General Formula for Error Propagation

\[ q = q(x, y, z) \]
\[ q_{\text{best}} = q(x_{\text{best}}, y_{\text{best}}, z_{\text{best}}) \]

\[ \delta q = \sqrt{\left( \frac{\partial q}{\partial x} \delta x \right)^2 + \left( \frac{\partial q}{\partial y} \delta y \right)^2 + \left( \frac{\partial q}{\partial z} \delta z \right)^2} \]

for independent random errors \( \delta x, \delta y, \) and \( \delta z \)

main formula for error propagation
always use this formula
Experiment 1: Measure Density of Earth

• Calculate average density $\rho$ and determine which elements constitute the major portion of the earth.

• Two measurements
  – (a) Earth’s Radius $R_e$. (challenging measurement)
  – (b) Local acceleration of gravity $g$. (fairly easy)

• Use Newton’s constant $G=6.67 \times 10^{-11}$ N m$^2$/kg$^2$

• Aim for 10% or better error on $\rho$.

\[
F = \frac{GMm}{r^2} \quad \text{Gravitational force}
\]

\[
g = \frac{F}{m} = \frac{GM}{R_e^2} = \frac{G(\frac{4}{3} \pi R_e^3 \rho)}{R_e^2} = \frac{4}{3} \pi G R_e \rho
\]

\[
\rho = \frac{3}{4\pi} \frac{g}{G R_e}
\]
What’s the Point

It's an experiment about optimizing measurement technique, error estimation, and error propagation.
What Element(s) make up the Earth

- Assume most of earth’s volume is one element.

<table>
<thead>
<tr>
<th>Element</th>
<th>Density</th>
</tr>
</thead>
<tbody>
<tr>
<td>rock</td>
<td>2.7</td>
</tr>
<tr>
<td>aluminum</td>
<td>2.7</td>
</tr>
<tr>
<td>zinc</td>
<td>7.14</td>
</tr>
<tr>
<td>iron</td>
<td>7.20</td>
</tr>
<tr>
<td>nickel</td>
<td>8.85</td>
</tr>
<tr>
<td>copper</td>
<td>8.89</td>
</tr>
<tr>
<td>silver</td>
<td>10.5</td>
</tr>
<tr>
<td>lead</td>
<td>11.34</td>
</tr>
<tr>
<td>mercury</td>
<td>13.60</td>
</tr>
<tr>
<td>gold</td>
<td>19.3</td>
</tr>
</tbody>
</table>
Measure Earth’s Radius using $\Delta t$ Sunset

From right triangle:

$$\cos(\theta) = \frac{R_e}{R_e + h} = \frac{1}{1 + \frac{h}{R_e}} \approx 1 - \frac{h}{R_e}$$

For small $\theta$:

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2}$$

Equating:

$$\frac{\theta^2}{2} = \frac{h}{R_e}$$

$$\theta = \sqrt{\frac{2h}{R_e}}$$

Assume we are at equator

View from North Pole

Sunset at height $h$

Apparent motion of the sun as it sets.

Sunset on earth’s surface (tangent)

$h \ll R_e$
\( \theta \) Increases at Earth Rotates

Earth makes (nearly) one rotation per day. Angular frequency is \( 2\pi \) radians per day.

\( \omega \) (omega) = earth’s angular frequency.

\[
\omega = \frac{2\pi \text{ radians}}{\text{day}} \left( \frac{24 \text{ hours}}{\text{day}} \right) \left( \frac{60 \text{ minutes}}{\text{hour}} \right) \left( \frac{60 \text{ seconds}}{\text{minute}} \right) = 7.27 \times 10^{-5} \frac{\text{radians}}{\text{second}}
\]

\[
\theta = \omega t = \sqrt{\frac{2h}{R_e}} \quad \theta \text{ (theta) = angle earth rotates after true sunset.}
\]

\[
t = \frac{1}{\omega} \sqrt{\frac{2h}{R_e}} \quad \text{Solving for } t, \text{ we get the time delay of the sunset at height } h \text{ (since the true sunset).}
\]
Correct for Latitude and Earth’s Axis

La Jolla latitude

\[ \lambda = 32.87^\circ \]

\[ \lambda_s = -23.4^\circ \sin\left(\frac{2\pi d}{365}\right) \]

Solar latitude varies.

d=days since Sept. 22 (or March 20).

This formula accounts for our latitude and for the angle of the earth’s axis from the plane of its orbit.

\[ t = \frac{1}{\omega} \sqrt{\frac{2h}{\Re \left[ \cos^2(\lambda) \cos^2(\lambda_s) - \sin^2(\lambda) \sin^2(\lambda_s) \right]} \equiv \frac{1}{\omega} \sqrt{\frac{2Ch}{\Re}} \]
Measuring the Height of the Cliff

- The formula we derived is for height above sea level.
- Strings, protractors, and rulers will be available.
- Be sure to understand how well heights must be measured before you do the experiment.
- Each pair of experimenters should get their own measurements.

\[ h = l \cos(\theta) \]

\( h \) is the height above sea level, \( l \) is the horizontal distance, and \( \theta \) is the angle of elevation.
• The experimenter on the beach also views the sunset from above sea level.

• When you check the error propagation you will find that the measurement of the earth’s radius is quite sensitive to the $h_2$ measurement.
The Equation for Experiment 1a

\[ t = \frac{1}{\omega} \sqrt{\frac{2Ch}{R_e}} \]

From previous page.

\[ \Delta t = t_1 - t_2 = \frac{1}{\omega} \sqrt{\frac{2C}{R_e}} \left( \sqrt{h_1} - \sqrt{h_2} \right) \]

Time difference between the two sunset observers.

\[ C \equiv \frac{1}{\cos^2(\lambda)\cos^2(\lambda_s) - \sin^2(\lambda)\sin^2(\lambda_s)} \]

Season dependant factor slightly greater than 1.

use this formula for your error analysis

\[ R_e = \frac{2C}{\omega^2} \left( \frac{\sqrt{h_1} - \sqrt{h_2}}{\Delta t} \right)^2 \]
**Propagating Errors for** $R_e$

\[ R_e = \frac{2C}{\omega^2} \left( \frac{\sqrt{h_1} - \sqrt{h_2}}{\Delta t} \right)^2 \quad \text{basic formula} \]

Propagating errors (use shorthand for addition in quadrature)

\[ \sigma_{R_e} = \frac{\partial R_e}{\partial \Delta t} \sigma_{\Delta t} \oplus \frac{\partial R_e}{\partial h_1} \sigma_{h_1} \oplus \frac{\partial R_e}{\partial h_2} \sigma_{h_2} \]

\[ \sigma_{R_e} = \frac{2R_e}{\Delta t} \sigma_{\Delta t} \oplus \frac{R_e}{\sqrt{h_1 (\sqrt{h_1} - \sqrt{h_2})}} \sigma_{h_1} \oplus \frac{R_e}{\sqrt{h_2 (\sqrt{h_1} - \sqrt{h_2})}} \sigma_{h_2} \]

Note that error blows up at $h_1=0$ and at $h_2=0$. 
Cliffs West of Muir Campus

At the bottom of the asphalt road is a reasonable place to measure.

Must return there at sunset.

Do not go too near the cliffs.

Do not drop or kick objects below on the beach.

Wear walking shoes.

It may be cold in the evening.
Weather plays a role. Completely clear days are best.

sunset time – a moment when the last point of the Sun disappears
Measuring $g$ with a Pendulum

- Period can be measured with electronic timer over one cycle or with a stopwatch over many cycles.
- Frictional forces play a role for light weights.
- Small oscillations are good.
- Heavy weights may cause coupling to other oscillators like unstable stand.
- Short strings may cause moment of inertia to become important.

$$T = 2\pi \sqrt{\frac{l}{g}}$$

**Period of pendulum**
Propagating Errors for Experiment 1

\[ \rho = \frac{3}{4\pi} \frac{g}{GR_e} \]  
**Formula for density.**

\[ \sigma_{\rho} = \frac{3}{4\pi} \frac{1}{GR_e} \sigma_g \oplus \frac{-3}{4\pi} \frac{g}{GR_e^2} \sigma_{R_e} \]  
**Take partial derivatives and add errors in quadrature**

\[ \frac{\sigma_\rho}{\rho} = \frac{\sigma_g}{g} \oplus \frac{\sigma_{R_e}}{R_e} \]
Statistical analysis

\[ T = 2\pi \sqrt{\frac{l}{g}} \]  
Period of pendulum

uncertainty

error propagation  
statistical analysis

two methods to find uncertainty
The mean

\[ x_1, x_2, \ldots, x_N \quad N \text{ measurements of the quantity } x \]

\[ x_{\text{best}} = \bar{x} \quad \text{the best estimate for } x \rightarrow \text{the average or mean} \]

\[ \bar{x} = \frac{x_1 + x_2 + \ldots + x_N}{N} = \frac{\sum x_i}{N} \]

\[ \sum_{i=1}^{N} x_i = \sum_i x_i = \sum_i x_i = x_1 + x_2 + \ldots + x_N \quad \text{sigma notation} \]

common abbreviations
The standard deviation

\[ d_i = x_i - \bar{x} \quad \text{deviation of } x_i \text{ from } \bar{x} \]

\[ \sigma_x = \sqrt{\frac{1}{N} \sum (x_i - \bar{x})^2} \quad \text{average uncertainty of the measurements } x_1, \ldots, x_N \]

uncertainty in any one measurement of \( x \) → \( \delta x = \sigma_x \)

68% of measurements will fall in the range \( x_{true} \pm \sigma_x \)
The standard deviation of the mean

\[
\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}
\]

is the standard deviation of the mean

\[
\delta x = \sigma_{\bar{x}}
\]

uncertainty in \(\bar{x}\)

based on the \(N\) measured values \(x_1, \ldots, x_N\) we can state our final answer for the value of \(x\):

\[
(value \ of \ x) = x_{\text{best}} \pm \delta x
\]

\[
x_{\text{best}} = \bar{x}
\]

\[
\delta x = \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}
\]

\[
(value \ of \ x) = \bar{x} \pm \sigma_{\bar{x}}
\]
Example
We make measurements of the period of a pendulum 3 times and find the results:
\( T = 2.0, 2.1, \text{ and } 2.2 \text{ s.} \)
(a) What is the mean period?
(b) What is the RMS error (the standard deviation) in the period?
(c) What is the error in the mean period (the standard deviation of the mean)?
(d) What is the best estimate for the period and the uncertainty in the best estimate.

\[
\bar{x} = \frac{1}{N} \sum x_i \quad \sigma_x = \sqrt{\frac{1}{N-1} \sum (x_i - \bar{x})^2} \quad \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}} \quad (\text{value of } x) = \bar{x} \pm \sigma_{\bar{x}}
\]

\[
\bar{T} = \frac{1}{N} \sum T_i = \frac{1}{3} (2 + 2.1 + 2.2) = 2.1 \text{ s}
\]
\[
\sigma_T = \sqrt{\frac{1}{N-1} \sum (T_i - \bar{T})^2} = \sqrt{\frac{1}{2} [(2 - 2.1)^2 + (2.1 - 2.1)^2 + (2.2 - 2.1)^2]} = \sqrt{\frac{1}{2} [0.1^2 + 0.1^2]} = 0.1 \text{ s}
\]
\[
\sigma_T = \frac{\sigma_T}{\sqrt{N}} = \frac{0.1}{\sqrt{3}} = 0.057735 \text{ s} \rightarrow 0.06 \text{ s}
\]
\[
T = \bar{T} \pm \sigma_T = 2.10 \pm 0.06 \text{ s}
\]
Systematic errors

\[ \delta x = \sqrt{(\delta x_{\text{ran}})^2 + (\delta x_{\text{sys}})^2} \]

random component \quad systematic component