PHYSICS 140B : STATISTICAL PHYSICS
FINAL EXAMINATION

(1) Provide clear, accurate, and substantial answers for each of the following:

(a) For a fermionic system of number density n and with single particle dispersion $\varepsilon(\boldsymbol{k})$, where $\boldsymbol{k}$ is the wavevector, what is the definition of the Fermi energy and the Fermi surface? `[5 points]`

(b) Write down the symmetric transfer matrix R for the one-dimensional spin-1 Ising Hamiltonian,

$$\hat{H} = -J \sum_n S_n S_{n+1} \quad ,$$

where each $S_n \in \{-1, 0, +1\}$. `[5 points]`

(c) For the cluster γ shown in Fig. 1, identify the symmetry factor s_γ, the lowest order virial coefficient B_j to which γ contributes, and write an expression for the cluster integral $b_\gamma(T)$ in terms of the Mayer function $f(r)$. `[5 points]`

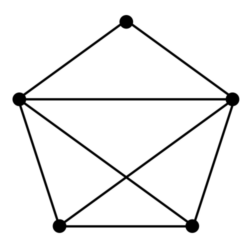

Figure 1: The connected cluster γ for problem 1c.

(d) Describe the physics of spinodal decomposition, phase separation, and the Maxwell construction. Include a sketch of $p(v,T)$ *versus* v to illustrate your description. `[5 points]`

(e) What does it mean to say that for the Landau free energy density (with $b > 0$)

$$f(m) = \tfrac{1}{2} a m^2 - \tfrac{1}{3} y m^3 + \tfrac{1}{4} b m^4 \quad ,$$

that "a first order transition preempts the second order transition"? `[5 points]`

(2) Consider the equation of state

$$p(T,v) = \frac{RT}{v-b} \, \exp\!\left(-\frac{a}{RTv^2}\right) \quad ,$$

where v is the volume per mole.

(a) Find v_c. `[5 points]`

(b) Find T_c. `[5 points]`

(c) Find p_c. `[5 points]`

(d) Defining the dimensionless quantities $\bar{p} \equiv p/p_{\text{c}}$, $\bar{T} \equiv T/T_{\text{c}}$, and $\bar{v} \equiv v/v_{\text{c}}$, write the equation of state $\bar{p} = \bar{p}(\bar{T}, \bar{v})$. Show that $\bar{p}(\bar{T} = 1, \bar{v} = 1) = 1$. `[10 points]`

(3) Consider a system consisting of mobile ions of charge $+Ze > 0$ and electrons of charge $-e < 0$. Let the ion mass be m_+ and the electron mass be m_-. The average number density of ions is n_+.

(a) Let $z_\pm$ be the fugacities for the ions $(+)$ and electrons $(-)$. Within Debye-Hückel theory, what is the formula for the charge density $\rho(\boldsymbol{r})$? *Hint: Your formula should involve the local potential* $\phi(\boldsymbol{r})$. `[5 points]`

(b) Assuming overall charge neutrality, what is the number density n_- of electrons? What is the relation between the number densities $n_\pm$, the fugacities $z_\pm$, and the masses $m_\pm$ at temperature T? *Hint: At* $|\boldsymbol{r}| \to \infty$*, take* $\phi(\boldsymbol{r}) \to 0$. `[5 points]`

(c) What is the full nonlinear self-consistent equation for $\phi(\boldsymbol{r})$? `[5 points]`

(d) Assuming $|e\phi(\boldsymbol{r})| \ll k_{\text{B}}T$, the linearized self-consistent equation for $\phi(\boldsymbol{r})$ in the presence of an external charge distribution $\rho_{\text{ext}}(\boldsymbol{r}) = Q\,\delta(\boldsymbol{r})$ is

$$\nabla^2 \phi = \kappa_{\text{D}}^2\, \phi - 4\pi Q\, \delta(\boldsymbol{r}) \quad ,$$

where κ_{D} is the Debye screening wavevector. Find an expression for κ_{D}. `[5 points]`

(e) In $d = 3$ dimensions, again assuming $|e\phi(\boldsymbol{r})| \ll k_{\text{B}}T$, what is the total charge distribution $\rho_{\text{tot}}(\boldsymbol{r})$ in the presence of the external charge Q? `[5 points]`

(4) Consider a four-state Ising model on a cubic lattice with Hamiltonian

$$\hat{H} = -J \sum_{\langle ij \rangle} S_i S_j - H \sum_i S_i \quad ,$$

where each spin variable S_i takes on one of four possible values: $S_i \in \{-2, -1, +1, +2\}$, and the first sum is over all nearest-neighbor pairs of the lattice (*i.e.* over all unique links). Note there is no $S_i = 0$ state.

(a) What is the mean field Hamiltonian $\hat{H}_{\text{MF}}$? `[5 points]`

(b) Find the mean field free energy per site $f(\theta, h, m)$, where $m = \langle S_i \rangle$, $\theta = k_{\text{B}}T/zJ$, $h = H/zJ$, and $f = F/NzJ$. Here z is the lattice coordination number. `[5 points]`

(c) Find the mean field equation relating m, θ, and h. `[5 points]`

(d) Expand f to fourth order in m, retaining terms only to first order in h, and working to lowest order in $\theta - \theta_{\text{c}}$. What is θ_{c}? `[5 points]`

(e) If $J/k_{\text{B}} = 100\,\text{K}$, what is the critical temperature T_{c}? `[5 points]`