PHYSICS 140B : STATISTICAL PHYSICS
HW ASSIGNMENT #1 SOLUTIONS

(1) For the Dieterici equation of state,

$$p(V - Nb) = Nk_{\mathrm{B}}T\, e^{-Na/Vk_{\mathrm{B}}T} \quad ,$$

find the virial coefficients $B_2(T)$ and $B_3(T)$.

Solution :

We first write the equation of state as $p = p(n,T)$ where $n = N/V$:

$$p = \frac{nk_{\mathrm{B}}T}{1-bn}\, e^{-an/k_{\mathrm{B}}T} \quad .$$

Next, we expand in powers of the density n:

$$\begin{aligned} p &= nk_{\mathrm{B}}T\,\big(1 + bn + b^2n^2 + \ldots\big)\,\big(1 - \beta an + \tfrac{1}{2}\beta^2a^2n^2 + \ldots\big) \\ &= nk_{\mathrm{B}}T\Big[1 + \big(b - \beta a\big)\,n + \big(b^2 - \beta ab + \tfrac{1}{2}\beta^2a^2\big)\,n^2 + \ldots\Big] \\ &= nk_{\mathrm{B}}T\Big[1 + B_2\,n + B_3\,n^2 + \ldots\Big] \quad , \end{aligned}$$

where $\beta = 1/k_{\mathrm{B}}T$. We can now read off the virial coefficients:

$$B_2(T) = b - \frac{a}{k_{\mathrm{B}}T} \qquad , \qquad B_3(T) = b^2 - \frac{ab}{k_{\mathrm{B}}T} + \frac{a^2}{2k_{\mathrm{B}}^2T^2} \quad .$$

(2) Consider a gas of particles with dispersion $\varepsilon(\boldsymbol{k}) = \varepsilon_0\,|\boldsymbol{k}\ell|^{5/2}$, where ε_0 is an energy scale and ℓ is a length scale. Find the density of states $g(\varepsilon)$ in $d = 2$ and $d = 3$ dimensions.

Solution :

(a) For $\varepsilon(\boldsymbol{k}) = \varepsilon_0\,|\boldsymbol{k}\ell|^{\alpha}$ we have

$$\begin{aligned} g(\varepsilon) = \int\!\frac{d^d\!k}{(2\pi)^d}\,\delta\big(\varepsilon - \varepsilon(\boldsymbol{k})\big) &= \frac{\Omega_d}{(2\pi)}\int\limits_0^\infty\! dk\; k^{d-1}\,\frac{\delta\big(k - (\varepsilon/\varepsilon_0)^{1/\alpha}/\ell\big)}{\alpha\varepsilon_0\ell^\alpha\,k^{\alpha-1}} \\ &= \frac{\Omega_d}{(2\pi)^d}\,\frac{1}{\alpha\varepsilon_0\ell^d}\left(\frac{\varepsilon}{\varepsilon_0}\right)^{\frac{d}{\alpha}-1}\Theta(\varepsilon) \quad . \end{aligned}$$

Thus, for $\alpha = \frac{5}{2}$,

$$g_{d=2}(\varepsilon) = \frac{1}{5\pi\varepsilon_0\ell^2}\left(\frac{\varepsilon}{\varepsilon_0}\right)^{-1/5}\Theta(\varepsilon) \qquad , \qquad g_{d=3}(\varepsilon) = \frac{1}{5\pi\varepsilon_0\ell^3}\left(\frac{\varepsilon}{\varepsilon_0}\right)^{1/5}\Theta(\varepsilon) \quad .$$

(3) For the dispersion $\varepsilon(\boldsymbol{k}) = A\,|\boldsymbol{k}|^4$ obtain expressions for the second virial coefficient $B_2(T)$ for the Bose-Einstein and Fermi-Dirac cases. Assume $d = 3$ dimensions.

Solution :

From §5.3 off the lecture notes, we have

$$n(T,z) = \sum_{j=1}^{\infty} C_j(T)\, z^j \qquad , \qquad p(T,z) = k_{\rm B}T \sum_{j=1}^{\infty} j^{-1} C_j(T)\, z^j \quad ,$$

whence

$$p(T,n) = n k_{\rm B}T \Big(1 + B_2(T)n + B_3(T)n^2 + \ldots\Big)$$

with

$$B_2 = -\frac{C_2}{2C_1^2} \qquad , \qquad B_3 = \frac{C_2^2}{C_1^4} - \frac{2C_3}{C_1^3} \quad .$$

After obtaining the density of states,

$$g(\varepsilon) = \frac{\varepsilon^{-1/4}}{8\pi^2 A^{3/4}} \quad ,$$

we find that the $C_j(T)$ are given by

$$C_j(T) = (\pm 1)^{j-1} \int\limits_{-\infty}^{\infty} \! d\varepsilon\, g(\varepsilon)\, e^{-j\varepsilon/k_{\rm B}T} = (\pm 1)^{j-1}\, \frac{\Gamma\big(\frac{3}{4}\big)}{8\pi^2} \left(\frac{k_{\rm B}T}{A}\right)^{3/4} j^{-3/4} = (\pm 1)^{j-1}\, \frac{j^{-3/4}}{\Omega} \quad ,$$

where

$$\Omega = \frac{8\pi^2}{\Gamma\big(\frac{3}{4}\big)} \left(\frac{A}{k_{\rm B}T}\right)^{3/4}$$

has dimensions of volume. Thus,

$$C_1 = \Omega^{-1} \qquad , \qquad C_2 = \pm 2^{-3/4}\, \Omega^{-1} \qquad , \qquad C_3 = 3^{-3/4}\, \Omega^{-1}$$

and

$$B_2 = \mp 2^{-7/4}\, \Omega \qquad , \qquad B_3 = \big(2^{-3/2} - 2 \cdot 3^{-3/4}\big)\, \Omega^2 \quad .$$

(4) A gas of quantum particles with photon statistics in $d = 3$ dimensions has dispersion $\varepsilon(\boldsymbol{k}) = A\,|\boldsymbol{k}|^{3/2}$.

(a) Find the single particle density of states per unit volume $g(\varepsilon)$.

(b) Repeat the arguments of §5.5.2 in the lecture notes for this dispersion.

(c) Assuming our known values for the surface temperature of the sun, the radius of the earth-sun orbit, and the radius of the earth, what would you expect the surface temperature of the earth to be if the sun radiated particles with this dispersion instead of photons? (Hint: study §5.5.5 of the lecture notes.)

Solution :

(a) The general expression for $g(\varepsilon)$ is obtained by setting

$$g(\varepsilon)\,d\varepsilon = \mathsf{g}\,\frac{d^dk}{(2\pi)^d} = \frac{\mathsf{g}\,\Omega_d}{(2\pi)^d}\,k^{d-1}\,dk \quad\Rightarrow\quad g(\varepsilon) = \frac{\mathsf{g}\Omega_d}{(2\pi)^d}\,\frac{k^{d-1}}{d\varepsilon/dk} \quad ,$$

where g is the internal degeneracy and, for $\varepsilon(k) = A\,k^\alpha$, we have $d\varepsilon/dk = \alpha A\,k^{\alpha-1}$ and thus

$$g(\varepsilon) = \frac{\mathsf{g}\Omega_d}{(2\pi)^d}\,\frac{k^{d-\alpha}}{\alpha A} = \frac{1}{(2\pi)^d}\,\frac{\mathsf{g}\Omega_d}{\alpha A^{d/\alpha}}\,\varepsilon^{\frac{d}{\alpha}-1} \quad .$$

With $\mathsf{g} = 1$, $\alpha = \frac{3}{2}$, and $d = 3$ we have

$$g(\varepsilon) = \frac{\varepsilon}{3\pi^2 A^2} \quad ,$$

with $\varepsilon \geq 0$.

(b) Scaling volume by λ scales the lengths by $\lambda^{1/3}$, the quantized wavevectors by $\lambda^{-1/3}$, and the energy eigenvalues by $\lambda^{1/2}$, since $\varepsilon \propto k^{3/2}$. Thus,

$$p = -\left(\frac{\partial E}{\partial V}\right)_S = \frac{E}{2V} \quad ,$$

which says

$$\left(\frac{\partial E}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p = 2p \quad\Rightarrow\quad p(T) = \mathcal{C}\,T^3 \quad .$$

Indeed,

$$p(T) = -k_{\rm B}T\int\limits_{-\infty}^{\infty} d\varepsilon\; g(\varepsilon)\,\ln\!\big(1 - e^{-\varepsilon/k_{\rm B}T}\big) = -\frac{(k_{\rm B}T)^3}{3\pi^2 A^2}\int\limits_0^\infty du\; u\,\ln\!\big(1 - e^{-u}\big) = \frac{\zeta(3)}{3\pi^2 A^2}\,(k_{\rm B}T)^3 \quad .$$

(c) See §5.5.5 of the Lecture Notes. Assume a dispersion of the form $\varepsilon(k)$ for the (non-conserved) bosons. Then the energy current incident on a differential area dA of surface normal to $\hat{\boldsymbol{z}}$ is

$$dP = dA\cdot\int\!\frac{d^3k}{(2\pi)^3}\,\Theta(\cos\theta)\cdot\varepsilon(k)\cdot\frac{1}{\hbar}\frac{\partial\varepsilon(k)}{\partial k_z}\cdot\frac{1}{e^{\varepsilon(k)/k_{\rm B}T} - 1} \quad .$$

Note that

$$\frac{\partial\varepsilon(k)}{\partial k_z} = \frac{k_z}{k}\,\frac{\partial\varepsilon}{\partial k} = \cos\theta\,\varepsilon'(k) \quad .$$

Now let us assume a power law dispersion $\varepsilon(k) = Ak^\alpha$. Changing variables to $t = Ak^\alpha/k_{\rm B}T$, we find

$$\frac{dP}{dA} = \sigma\,T^{2+\frac{2}{\alpha}} \quad ,$$

where

$$\sigma = \zeta\big(2+\tfrac{2}{\alpha}\big)\,\Gamma\big(2+\tfrac{2}{\alpha}\big)\cdot\frac{\mathsf{g}\,k_{\rm B}^{2+\frac{2}{\alpha}}\,A^{-\frac{2}{\alpha}}}{8\pi^2\hbar}\quad .$$

One can check that for $\mathsf{g}=2$, $A=\hbar c$, and $\alpha=1$ that this result reduces to Stefan's Law.

Equating the power incident on the earth to that radiated by the earth,

$$4\pi R_\odot^2\cdot\sigma T_\odot^{2(1+\alpha^{-1})}\cdot\frac{\pi R_{\rm e}^2}{4\pi a_{\rm e}^2}=4\pi R_{\rm e}^2\cdot\sigma T_{\rm e}^{2(1+\alpha^{-1})}\quad ,$$

which yields

$$T_{\rm e}=\left(\frac{R_\odot}{2a_{\rm e}}\right)^{\frac{\alpha}{\alpha+1}}T_\odot\quad .$$

Plugging in the appropriate constants and setting $\alpha=\frac{3}{2}$, we obtain $T_{\rm e}=152\,{\rm K}$. Brrr!