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Complex Number Analysis for Optics

We review elements of complex number analysis that are useful in AC circuit theory and in optics, in particular in the discussion of diffraction. These applications are usually done using phasors, a concept that skirts the use of complex numbers. We will make use of some elementary complex number analysis instead.

1 Complex Number Analysis

A complex number z can be represented as

$$z = x + iy \tag{1}$$

Here x and y are real numbers and $i = \sqrt{-1}$ and x (y) are called the real (imaginary) part of z.

Thus $\sqrt{-9} = 3i$. A real number x can be represented graphically by a point at coordinate x on the one-dimensional "number axis". A complex number can be represented graphically by a point at coordinates x, y in the two-dimensional "complex plane". One can do algebra with complex numbers such as addition, subtraction, multiplication, and division. For example $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$. This can be represented too in the two-dimensional x, y plane.

The magnitude or absolute value of a complex number is defined as the (positive) distance between the point x, y and the origin $\mathcal{O}$ of the coordinate system, in analogy with the definition of the magnitude or absolute value of a real number. The absolute value of z is written as $|z|$, as is done for the absolute value of a real number. Thus we have

$$|z| = \sqrt{x^2 + y^2} \tag{2}$$

The complex conjugate z^* of a complex number z or of a complex expression is obtained by replacing $i \rightarrow -i$ everywhere. Thus

$$z^* = x - iy \tag{3}$$

and

$$|z| = \sqrt{z^* z} = \sqrt{(x - iy)(x + iy)} = \sqrt{x^2 + y^2} \tag{4}$$

One can show, using Taylor series for $\cos\phi, \sin\phi$ and $\exp(i\phi)$, that

$$\mathrm{e}^{i\phi} = \cos\phi + i\sin\phi \tag{5}$$

The magnitude of $\exp(i\phi)$ is given by

$$|\mathrm{e}^{i\phi}| = |\cos\phi + i\sin\phi| = \sqrt{\cos^2\phi + \sin^2\phi} = 1 \tag{6}$$

Therefore, a complex number can also be represented as

$$z = |z|\,\mathrm{e}^{i\phi} = |z|\,(\cos\phi + i\sin\phi) \tag{7}$$

In the complex plane, ϕ is the angle between the line connecting $\mathcal{O}$ and the point x, y and the horizontal x-axis. This angle is called the "phase" of the complex number $z = x + iy$.

Relation (7) can be inverted to get $\cos\phi$ and $\sin\phi$ as

$$\cos\phi = \frac{\mathrm{e}^{i\phi} + \mathrm{e}^{-i\phi}}{2} \quad \text{and} \quad \sin\phi = \frac{\mathrm{e}^{i\phi} - \mathrm{e}^{-i\phi}}{2i} \tag{8}$$

The relations (8) are useful when one needs the Taylor series' expansion of the cosine or sine functions and has memorized only the Taylor series' expansion for e^x.

2 Application to Diffraction

The study of diffraction in optics requires the summation of a series such as

$$\sum_{k=0}^{n-1} a\cos(\alpha + k\beta) \quad \text{or} \quad \sum_{k=0}^{n-1} a\sin(\alpha + k\beta) \tag{9}$$

where the a represent the amplitude, a real number. There are exactly n terms in the series (9). We do these sums by representing the cosine or sine functions by an exponential with a complex exponent such as (7) with the understanding that one uses only the real part or only the imaginary part (according to once's taste). A similar situation is encountered in AC circuit theory where one can represent cosine or sine functions (voltages and currents) by exponentials with a complex exponent.

Instead of (9) we consider

$$S_n = \sum_{k=0}^{n-1} a\,\mathrm{e}^{i(\alpha+k\beta)} \tag{10}$$

We can write (9) as

$$S_n = a\,\mathrm{e}^{i\alpha} \sum_{k=0}^{n-1} \mathrm{e}^{ik\beta} \tag{11}$$

The sum is a geometric series of the form $1 + r + r^2 + \cdots$ with n terms and $r = \exp(i\beta)$ so $|r| < 1$. This series can be written

$$\Sigma_n = \sum_{k=0}^{n-1} r^k \tag{12}$$

and can be summed to give

$$\Sigma_n = \frac{1 - r^n}{1 - r} \tag{13}$$

This relation can be derived by writing the series for Σ_n and $r\Sigma_n$ below each other and subtract. All terms cancel except the first and the last ones and one obtains (13).

Using relations (13) and (14) we find for S_n of (10)

$$S_n = a\,\mathrm{e}^{i\alpha} \frac{1 - \mathrm{e}^{in\beta}}{1 - \mathrm{e}^{i\beta}} \tag{14}$$

S_n is a complex number. To calculate its magnitude we use (4) to get

$$|S_n|^2 = S_n^* S_n = a\,\mathrm{e}^{-i\alpha} \frac{1 - \mathrm{e}^{-in\beta}}{1 - \mathrm{e}^{-i\beta}}\, a\,\mathrm{e}^{i\alpha} \frac{1 - \mathrm{e}^{in\beta}}{1 - \mathrm{e}^{i\beta}} \tag{15}$$

This expression can be simplified to

$$
\begin{aligned}
|S_n|^2 &= a^2 \frac{1 - \mathrm{e}^{in\beta} - \mathrm{e}^{-in\beta} + 1}{1 - \mathrm{e}^{i\beta} - \mathrm{e}^{-in\beta} + 1} && (16) \\
&= a^2 \frac{2 - 2\cos n\beta}{2 - 2\cos\beta} && (17) \\
&= a^2 \frac{1 - \cos n\beta}{1 - \cos\beta} && (18) \\
&= a^2 \left(\frac{\sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \right)^2 && (19)
\end{aligned}
$$

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where we used (8) in (16) and went to half the argument of the cosine using $\cos 2\alpha = 1 - 2\sin^2\alpha$ in (18). This relation is used to calculate the intensity of electromagnetic waves emanating from one or more slits that reach a particular point in space. This will be discussed in detail in the lectures.