$ is assumed to be analytic within and on the contour C.

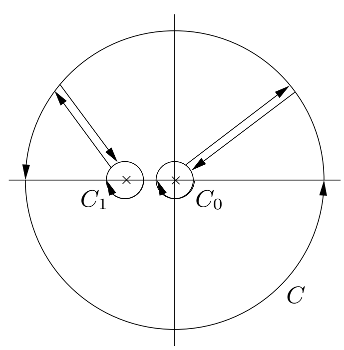
Note that whenever a contour C is written as a subscript on an integral, it is assumed to be followed in the counterclockwise direction. Thus a clockwise contour will be written $-C$ and $\oint_{-C} = -\oint_C$.

1.

$$I = \oint_C \frac{3z + 2}{z(z+1)^3}\,dz$$

where C is the circle $|z| = 3$. To do this integral, deform the contour around the poles at $z = 0$ and $z = -1$ and use (1) to write

$$0 = \oint_C + \oint_{-C_0} + \oint_{-C_1} = \oint_C - \oint_{C_0} - \oint_{C_1}\ .$$



In other words,

$$I = \oint_C \frac{3z+2}{z(z+1)^3}\,dz = \oint_{C_0} \frac{(3z+2)/(z+1)^3}{z}\,dz + \oint_{C_1} \frac{(3z+2)/z}{(z+1)^3}\,dz\,.$$

We now evaluate the integral over C_0 using (2) and the integral over C_1 using (9) to obtain

$$\begin{aligned} I &= 2\pi i \frac{3z+2}{(z+1)^3}\bigg|_{z=0} + \frac{2\pi i}{2!}\frac{d^2}{dz^2}(3z+2)/z\bigg|_{z=-1} \\ &= 2\pi i(2) + \pi i(-4) \\ &= 0\,. \end{aligned}$$

2.

$$I = \int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$$

where $a > 1$. To evaluate integrals where the integrand is a rational function of the trigonometric functions (i.e., a ratio of polynomials in $\sin\theta$ and $\cos\theta$), we let $z = e^{i\theta}$ and use

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$$

and

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$$

together with $d\theta = -idz/z$ and the contour C defined by $|z| = 1$. So we have

$$I = \oint_C \frac{-idz/z}{a+\frac{z+z^{-1}}{2}} = \oint_C \frac{-2idz}{z^2+2az+1}\,.$$

Using the quadratic formula, the denominator can be factored as $(z-\alpha)(z-\beta)$ where

$$\alpha = -a + \sqrt{a^2-1} \qquad \text{and} \qquad \beta = -a - \sqrt{a^2-1}\,.$$

Since $a > 1$, we see that for $a = 1$ we have $\alpha = -a = -1$ and $\beta = -a = -1$, and as $a \to \infty$ we have $\alpha \to 0$ and $\beta \to -\infty$ and thus $|\alpha| < 1$ and $|\beta| > 1$. Therefore only the pole at α is enclosed by C, and using (2) we have

$$I = \oint_C \frac{-2idz}{(z-\alpha)(z-\beta)} = \oint_C \frac{-2i/(z-\beta)}{z-\alpha}\,dz = 2\pi i \frac{-2i}{\alpha-\beta} = \frac{2\pi}{\sqrt{a^2-1}}\,.$$

3.

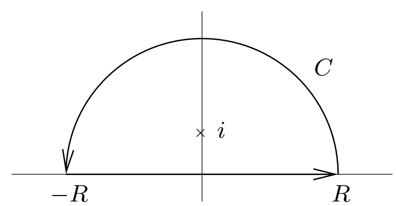
$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}\,.$$

This is an example of a real integral of the form $\int_{-\infty}^{\infty} P(x)\,dx$ where $P(x)$ is the ratio of two polynomials and the degree of the denominator is at least two more than the degree of the numerator (so that $|P(z)\,dz| \to 0$ as $|z| \to \infty$). We will first consider the range on the integral to be from $-R$ to R, and then take the limit as $R \to \infty$.

Looking at this in the complex plane and creating a closed contour C by choosing the path as shown, we then consider the integral

$$\oint_C \frac{dz}{(z^2+1)^2} = \oint_C \frac{dz}{(z+i)^2(z-i)^2}$$

which encloses the pole at $z = i$. (We could have closed the contour in the lower half-plane which would then have enclosed the pole at $z = -i$. But be sure to note that then the contour would have been clockwise so we would pick up an additional $(-)$ sign.)



We have

$$\oint_C = \int_{-R}^{R} + \int_{\text{semicircle}}$$

where the integral over the semicircle vanishes as $|z| \to \infty$. This is because on the semicircle we have $z = Re^{i\theta}$ so $dz = iRe^{i\theta}d\theta$ and

$$\left| \frac{dz}{(z+i)^2(z-i)^2} \right| \to \frac{R}{R^4} d\theta \to 0\,.$$

Using (9) we then obtain

$$\begin{aligned} I &= \lim_{R\to\infty} \oint_C \frac{dz}{(z^2+1)^2} = \lim_{R\to\infty} \oint_C \frac{(z+i)^{-2}}{(z-i)^2}\,dz \\ &= 2\pi i \frac{d}{dz}(z+i)^{-2}\bigg|_{z=i} = \frac{\pi}{2}\,. \end{aligned}$$

4.

$$I = \int_{-\infty}^{\infty} \frac{e^{ikr}}{k^2+\mu^2}\,dk$$

where $r > 0$. This is a Fourier integral $\int P(k)e^{ikr}\,dk$ of a rational function $P(k)$, which we assume has no poles on the real axis. We further assume that $P(k)$ has a zero at least of order two at infinity so that $|P(z)dz| \to 0$ as $|z| \to \infty$. This means that we can close the contour as in the previous example, and the integral over the semicircle will then vanish.

Writing $k = k_{\text{Re}} + ik_{\text{Im}}$ we have

$$e^{ikr} = e^{ik_{\text{Re}}r}e^{-k_{\text{Im}}r}$$

so for $r > 0$ we must close the contour in the upper half-plane (where $k_{\text{Im}} > 0$). Noting that $\left|e^{ikr}\right| = e^{-k_{\text{Im}}r} \leq 1$ and $\left|dk/(k^2+\mu^2)\right| \to 0$ as $R \to \infty$, we indeed have the integral vanishing over the semicircle at infinity so that by (2)

$$I = \oint_C \frac{e^{ikr}/(k+i\mu)}{k-i\mu}\,dk = 2\pi i \frac{e^{ikr}}{k+i\mu}\bigg|_{k=i\mu} = \pi\frac{e^{-\mu r}}{\mu}\,.$$

As a side remark, note that $e^{ikr} = \cos kr + i\sin kr$ together with the fact that $\sin kr$ is an odd function of k (so its integral over an even interval vanishes) shows that

$$I = \int_{-\infty}^{\infty} \frac{\cos kr}{k^2+\mu^2}\,dk$$

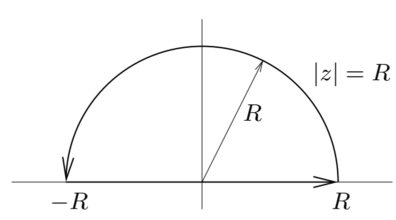
which is independent of the sign of r. Hence we have really shown that

$$I = \int_{-\infty}^{\infty} \frac{e^{ikr}}{k^2+\mu^2}\,dk = \pi\frac{e^{-\mu|r|}}{\mu}\,.$$

5. Now we want to consider a variation of the previous example where we again have an integral of the form $\int P(k)e^{ikr}\,dk$ but now we assume that the rational function $P(k)$ has a zero of only order one at infinity. That these integrals exist depends on the following important result, known as **Jordan's lemma**.

Theorem. If for $R \to \infty$ we have $|P(z)| \to 0$ uniformly in θ for $0 < \theta < \pi$, then

$$\lim_{R\to\infty} \int_{|z|=R} P(z)e^{ikz}\,dz = 0 \qquad \text{for } k > 0\,.$$



Proof. Let $M(R)$ be the maximum value of $|P(z)|$ on the semicircle $|z| = R$. To say that $|P(z)| \to 0$ uniformly means that $|P(z)| \leq M(R)$ independently of θ, where

$\lim_{R\to\infty} M(R) = 0$. If we let

$$I = \int_{|z|=R} P(z)e^{ikz}\,dz$$

then (with $z = Re^{i\theta} = R\cos\theta + iR\sin\theta$)

$$\begin{aligned} |I| &\le M(R)\int_0^\pi \left|e^{ikR\cos\theta - kR\sin\theta}\right| \left|iRe^{i\theta}\right| d\theta \\ &= M(R)R\int_0^\pi e^{-kR\sin\theta}\,d\theta\,. \end{aligned}$$

Now we break up the integral as $\int_0^\pi = \int_0^{\pi/2} + \int_{\pi/2}^\pi$. It is easiest to see from a plot that $\sin\theta \ge 2\theta/\pi$ for $0 \le \theta \le \pi/2$, and similarly that $\cos\varphi \ge 1 - 2\varphi/\pi$ for $0 \le \varphi \le \pi/2$. (Alternatively, the function $f(\theta) := \sin\theta - 2\theta/\pi$ equals zero at $\theta = 0$ and $\theta = \pi/2$, and is positive at $\theta = \pi/4$. So if it were ever less than or equal to zero anywhere on the interval $[0, \pi/2]$, then it would have to have a minimum somewhere on this interval. But $f(\theta)$ can't possibly have a minimum here because $f''(\theta) = -\sin\theta$ is less than or equal to zero everywhere on $[0, \pi/2]$. Therefore $f(\theta) \ge 0$ everywhere on $[0, \pi/2]$. A similar argument applies to the cosine relationship.) In any case, we then have

$$\int_0^{\pi/2} e^{-kR\sin\theta}\,d\theta \le \int_0^{\pi/2} e^{-2kR\theta/\pi}\,d\theta = \frac{\pi}{2kR}\bigl(1 - e^{-kR}\bigr)$$

and (letting $\varphi = \theta - \pi/2$ or $\theta = \varphi + \pi/2$)

$$\begin{aligned} \int_{\pi/2}^\pi e^{-kR\sin\theta}\,d\theta &= \int_0^{\pi/2} e^{-kR\cos\varphi}\,d\varphi \le \int_0^{\pi/2} e^{-kR(1-2\varphi/\pi)}\,d\varphi \\ &= \frac{\pi}{2kR}\bigl(1 - e^{-kR}\bigr)\,. \end{aligned}$$

(Actually, this result should have been expected from the symmetry of the sine function around $\pi/2$.) Therefore

$$|I| \le \frac{\pi M(R)}{k}\bigl(1 - e^{-kR}\bigr)$$

which goes to 0 as $R \to \infty$ (because $k > 0$ and $M(R) \to 0$ as $R \to \infty$). QED

(In the case where $k < 0$ we must close the contour in the lower half-plane.)

What this result tells us is that if we have an integral of the form $\int_{-\infty}^{\infty} P(k)e^{ikr}\,dk$ where $P(k)$ is a rational function with a zero of order one at infinity, then we can close the contour in the complex k-plane and evaluate the closed contour in the usual manner. Since the part of the integral along the semicircle vanishes, this then gives us the desired result.

As a specific example, consider the integral (where $k = |\mathbf{k}|$ and $r = |\mathbf{r}|$)

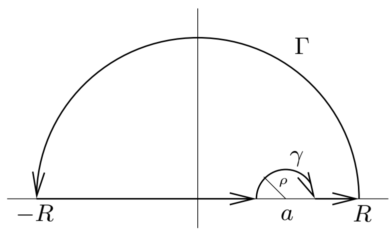
$$I = \int_{\text{all space}} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2 + \mu^2}\, d^3k = \int_0^\infty k^2\, dk \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi\, \frac{e^{ikr\cos\theta}}{k^2+\mu^2}$$

$$= \frac{2\pi}{ir}\int_0^\infty k\, \frac{e^{ikr} - e^{-ikr}}{k^2+\mu^2}\, dk = \frac{2\pi}{ir}\int_{-\infty}^{\infty} \frac{ke^{ikr}}{k^2+\mu^2}\, dk\, .$$

The integrand has poles at $k = \pm i\mu$. Applying Jordan's lemma by closing the contour in the upper half-plane (since $r > 0$) we have

$$\int \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2+\mu^2}\, d^3k = \frac{2\pi}{ir} \oint \frac{ke^{ikr}/(k+i\mu)}{k - i\mu} = 2\pi i \frac{2\pi}{ir} \frac{ke^{ikr}}{k+i\mu}\bigg|_{k=i\mu} = 2\pi^2 \frac{e^{-\mu r}}{r}\, .$$

6. Now let's take a look at what happens when the integrand has poles that lie on the real axis, as in the figure below.



So, suppose $Q(z)$ has poles of finite order in the upper half-plane, and simple poles (i.e., poles of order one) on the real axis. Also assume that $Q(z)$ is a rational function with behavior at infinity as in examples 3, 4 or 5. Referring to the above figure in the case of a single pole on the real axis, the entire enclosed contour C consists of the large semicircle Γ, the segments on the real axis from $-R$ to $a-\rho$ and $a+\rho$ to R, and the small semicircle γ. Note that γ goes around the pole at a in such a way that the pole is not included in C. We also let R be so large that all other poles of $Q(z)$ are included inside C, and large enough that the integral over Γ approaches zero as $R \to \infty$. Then we have

$$\oint_C Q(z)\, dz = \left(\int_\Gamma + \int_{-R}^{a-\rho} + \int_\gamma + \int_{a+\rho}^{R}\right) Q(z)\, dz = 2\pi i \sum_{y>0} \operatorname{Res} Q(z)$$

where $\sum_{y>0} \operatorname{Res} Q(z)$ means to sum over all residues of $Q(z)$ in the upper half-plane. (Recall that the residue of the function $f(z)$ at the point z_j is the number $R_j = (1/2\pi i) \oint_{C_j} f(z)\, dz$.)

Taking the limit $\rho \to 0$, the two integrals along the real axis may be combined into what is called the **Cauchy principal value**

$$P\int_{-R}^{R} Q(x)\, dx := \lim_{\rho\to 0}\left(\int_{-R}^{a-\rho} Q(x)\, dx + \int_{a+\rho}^{R} Q(x)\, dx\right).$$

For example, consider the integral

$$I = \int_{-R}^{R} \frac{dx}{x} \,.$$

Because of the singularity at $x = 0$, this integral is not defined; taking the principal value lets us evaluate such integrals by skipping over the singularity. We have

$$\begin{aligned} P\int_{-R}^{R} \frac{dx}{x} &= \lim_{\rho\to 0}\left(\int_{-R}^{-\rho} \frac{dx}{x} + \int_{\rho}^{R} \frac{dx}{x}\right) \\ &= \lim_{\rho\to 0}\left(\int_{R}^{\rho} \frac{dx}{x} + \int_{\rho}^{R} \frac{dx}{x}\right) \qquad \text{(let } x \to -x \text{ in the first integral)} \\ &= 0 \,. \qquad \text{(since } \int_{R}^{\rho} = -\int_{\rho}^{R}\text{)} \end{aligned}$$

Thus the principal value of the integral I is equal to zero. This makes sense since I is the integral of an odd function over an even interval.

Returning to our integral $\oint_C Q(z)\,dz$, taking the limit $R \to \infty$ and remembering that $\int_\Gamma \to 0$ leaves us with

$$\oint_C Q(z)\,dz = P\int_{-\infty}^{\infty} Q(x)\,dx + \int_\gamma Q(z)\,dz = 2\pi i \sum_{y>0} \text{Res}\; Q(z) \,.$$

To evaluate the integral over γ, we let $z = a + \rho e^{i\theta}$ so that

$$\int_\gamma Q(z)\,dz = \int_\pi^0 Q(a+\rho e^{i\theta}) i\rho\, e^{i\theta}\, d\theta \,.$$

By assumption, $Q(z)$ has a simple pole at $z = a$, and therefore we may write it in the general form

$$Q(z) = \frac{f(z)}{z-a}$$

where $f(z)$ is analytic in a neighborhood of $z = a$.

As to the function $f(z)$, expanding in a Taylor series we have

$$f(z) = f(a+\rho e^{i\theta}) = f(a) + \text{terms proportional to } \rho$$

so that with $z - a = \rho e^{i\theta}$ we obtain in the limit $\rho \to 0$

$$\int_\gamma Q(z)\,dz = \int_\pi^0 \frac{f(a)}{\rho e^{i\theta}}\, i\rho e^{i\theta}\, d\theta = -i\pi f(a) \,.$$

Noting that $f(a)$ is the residue of the function $f(z)/(z-a)$ at $z = a$, we finally obtain (generalizing to a finite number of simple poles on the real axis)

$$P\int_{-\infty}^{\infty} Q(x)\,dx = 2\pi i \sum_{y>0} \text{Res}\; Q(z) + i\pi \sum_{y=0} \text{Res}\; Q(z) \tag{4}$$

where $\sum_{y=0} \operatorname{Res} Q(z)$ denotes the sum of the residues of $Q(z)$ at each of its simple poles along the real axis.

As an example, consider

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx\,.$$

Since $\lim_{x\to 0}(\sin x)/x = 1$, there is no pole at $x = 0$. However, in order to evaluate the integral I we consider

$$P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx\,.$$

This integrand only has a single pole and it lies on the real axis at $z = x = 0$. Noting that

$$\frac{e^{iz}}{z} = \frac{1}{z} + i + \frac{z}{2} + \cdots$$

we apply (4) to find

$$P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx = i\pi \sum_{y=0} \operatorname{Res} \frac{e^{iz}}{z} = i\pi\,.$$

Taking real and imaginary parts we obtain

$$P\int_{-\infty}^{\infty} \frac{\cos x}{x}\, dx = 0$$

(again, this is obvious because it's the integral of an odd function over an even interval) and

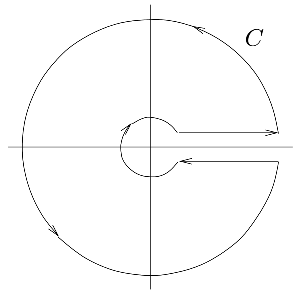
$$P\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi\,.$$

7. We now turn our attention to integrals of the form

$$\int_{0}^{\infty} x^{\lambda-1} P(x)\, dx$$

where the rational function $P(z)$ is analytic at $z = 0$ and has no poles along the positive real axis, and where $\left|z^{\lambda} P(z)\right| \to 0$ uniformly as $|z| \to 0$ and as $|z| \to \infty$.

To make this case different from what we have done so far, we assume further that λ is not an integer. This means that we must choose a branch cut for $z^{\lambda-1}$ which we take to be along the positive real axis. (While the branch cut is arbitrary, the branch point $z = 0$ itself is not.) Thus we consider integrals of the form $\oint_C z^{\lambda-1} P(z)\, dz$ where the contour C is as shown below.



Note that because of the branch cut, the integrand is discontinuous across the positive real axis, and hence the two paths shown do not cancel each other. Since in general $z = re^{i\theta}$, we choose the phase of $z^{\lambda-1}$ to be 0 along the line just above the real axis, and 2π just below. In other words, we have $z^{\lambda-1} = x^{\lambda-1}$ just above the axis and $z^{\lambda-1} = x^{\lambda-1}e^{i2\pi(\lambda-1)}$ just below it.

Because of our condition that $\left|z^{\lambda}P(z)\right| \to 0$ uniformly as $|z| \to 0$ and as $|z| \to \infty$, we see that

$$\left|z^{\lambda-1}P(z)\right| = \frac{\left|z^{\lambda}P(z)\right|}{|z|} \to 0 \qquad \text{as } |z| \to \infty$$

and

$$\left|z^{\lambda-1}P(z)\right| |dz| = \frac{\left|z^{\lambda}P(z)\right|}{|\rho e^{i\theta}|} \left|i\rho e^{i\theta} d\theta\right| \to 0 \qquad \text{as } |z| \to 0$$

and hence the integrals over the large and small circles both vanish. This leaves us with

$$\begin{aligned}
\oint_C z^{\lambda-1}P(z)\,dz &= \int_{\infty}^{0} e^{i2\pi(\lambda-1)}x^{\lambda-1}P(x)\,dx + \int_0^{\infty} x^{\lambda-1}P(x)\,dx \\
&= -\int_0^{\infty} e^{i2\pi(\lambda-1)}x^{\lambda-1}P(x)\,dx + \int_0^{\infty} x^{\lambda-1}P(x)\,dx \\
&= \left[1 - e^{i2\pi(\lambda-1)}\right] \int_0^{\infty} x^{\lambda-1}P(x)\,dx
\end{aligned}$$

But $e^{-i2\pi} = 1$ so that

$$1 - e^{i2\pi(\lambda-1)} = 1 - e^{i2\pi\lambda} = e^{i\pi\lambda}(e^{-i\pi\lambda} - e^{i\pi\lambda}) = -2ie^{i\pi\lambda}\sin\pi\lambda$$

and hence

$$\oint_C z^{\lambda-1}P(z)\,dz = -2ie^{i\pi\lambda}\sin\pi\lambda \int_0^{\infty} x^{\lambda-1}P(x)\,dx\,.$$

On the other hand, we also know that

$$\oint_C z^{\lambda-1}P(z)\,dz = 2\pi i \sum_C \text{Res}\,[z^{\lambda-1}P(z)]$$

where the sum is over all of the residues inside the contour C. Equating this to our previous result we have

$$\int_0^\infty x^{\lambda-1}P(x)\,dx = \frac{-\pi}{e^{i\pi\lambda}\sin\pi\lambda}\sum_C \operatorname{Res}\left[z^{\lambda-1}P(z)\right].$$

Since $e^{-i\pi\lambda} = (e^{-i\pi})^\lambda = (-1)^\lambda$ and $-1 = (-1)^{-1}$, we can write our final result as

$$\int_0^\infty x^{\lambda-1}P(x)\,dx = \frac{(-1)^{\lambda-1}\pi}{\sin\pi\lambda}\sum_C \operatorname{Res}\left[z^{\lambda-1}P(z)\right]. \tag{5}$$

As a specific example, let us evaluate integrals of the type

$$\int_0^\infty \frac{x^{\lambda-1}}{1+x}\,dx \qquad \text{for } 0<\lambda<1\,.$$

For this range of λ we clearly have

$$\left|\frac{z^\lambda}{1+z}\right| \to 0$$

both as $|z|\to\infty$ and as $|z|\to 0$. Since the only pole of the integrand is at $z=-1$ and it lies inside C, equation (5) yields

$$\int_0^\infty \frac{x^{\lambda-1}}{1+x}\,dx = \frac{(-1)^{\lambda-1}\pi}{\sin\pi\lambda}(-1)^{\lambda-1} = \frac{\pi}{\sin\pi\lambda}\,.$$

Now let us take a look at how contour integration can be used to sum series. As a first example, we want to prove that

$$\sum_{n=1}^\infty \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}. \tag{6}$$

This sum arises in, for example, the evaluation of Fermi-Dirac integrals of the form

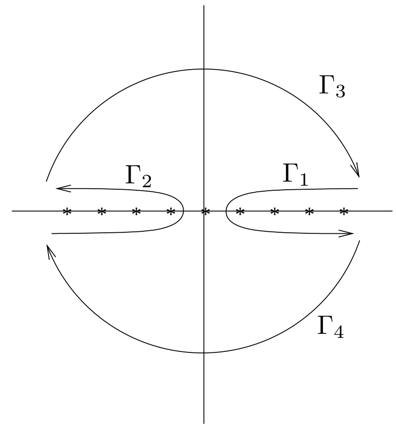
$$I[f] = \int_0^\infty \frac{f(\varepsilon)\,d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}$$

in a neighborhood of $T=0$.

Noting that the function $\sin\pi x$ has zeros at $x=n$ for $n=0,\pm1,\pm2,\ldots$, we will do this by evaluating the integral

$$\oint \frac{1}{z^2}\frac{1}{\sin\pi z}\,dz$$

around the contour shown below. (If we wanted to evaluate $\sum 1/n^2$ then we would use $\tan\pi z$ instead of $\sin\pi z$.)



We first need the Taylor series expansion of $f(z) = \sin \pi z$ about $z = n$. In other words, letting $z - n = \xi$ we have

$$f(z) = f(n + \xi) = f(n) + \xi f'(n) + (\xi^2/2)f''(n) + (\xi^3/6)f'''(n) + \cdots$$

where $f'(z) = \pi \cos \pi z$, $f''(z) = -\pi^2 \sin \pi z$ and $f'''(z) = -\pi^3 \cos \pi z$. Since $\cos n\pi = (-1)^n$ we have (to third order)

$$\begin{aligned}\sin \pi z = \sin \pi(n + \xi) &= (-1)^n \pi(z - n) - (-1)^n \pi^3 \frac{(z - n)^3}{3!} \\ &= (-1)^n \pi(z - n) \left[1 - \frac{\pi^2}{6}(z - n)^2\right].\end{aligned}$$

Using the approximation $1/(1 - w) = 1 + w$ for $w \ll 1$ we then have

$$\frac{1}{\sin \pi z} = \frac{(-1)^n}{\pi} \frac{1}{z - n} \left[1 + \frac{\pi^2}{6}(z - n)^2\right]. \tag{7}$$

Now consider path Γ_1 which is to be considered closed at infinity. If C_n is a small closed circle around the point n on the real axis (*taken in the standard counterclockwise direction*), then the Cauchy integral theorem tells us that

$$I_1 = \oint_{\Gamma_1} \frac{1}{z^2} \frac{1}{\sin \pi z}\, dz = \sum_{n=1}^{\infty} \oint_{C_n} \frac{1}{z^2} \frac{1}{\sin \pi z}\, dz.$$

Using equation (7) we have

$$\oint_{C_n} \frac{1}{z^2} \frac{1}{\sin \pi z}\, dz = \frac{(-1)^n}{\pi} \oint_{C_n} \frac{\frac{1}{z^2}\left[1 + \frac{\pi^2}{6}(z - n)^2\right]}{z - n}\, dz$$

where the numerator is analytic throughout C_n. Applying the Cauchy integral formula (2) and evaluating the numerator at $z = n$ yields

$$\oint_{C_n} \frac{1}{z^2} \frac{1}{\sin \pi z}\, dz = 2\pi i \frac{(-1)^n}{\pi} \frac{1}{n^2} = (2i) \frac{(-1)^n}{n^2}.$$

(Note that higher order terms in $z-n$ would also vanish.) This leaves us with

$$I_1 = (2i)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.$$

Clearly, $I_2 = \oint_{\Gamma_2} = \oint_{\Gamma_1} = I_1$ since $(-n)^2 = n^2$, and therefore

$$I_1 + I_2 = (4i)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.$$

We now want to close Γ_1 and Γ_2 with Γ_3 and Γ_4. If we can do this, then by the Cauchy integral theorem again we have

$$\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4} = \oint_{-C_0} = -\oint_{C_0}$$

where C_0 is a small (counterclockwise) circle around the origin. We will show $\int_{\Gamma_3} = \int_{\Gamma_4} = 0$ so that

$$(4i)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\oint_{C_0}\frac{1}{z^2}\frac{1}{\sin\pi z}\,dz. \tag{8}$$

Let $z = Re^{i\theta}$ on Γ_3. (This result will also hold exactly the same on Γ_4.) Then $dz = iRe^{i\theta}d\theta$ and

$$\lim_{R\to\infty}\left|\frac{dz}{z^2}\right| = \lim_{R\to\infty}\frac{d\theta}{R} = 0.$$

We also have

$$\begin{aligned}\sin\pi z = \frac{1}{2i}(e^{i\pi z} - e^{-i\pi z}) &= \frac{1}{2i}\left[e^{i\pi R(\cos\theta + i\sin\theta)} - e^{-i\pi R(\cos\theta + i\sin\theta)}\right] \\ &= \frac{1}{2i}\left[e^{i\pi R\cos\theta}e^{-\pi R\sin\theta} - e^{-i\pi R\cos\theta}e^{\pi R\sin\theta}\right].\end{aligned}$$

In the upper half-plane we have $0 < \theta < \pi$ so $\sin\theta > 0$, and in the lower half-plane $\pi < \theta < 2\pi$ so $\sin\theta < 0$. In either case, one of the exponentials in this expression for $\sin\pi z$ goes to 0 and the other goes to infinity as $R \to \infty$, and therefore

$$\lim_{R\to\infty}\left|\frac{1}{\sin\pi z}\right| \to 0.$$

So now

$$\left|\int_{\Gamma_3}\frac{1}{z^2}\frac{1}{\sin\pi z}\,dz\right| \le \int_{\Gamma_3}\left|\frac{1}{z^2}\right|\left|\frac{1}{\sin\pi z}\right||dz| \to 0$$

as $R \to \infty$ which is what we wanted to show.

Now we need to evaluate the integral in (8). In the case where $n = 0$, equation (7) shows that

$$\frac{1}{z^2}\frac{1}{\sin \pi z} = \frac{1}{z^2}\frac{1}{\pi z}\left[1 + \frac{\pi^2 z^2}{6}\right] = \frac{1}{\pi z^3} + \frac{\pi}{6z}.$$

It's obvious from this that the residue of the integrand is $\pi/6$ because it's the coefficient of $1/z$, and thus by the residue theorem we have

$$-\oint_{C_0} \frac{1}{z^2}\frac{1}{\sin \pi z}\, dz = -2\pi i \left(\frac{\pi}{6}\right) = -i\frac{\pi^2}{3}.$$

Alternatively, we can evaluate the integral directly. For the $1/z^3$ term, using the formula for the derivative of an analytic function

$$f^{(m)}(z_0) = \frac{m!}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{m+1}} dz \tag{9}$$

we see that $\oint dz/z^3 = 0$ because the derivative of the constant function $f(z) = 1$ is identically zero. Then for the $1/z$ term we let $z = re^{i\theta}$ again so that

$$-\oint_{C_0} \frac{\pi}{6z} dz = -\frac{\pi}{6}\int_0^{2\pi} \frac{ire^{i\theta} d\theta}{re^{i\theta}} = -i\frac{\pi^2}{3}.$$

This last method also shows directly that $\oint dz/z^3 = 0$ because $\int_0^{2\pi} e^{-2i\theta} d\theta = 0$.

In any case, we have now shown that equation (8) becomes

$$(4i)\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\oint_{C_0} \frac{1}{z^2}\frac{1}{\sin \pi z}\, dz = -i\frac{\pi^2}{3}$$

or

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

as claimed.

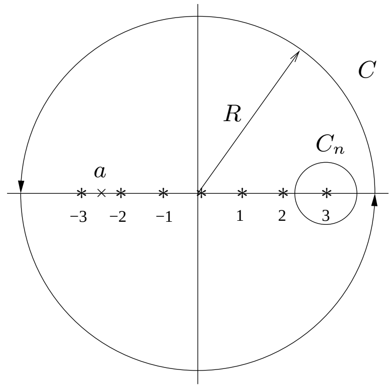
As a second example, let us evaluate the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n + a)^2} \tag{10}$$

where $a \notin \mathbb{Z}$. Now we will consider the integral

$$I = \oint_C \frac{\pi}{(z + a)^2 \tan \pi z}\, dz$$

where C is a large circle of radius R centered in the complex plane.



By the Cauchy integral theorem again we have

$$\oint_C = \sum_{n=-N}^{N} \oint_{C_n} + \oint_{C_a}$$

where each C_n is taken counterclockwise, and N is the largest integer less than R and $a < N$. We will eventually let $R \to \infty$.

We first do the integral over C_a. From equation (9) we see that

$$2\pi i f'(z_0) = \oint_C \frac{f(z)}{(z-z_0)^2}\, dz$$

and hence (remember that a is not an integer)

$$\oint_{C_a} \frac{\pi/\tan \pi z}{(z+a)^2}\, dz = 2\pi i \, \frac{d}{dz}\pi \cot \pi z\bigg|_{z=-a} = -2\pi^3 i \csc^2 \pi a\,.$$

For the integrals over C_n, we expand $f(z) = \tan \pi z$ about $z = n$ as we did above for $\sin \pi z$. Here we have $f'(z) = \pi \sec^2 \pi z$, $f''(z) = 2\pi^2 \sec^2 \pi z \tan \pi z$ and $f'''(z) = 2\pi^3 \sec^4 \pi z + 4\pi^2 \sec^2 \pi z \tan^2 \pi z$ so that $f(n) = f''(n) = 0$ while $f'(n) = \pi$ and $f'''(n) = 2\pi^3$. Then

$$\begin{aligned} \tan \pi z = \tan \pi(n+\xi) &= \pi(z-n) + \frac{(z-n)^3}{3!} 2\pi^3 \\ &= \pi(z-n)\left[1 + \frac{\pi^2}{3}(z-n)^2\right] \end{aligned}$$

so that (now using $1/(1+w) = 1 - w + w^2 - \cdots$)

$$\frac{1}{\tan \pi z} = \frac{1}{\pi(z-n)}\left[1 - \frac{\pi^2}{3}(z-n)^2\right] .$$

Applying the Cauchy integral formula (2) gives us

$$\begin{aligned}\oint_{C_n} \frac{\pi/(z+a)^2}{\tan \pi z}\, dz &= \oint_{C_n} \frac{[\pi/(z+a)^2][1-\pi^2(z-n)^2/3]}{\pi(z-n)}\, dz \\ &= 2\pi i \frac{1}{(z+a)^2}\left[1-\frac{\pi^2}{3}(z-n)^2\right]\bigg|_{z=n} \\ &= \frac{2\pi i}{(n+a)^2}\,.\end{aligned}$$

Putting together our results we have so far

$$I = \oint_C \frac{\pi}{(z+a)^2 \tan \pi z}\, dz = -2\pi^3 i \csc^2 \pi a + \sum_{n=-N}^{N} \frac{2\pi i}{(n+a)^2}\,.$$

Now we let $R \to \infty$ which is the same as letting $N \to \infty$ in the sum. I claim that in this limit the integral goes to zero. To see this, observe that we can write

$$\begin{aligned}\frac{1}{\tan \pi z} &= \frac{\cos \pi z}{\sin \pi z} = i\,\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}} = i\,\frac{e^{i\pi R(\cos\theta + i\sin\theta)}+e^{-i\pi R(\cos\theta+i\sin\theta)}}{e^{i\pi R(\cos\theta+i\sin\theta)}-e^{-i\pi R(\cos\theta+i\sin\theta)}} \\ &= i\,\frac{e^{i\pi R\cos\theta}e^{-\pi R\sin\theta}+e^{-i\pi R\cos\theta}e^{\pi R\sin\theta}}{e^{i\pi R\cos\theta}e^{-\pi R\sin\theta}-e^{-i\pi R\cos\theta}e^{\pi R\sin\theta}}\,.\end{aligned}$$

Now note that for $0 < \theta < \pi$ (i.e. the upper half-plane) we have $\sin\theta > 0$ so as $R \to \infty$ one term in the numerator and one term in the denominator go to zero, while the other term in the numerator and the other term in the denominator blow up. But their ratio clearly goes to -1 so that in the upper half-plane we have $\cot \pi z \to -i$. Similarly, for $\pi < \theta < 2\pi$ (the lower half-plane) we obtain $\cot \pi z \to +i$. In either case, we also have $\left|dz/(z+a)^2\right| \to 0$ so the absolute value of the integrand in I goes to zero as claimed. Thus we are left with

$$0 = -2\pi^3 i \csc^2 \pi a + \sum_{n=-N}^{N} \frac{2\pi i}{(n+a)^2}$$

or

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \pi^2 \csc^2 \pi a = \frac{\pi^2}{\sin^2 \pi a}\,.$$