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Chapter 13

Applications of BCS Theory

13.1 Quantum XY Model for Granular Superconductors

Consider a set of superconducting grains, each of which is large enough to be modeled by BCS theory, but small enough that the self-capacitance (*i.e.* Coulomb interaction) cannot be neglected. The Coulomb energy of the $j^{\rm th}$ grain is written as

$$\hat{U}_j = \frac{2e^2}{C_j}\big(\hat{M}_j - \bar{M}_j\big)^2 \quad , \tag{13.1}$$

where $\hat{M}_j$ is the operator which counts the number of Cooper pairs on grain j, and $\bar{M}_j$ is the mean number of pairs in equilibrium, which is given by half the total ionic charge on the grain. The capacitance C_j is a geometrical quantity which is proportional to the radius of the grain, assuming the grain is roughly spherical. For very large grains, the Coulomb interaction is negligible. It should be stressed that here we are accounting for only the long wavelength part of the Coulomb interaction, which is proportional to $4\pi\big|\delta\hat{\rho}(q_{\rm min})\big|^2/q^2_{\rm min}$, where $q_{\rm min} \sim 1/R_j$ is the inverse grain size. The remaining part of the Coulomb interaction is included in the BCS part of the Hamiltonian for each grain.

We assume that $\hat{K}_{{\rm BCS},j}$ describes a simple s-wave superconductor with gap $\Delta_j = |\Delta_j|\, e^{i\phi_j}$. We saw in chapter 3 how ϕ_j is conjugate to the Cooper pair number operator $\hat{M}_j$, with

$$\hat{M}_j = \frac{1}{i}\,\frac{\partial}{\partial\phi_j} \quad . \tag{13.2}$$

The operator which adds one Cooper pair to grain j is therefore $e^{i\phi_j}$, because

$$\hat{M}_j\, e^{i\phi_j} = e^{i\phi_j}\,(\hat{M}_j + 1) \quad . \tag{13.3}$$

Thus, accounting for the hopping of Cooper pairs between neighboring grains, the effective Hamiltonian for a granular superconductor should be given by

$$\hat{H}_{\text{gr}} = -\tfrac{1}{2}\sum_{i,j} J_{ij}\big(e^{i\phi_i}\,e^{-i\phi_j} + e^{-i\phi_i}\,e^{i\phi_j}\big) + \sum_i \frac{2e^2}{C_j}\big(\hat{M}_j - \bar{M}_j\big)^2 \quad , \tag{13.4}$$

where J_{ij} is the hopping matrix element for the Cooper pairs, here assumed to be real.

Before we calculate J_{ij}, note that we can eliminate the constants $\bar{M}_i$ from the Hamiltonian via the unitary transformation $\hat{H}_{\text{gr}} \to \hat{H}'_{\text{gr}} = V^\dagger \hat{H}_{\text{gr}} V$, where $V = \prod_j e^{i\,[\bar{M}_j]\,\phi_j}$, where $[\bar{M}_j]$ is defined as the integer nearest to $\bar{M}_j$. The difference, $\delta\bar{M}_j = \bar{M}_j - [\bar{M}_j]$, cannot be removed. This transformation commutes with the hopping part of $\hat{H}_{\text{gr}}$, so, after dropping the prime on $\hat{H}'_{\text{gr}}$, we are left with

$$\hat{H}_{\text{gr}} = \sum_j \frac{2e^2}{C_j}\left(\frac{1}{i}\,\frac{\partial}{\partial\phi_j} - \delta\bar{M}_j\right)^2 - \sum_{i,j} J_{ij}\cos(\phi_i - \phi_j) \quad . \tag{13.5}$$

In the presence of an external magnetic field,

$$\hat{H}_{\text{gr}} = \sum_j \frac{2e^2}{C_j}\left(\frac{1}{i}\,\frac{\partial}{\partial\phi_j} - \delta\bar{M}_j\right)^2 - \sum_{i,j} J_{ij}\cos(\phi_i - \phi_j - \mathcal{A}_{ij}) \quad , \tag{13.6}$$

where

$$\mathcal{A}_{ij} = \frac{2e}{\hbar c}\int\limits_{\boldsymbol{R}_i}^{\boldsymbol{R}_j} d\boldsymbol{l}\cdot\boldsymbol{A} \tag{13.7}$$

is a lattice vector potential, with $\boldsymbol{R}_i$ the position of grain i.

13.1.1 No disorder

In a perfect lattice of identical grains, with $J_{ij} = J$ for nearest neighbors, $\delta\bar{M}_j = 0$ and $2e^2/C_j = U$ for all j, we have

$$\hat{H}_{\text{gr}} = -U\sum_i \frac{\partial^2}{\partial\phi_i^2} - 2J\sum_{\langle ij\rangle}\cos(\phi_i - \phi_j) \quad , \tag{13.8}$$

where $\langle ij\rangle$ indicates a nearest neighbor pair. This model, known as the *quantum rotor model*, features competing interactions. The potential energy, proportional to U, favors each grain being in a state $\psi(\phi_i) = 1$, corresponding to $M = 0$, which minimizes the Coulomb interaction. However, it does a poor job with the hopping, since $\big\langle\cos(\phi_i - \phi_j)\big\rangle = 0$ in this state. The kinetic (hopping) energy, proportional to J, favors that all grains be coherent with $\phi_i = \alpha$ for all i, where α is a constant. This state has significant local charge fluctuations which cost Coulomb energy – an infinite amount, in fact! Some sort of compromise must be reached. One important issue is whether the ground state exhibits a finite order parameter $\langle e^{i\phi_i}\rangle$.

The model has been simulated numerically using a cluster Monte Carlo algorithm[1], and is known to exhibit a quantum phase transition between superfluid and insulating states at a critical value of J/U. The superfluid state is that in which $\langle e^{i\phi_i}\rangle \neq 0$.

13.1.2 Self-consistent harmonic approximation

The self-consistent harmonic approximation (SCHA) is a variational approach in which we approximate the ground state wavefunction as a Gaussian function of the many phase variables $\{\phi_i\}$. Specifically, we write

$$\Psi[\phi] = \mathcal{C}\,\exp\!\big(-\tfrac{1}{4}\,A_{ij}\,\phi_i\,\phi_j\big) \quad , \tag{13.9}$$

where $\mathcal{C}$ is a normalization constant. The matrix elements A_{ij} is assumed to be a function of the separation $\boldsymbol{R}_i - \boldsymbol{R}_j$, where $\boldsymbol{R}_i$ is the position of lattice site i. We define the *generating function*

$$Z[\mathcal{J}] = \int\! D\phi\,\big|\Psi[\phi]\big|^2\,e^{-\mathcal{J}_i\,\phi_i} = Z[0]\,\exp\!\big(\tfrac{1}{2}\,\mathcal{J}_i\,A^{-1}_{ij}\,\mathcal{J}_j\big) \quad . \tag{13.10}$$

Here $\mathcal{J}_i$ is a *source field* with respect to which we differentiate in order to compute correlation functions, as we shall see. Here $D\phi = \prod_i d\phi_i$, and all the phase variables are integrated over the $\phi_i \in (-\infty,+\infty)$. Right away we see something is fishy, since in the original model there is a periodicity under $\phi_i \to \phi_i + 2\pi$ at each site. The individual basis functions are $\psi_n(\phi) = e^{in\phi}$, corresponding to $M = n$ Cooper pairs. Taking linear combinations of these basis states preserves the 2π periodicity, but this is not present in our variational wavefunction. Nevertheless, we can extract some useful physics using the SCHA.

The first order of business is to compute the correlator

$$\langle\,\Psi\,|\,\phi_i\,\phi_j\,|\,\Psi\,\rangle = \frac{1}{Z[0]}\,\frac{\partial^2 Z[\mathcal{J}]}{\partial\mathcal{J}_i\,\partial\mathcal{J}_j}\bigg|_{\mathcal{J}=0} = A^{-1}_{ij} \quad . \tag{13.11}$$

This means that

$$\langle\,\Psi\,|\,e^{i(\phi_i-\phi_j)}\,|\,\Psi\,\rangle = e^{-\langle(\phi_i-\phi_j)^2\rangle/2} = e^{-(A^{-1}_{ii}-A^{-1}_{ij})} \quad . \tag{13.12}$$

Here we have used that $\langle e^Q\rangle = e^{\langle Q^2\rangle/2}$ where Q is a sum of Gaussian-distributed variables. Next, we need

$$\begin{aligned}\langle\,\Psi\,|\,\frac{\partial^2}{\partial\phi_i^2}\,|\,\Psi\,\rangle &= -\langle\,\Psi\,|\,\frac{\partial}{\partial\phi_i}\,\tfrac{1}{2}\,A_{ik}\,\phi_k\,|\,\Psi\,\rangle \\ &= -\tfrac{1}{2}\,A_{ii} + \tfrac{1}{4}\,A_{ik}\,A_{li}\,\langle\,\Psi\,|\,\phi_k\,\phi_l\,|\,\Psi\,\rangle = -\tfrac{1}{4}\,A_{ii} \quad .\end{aligned} \tag{13.13}$$

[1]See F. Alet and E. Sørensen, *Phys. Rev. E* **67**, 015701(R) (2003) and references therein.

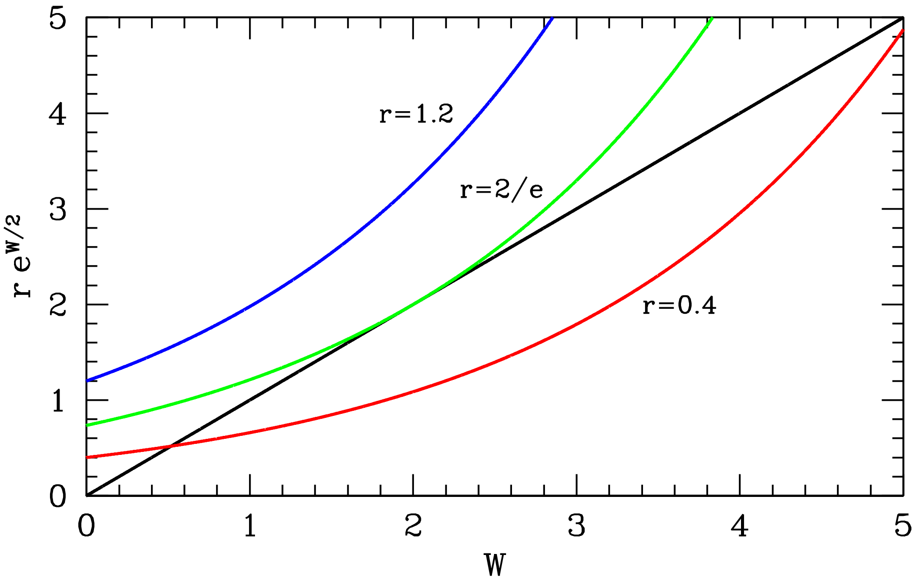

Figure 13.1: Graphical solution to the SCHA equation $W = r\,\exp\left(\frac{1}{2}W\right)$ for three representative values of r. The critical value is $r_c = 2/e = 0.73576$.

Thus, the variational energy per site is

$$\begin{aligned}\frac{1}{N}\,\langle\,\Psi\,|\,\hat{H}_{\rm gr}\,|\,\Psi\,\rangle &= \tfrac{1}{4}\,UA_{ii} - zJ\,e^{-(A^{-1}_{ii}-A^{-1}_{ij})}\\ &= \tfrac{1}{4}U\!\int\!\frac{d^dk}{(2\pi)^d}\,\hat{A}(\boldsymbol{k}) - zJ\,\exp\left\{-\!\int\!\frac{d^dk}{(2\pi)^d}\,\frac{1-\gamma_{\boldsymbol{k}}}{\hat{A}(\boldsymbol{k})}\right\}\quad,\end{aligned}\tag{13.14}$$

where z is the lattice coordination number ($N_{\rm links} = \frac{1}{2}zN$),

$$\gamma_{\boldsymbol{k}} = \frac{1}{z}\sum_{\boldsymbol{\delta}} e^{i\boldsymbol{k}\cdot\boldsymbol{\delta}}\tag{13.15}$$

is a sum over the z nearest neighbor vectors $\boldsymbol{\delta}$, and $\hat{A}(\boldsymbol{k})$ is the Fourier transform of A_{ij},

$$A_{ij} = \int\!\frac{d^dk}{(2\pi)^d}\,\hat{A}(\boldsymbol{k})\,e^{i(\boldsymbol{R}_i-\boldsymbol{R}_j)}\quad.\tag{13.16}$$

Note that $\hat{A}^*(\boldsymbol{k}) = \hat{A}(-\boldsymbol{k})$ since $\hat{A}(\boldsymbol{k})$ is the (discrete) Fourier transform of a real quantity.

We are now in a position to vary the energy in Eqn. 13.14 with respect to the variational parameters $\{\hat{A}(\boldsymbol{k})\}$. Taking the functional derivative with respect to $\hat{A}(\boldsymbol{k})$, we find

$$(2\pi)^d\,\frac{\delta(E_{\rm gr}/N)}{\delta\hat{A}(\boldsymbol{k})} = \tfrac{1}{4}\,U - \frac{1-\gamma_{\boldsymbol{k}}}{\hat{A}^2(\boldsymbol{k})}\cdot zJ\,e^{-W}\quad,\tag{13.17}$$

where

$$W = \int \frac{d^d k}{(2\pi)^d} \, \frac{1-\gamma_{\boldsymbol{k}}}{\hat{A}(\boldsymbol{k})} \quad . \tag{13.18}$$

We now have

$$\hat{A}(\boldsymbol{k}) = 2 \left(\frac{zJ}{U} \right)^{1/2} e^{-W/2} \sqrt{1-\gamma_{\boldsymbol{k}}} \quad . \tag{13.19}$$

Inserting this into our expression for W, we obtain the self-consistent equation

$$W = r\, e^{W/2} \qquad ; \qquad r = C_d \left(\frac{U}{4zJ} \right)^{1/2} \qquad , \qquad C_d \equiv \int \frac{d^d k}{(2\pi)^d} \, \sqrt{1-\gamma_{\boldsymbol{k}}} \quad . \tag{13.20}$$

One finds $C_{d=1} = 0.900316$ for the linear chain, $C_{d=2} = 0.958091$ for the square lattice, and $C_{d=3} = 0.974735$ on the cubic lattice.

The graphical solution to $W = r \, \exp\left(\frac{1}{2}W\right)$ is shown in Fig. 13.1. One sees that for $r > r_{\rm c} = 2/e \simeq 0.73576$, there is no solution. In this case, the variational wavefunction should be taken to be $\Psi = 1$, which is a product of $\psi_{n=0}$ states on each grain, corresponding to fixed charge $M_i = 0$ and maximally fluctuating phase. In this case we must restrict each $\phi_i \in [0, 2\pi]$. When $r < r_{\rm c}$, though, there are two solutions for W. The larger of the two is spurious, and the smaller one is the physical one. As J/U increases, *i.e.* r decreases, the size of $\hat{A}(\boldsymbol{k})$ increases, which means that A^{-1}_{ij} decreases in magnitude. This means that the correlation in Eqn. 13.12 is growing, and the phase variables are localized. The SCHA predicts a spurious first order phase transition; the real superfluid-insulator transition is continuous (second-order)[2].

13.1.3 Calculation of the Cooper pair hopping amplitude

Finally, let us compute J_{ij}. We do so by working to second order in perturbation theory in the *electron* hopping Hamiltonian

$$\hat{H}_{\rm hop} = -\frac{1}{(V_i \, V_j)^{1/2}} \sum_{\langle ij \rangle} \sum_{\boldsymbol{k},\boldsymbol{k}',\sigma} \Big(t_{ij}(\boldsymbol{k},\boldsymbol{k}') \, c^\dagger_{i,\boldsymbol{k},\sigma} \, c_{j,\boldsymbol{k}',\sigma} + t^*_{ij}(\boldsymbol{k},\boldsymbol{k}') \, c^\dagger_{j,\boldsymbol{k}',\sigma} \, c_{i,\boldsymbol{k},\sigma} \Big) \quad . \tag{13.21}$$

Here $t_{ij}(\boldsymbol{k},\boldsymbol{k}')$ is the amplitude for an electron of wavevector $\boldsymbol{k}'$ in grain j to hop to a state of wavevector $\boldsymbol{k}$ in grain i. To simplify matters we will assume the grains are identical in all respects other than their overall phases. We'll write the fermion destruction operators on grain i as $c_{\boldsymbol{k}\sigma}$ and those on grain j as $\tilde{c}_{\boldsymbol{k}\sigma}$. We furthermore assume $t_{ij}(\boldsymbol{k},\boldsymbol{k}') = t$ is real and independent of $\boldsymbol{k}$ and $\boldsymbol{k}'$. Only spin polarization, and not momentum, is preserved in the hopping process. Then

$$\hat{H}_{\rm hop} = -\frac{t}{V} \sum_{\boldsymbol{k},\boldsymbol{k}'} \left(c^\dagger_{\boldsymbol{k}\sigma} \, \tilde{c}_{\boldsymbol{k}'\sigma} + \tilde{c}^\dagger_{\boldsymbol{k}'\sigma} \, c_{\boldsymbol{k}\sigma} \right) \quad . \tag{13.22}$$

[2]That the SCHA gives a spurious first order transition was recognized by E. Pytte, *Phys. Rev. Lett.* **28**, 895 (1971).

Each grain is described by a BCS model. The respective Bogoliubov transformations are

$$\begin{aligned} c_{\boldsymbol{k}\sigma} &= \cos\vartheta_{\boldsymbol{k}}\,\gamma_{\boldsymbol{k}\sigma} - \sigma\sin\vartheta_{\boldsymbol{k}}\,e^{i\phi}\,\gamma^{\dagger}_{-\boldsymbol{k}\,-\sigma} \\ \tilde{c}_{\boldsymbol{k}\sigma} &= \cos\tilde{\vartheta}_{\boldsymbol{k}}\,\tilde{\gamma}_{\boldsymbol{k}\sigma} - \sigma\sin\tilde{\vartheta}_{\boldsymbol{k}}\,e^{i\tilde{\phi}}\,\tilde{\gamma}^{\dagger}_{-\boldsymbol{k}\,-\sigma} \quad . \end{aligned} \tag{13.23}$$

Second order perturbation says that the ground state energy $\mathcal{E}$ is

$$\mathcal{E} = \mathcal{E}_0 - \sum_n \frac{\big|\langle\, n \,|\, \hat{H}_{\text{hop}} \,|\, \text{G}\,\rangle\big|^2}{\mathcal{E}_n - \mathcal{E}_0} \quad , \tag{13.24}$$

where $|\,\text{G}\,\rangle = |\,\text{G}_i\,\rangle \otimes |\,\text{G}_j\,\rangle$ is a product of BCS ground states on the two grains. Clearly the only intermediate states $|\,n\,\rangle$ which can couple to $|\,\text{G}\,\rangle$ through a single application of $\hat{H}_{\text{hop}}$ are states of the form

$$|\,\boldsymbol{k}, \boldsymbol{k}', \sigma\,\rangle = \gamma^{\dagger}_{\boldsymbol{k}\sigma}\,\tilde{\gamma}^{\dagger}_{-\boldsymbol{k}'\,-\sigma}\,|\,\text{G}\,\rangle \quad , \tag{13.25}$$

and for this state

$$\langle\, \boldsymbol{k}, \boldsymbol{k}', \sigma \,|\, \hat{H}_{\text{hop}} \,|\, \text{G}\,\rangle = -\sigma\Big(\cos\vartheta_{\boldsymbol{k}}\sin\tilde{\vartheta}_{\boldsymbol{k}'}\,e^{i\tilde{\phi}} + \sin\vartheta_{\boldsymbol{k}}\cos\tilde{\vartheta}_{\boldsymbol{k}'}\,e^{i\phi}\Big) \tag{13.26}$$

The energy of this intermediate state is

$$E_{\boldsymbol{k},\boldsymbol{k}',\sigma} = E_{\boldsymbol{k}} + E_{\boldsymbol{k}'} + \frac{e^2}{C} \quad , \tag{13.27}$$

where we have included the contribution from the charging energy of each grain. Then we find[3]

$$\mathcal{E}^{(2)} = \mathcal{E}_0' - J\cos(\phi - \tilde{\phi}\,) \quad , \tag{13.28}$$

where

$$J = \frac{|t|^2}{V^2}\sum_{\boldsymbol{k},\boldsymbol{k}'} \frac{\Delta_{\boldsymbol{k}}}{E_{\boldsymbol{k}}} \cdot \frac{\Delta_{\boldsymbol{k}'}}{E_{\boldsymbol{k}'}} \cdot \frac{1}{E_{\boldsymbol{k}} + E_{\boldsymbol{k}'} + (e^2/C)} \quad . \tag{13.29}$$

For a general set of dissimilar grains,

$$J_{ij} = \frac{|t_{ij}|^2}{V_i V_j}\sum_{\boldsymbol{k},\boldsymbol{k}'} \frac{\Delta_{i,\boldsymbol{k}}}{E_{i,\boldsymbol{k}}} \cdot \frac{\Delta_{j,\boldsymbol{k}'}}{E_{j,\boldsymbol{k}'}} \cdot \frac{1}{E_{i,\boldsymbol{k}} + E_{j,\boldsymbol{k}'} + (e^2/2C_{ij})} \quad , \tag{13.30}$$

where $C_{ij}^{-1} = C_i^{-1} + C_j^{-1}$.

[3]There is no factor of two arising from a spin sum since we are summing over all $\boldsymbol{k}$ and $\boldsymbol{k}'$, and therefore summing over spin would overcount the intermediate states $|n\rangle$ by a factor of two.

13.2 Tunneling

We follow the very clear discussion in §9.3 of G. Mahan's *Many Particle Physics*. Consider two bulk samples, which we label left (L) and right (R). The Hamiltonian is taken to be

$$\hat{H} = \hat{H}_{\rm L} + \hat{H}_{\rm R} + \hat{H}_{\rm T} \quad , \tag{13.31}$$

where $\hat{H}_{\rm L,R}$ are the bulk Hamiltonians, and

$$\hat{H}_{\rm T} = -\sum_{i,j,\sigma} \big(T_{ij}\, c^\dagger_{{\rm L}\,i\,\sigma}\, c_{{\rm R}\,j\,\sigma} + T^*_{ij}\, c^\dagger_{{\rm R}\,j\,\sigma}\, c_{{\rm L}\,i\,\sigma} \big) \quad . \tag{13.32}$$

The indices i and j label single particle electron states (*not* Bogoliubov quasiparticles) in the two banks. As we shall discuss below, we can take them to correspond to Bloch wavevectors in a particular energy band. In a nonequilibrium setting we work in the grand canonical ensemble, with

$$\hat{K} = \hat{H}_{\rm L} - \mu_{\rm L}\, \hat{N}_{\rm L} + \hat{H}_{\rm R} - \mu_{\rm R}\, \hat{N}_{\rm R} + \hat{H}_{\rm T} \quad . \tag{13.33}$$

The difference between the chemical potentials is $\mu_{\rm R} - \mu_{\rm L} = eV$, where V is the voltage bias. The current flowing from left to right is

$$I(t) = e \left\langle \frac{d\hat{N}_{\rm L}}{dt} \right\rangle \quad . \tag{13.34}$$

Note that if $N_{\rm L}$ is increasing in time, this means an electron number current flows from right to left, and hence an electrical current (of fictitious positive charges) flows from left to right. We use perturbation theory in $\hat{H}_{\rm T}$ to compute $I(t)$. Note that expectations such as $\langle \Psi_{\rm L} | c_{{\rm L}i} | \Psi_{\rm L} \rangle$ vanish, while $\langle \Psi_{\rm L} |\, c_{{\rm L}i}\, c_{{\rm L}j}\, | \Psi_{\rm L} \rangle$ may not if $|\Psi_{\rm L}\rangle$ is a BCS state.

A few words on the labels i and j: We will assume the left and right samples can be described as perfect crystals, so i and j will represent crystal momentum eigenstates. The only exception to this characterization will be that we assume their respective surfaces are sufficiently rough to destroy conservation of momentum in the plane of the surface. Momentum perpendicular to the surface is also not conserved, since the presence of the surface breaks translation invariance in this direction. The matrix element T_{ij} will be dominated by the behavior of the respective single particle electron wavefunctions in the vicinity of their respective surfaces. As there is no reason for the respective wavefunctions to be coherent, they will in general disagree in sign in random fashion. We then expect the overlap to be proportional to $A^{1/2}$, where A is the junction area, on the basis of the Central Limit Theorem. Adding in the plane wave normalization factors, we therefore approximate

$$T_{ij} = T_{\boldsymbol{q},\boldsymbol{k}} \approx \left(\frac{A}{V_{\rm L} V_{\rm R}} \right)^{1/2} t\big(\xi_{{\rm L}\,\boldsymbol{q}}\,,\, \xi_{{\rm R}\,\boldsymbol{k}}\big) \quad , \tag{13.35}$$

where $\boldsymbol{q}$ and $\boldsymbol{k}$ are the wavevectors of the Bloch electrons on the left and right banks, respectively. Note that we presume spin is preserved in the tunneling process, although wavevector is not.

13.2.1 Perturbation theory

We begin by noting

$$\begin{aligned}\frac{d\hat{N}_{\rm L}}{dt} = \frac{i}{\hbar}\big[\hat{H},\hat{N}_{\rm L}\big] &= \frac{i}{\hbar}\big[\hat{H}_{\rm T}\,,\,\hat{N}_{\rm L}\big] \\ &= -\frac{i}{\hbar}\sum_{i,j,\sigma}\big(T_{ij}\,c^\dagger_{{\rm L}\,i\,\sigma}\,c_{{\rm R}\,j\,\sigma} - T^*_{ij}\,c^\dagger_{{\rm R}\,j\,\sigma}\,c_{{\rm L}\,i\,\sigma}\big) \quad .\end{aligned} \tag{13.36}$$

First order perturbation theory then gives

$$|\,\Psi(t)\,\rangle = e^{-i\hat{H}_0(t-t_0)/\hbar}\,|\,\Psi(t_0)\,\rangle - \frac{i}{\hbar}\,e^{-i\hat{H}_0t/\hbar}\int\limits_{t_0}^{t}\!dt_1\;\hat{H}_{\rm T}(t_1)\,e^{i\hat{H}_0t_0/\hbar}\,|\,\Psi(t_0)\,\rangle + \mathcal{O}\big(\hat{H}^2_{\rm T}\big) \quad , \tag{13.37}$$

where $\hat{H}_0 = \hat{H}_{\rm L} + \hat{H}_{\rm R}$ and

$$\hat{H}_{\rm T}(t) = e^{i\hat{H}_0t/\hbar}\,\hat{H}_{\rm T}\,e^{-i\hat{H}_0t/\hbar} \tag{13.38}$$

is the perturbation (hopping) Hamiltonian in the interaction representation. To lowest order in $\hat{H}_{\rm T}$, then,

$$\langle\,\Psi(t)\,|\,\hat{I}\,|\,\Psi(t)\,\rangle = -\frac{i}{\hbar}\int\limits_{t_0}^{t}\!dt_1\,\langle\,\tilde{\Psi}(t_0)\,|\,\big[\hat{I}(t)\,,\,\hat{H}_{\rm T}(t_1)\big]\,|\,\tilde{\Psi}(t_0)\,\rangle \quad , \tag{13.39}$$

where $|\,\tilde{\Psi}(t_0)\,\rangle = e^{i\hat{H}_0t_0/\hbar}\,|\,\Psi(t_0)\,\rangle$. Setting $t_0 = -\infty$, and averaging over a thermal ensemble of initial states, we have

$$I(t) = -\frac{i}{\hbar}\int\limits_{-\infty}^{t}\!dt'\,\big\langle\big[\hat{I}(t)\,,\,\hat{H}_{\rm T}(t')\big]\big\rangle \quad , \tag{13.40}$$

where $\hat{I}(t) = e\dot{\hat{N}}_{\rm L}(t) = (+e)\,e^{i\hat{H}_0t/\hbar}\,\dot{\hat{N}}_{\rm L}\,e^{-i\hat{H}_0t/\hbar}$ is the charge current flowing from *right* to *left*. Note that it is the electron charge $-e$ that enters here and not the Cooper pair charge, since $\hat{H}_{\rm T}$ describes electron hopping.

There remains a *caveat* which we have already mentioned. The chemical potentials $\mu_{\rm L}$ and $\mu_{\rm R}$ differ according to

$$\mu_{\rm R} - \mu_{\rm L} = eV \quad , \tag{13.41}$$

where V is the bias voltage, *i.e.* the voltage *drop* from left to right. If $V>0$, then $\mu_{\rm R} > \mu_{\rm L}$, which means an electron current flows from right to left, and an electrical current (*i.e.* the direction of positive charge flow) from left to right. We must work in an ensemble described by $\hat{K}_0$, where

$$\hat{K}_0 = \hat{H}_{\rm L} - \mu_{\rm L}\hat{N}_{\rm L} + \hat{H}_{\rm R} - \mu_{\rm R}\hat{N}_{\rm R} \quad . \tag{13.42}$$

We now separate $\hat{H}_{\rm T}$ into its component processes, writing $\hat{H}_{\rm T} = \hat{H}_{\rm T}^{+} + \hat{H}_{\rm T}^{-}$, with

$$\hat{H}_{\rm T}^{+} = -\sum_{i,j,\sigma} T_{ij}\, c^\dagger_{{\rm L}\,i\,\sigma}\, c^{\vphantom{\dagger}}_{{\rm R}\,j\,\sigma} \qquad , \qquad \hat{H}_{\rm T}^{-} = -\sum_{i,j,\sigma} T^*_{ij}\, c^\dagger_{{\rm R}\,j\,\sigma}\, c^{\vphantom{\dagger}}_{{\rm L}\,i\,\sigma} \quad . \tag{13.43}$$

Thus, $\hat{H}_{\rm T}^{+}$ describes hops from R to L, and $\hat{H}_{\rm T}^{-}$ from L to R. Note that $\hat{H}_{\rm T}^{-} = (\hat{H}_{\rm T}^{+})^\dagger$. Therefore $\hat{H}_{\rm T}(t) = \hat{H}_{\rm T}^{+}(t) + \hat{H}_{\rm T}^{-}(t)$, where[4]

$$\begin{aligned} \hat{H}_{\rm T}^{\pm}(t) &= e^{i(\hat{K}_0 + \mu_{\rm L}\hat{N}_{\rm L} + \mu_{\rm R}\hat{N}_{\rm R})t/\hbar}\, \hat{H}_{\rm T}^{\pm}\, e^{-i(\hat{K}_0 + \mu_{\rm L}\hat{N}_{\rm L} + \mu_{\rm R}\hat{N}_{\rm R})t/\hbar} \\ &= e^{\mp ieVt/\hbar}\, e^{i\hat{K}_0 t/\hbar}\, \hat{H}_{\rm T}^{\pm}\, e^{-i\hat{K}_0 t/\hbar} \quad . \end{aligned} \tag{13.44}$$

Note that the current operator is

$$\hat{I} = \frac{ie}{\hbar}\,\big[\hat{H}_{\rm T}\,,\,N_{\rm L}\big] = \frac{ie}{\hbar}\Big(\hat{H}_{\rm T}^{-} - \hat{H}_{\rm T}^{+}\Big) \quad . \tag{13.45}$$

We then have

$$\begin{aligned} I(t) &= \frac{e}{\hbar^2}\int\limits_{-\infty}^{t}\!dt'\,\Big\langle\big[e^{ieVt/\hbar}\,\hat{H}_{\rm T}^{-}(t) - e^{-ieVt/\hbar}\,\hat{H}_{\rm T}^{+}(t)\,,\,e^{ieVt'/\hbar}\,\hat{H}_{\rm T}^{-}(t') + e^{-ieVt'/\hbar}\,\hat{H}_{\rm T}^{+}(t')\big]\Big\rangle \\ &= I_{\rm N}(t) + I_{\rm J}(t) \quad , \end{aligned} \tag{13.46}$$

where

$$I_{\rm N}(t) = \frac{e}{\hbar^2}\int\limits_{-\infty}^{\infty}\! dt'\,\Theta(t-t')\,\Big\{e^{+i\Omega(t-t')}\,\Big\langle\big[\hat{H}_{\rm T}^{-}(t)\,,\,\hat{H}_{\rm T}^{+}(t')\big]\Big\rangle - e^{-i\Omega(t-t')}\,\Big\langle\big[\hat{H}_{\rm T}^{+}(t)\,,\,\hat{H}_{\rm T}^{-}(t')\big]\Big\rangle\Big\} \tag{13.47}$$

and

$$I_{\rm J}(t) = \frac{e}{\hbar^2}\int\limits_{-\infty}^{\infty}\! dt'\,\Theta(t-t')\,\Big\{e^{+i\Omega(t+t')}\,\Big\langle\big[\hat{H}_{\rm T}^{-}(t)\,,\,\hat{H}_{\rm T}^{-}(t')\big]\Big\rangle - e^{-i\Omega(t+t')}\,\Big\langle\big[\hat{H}_{\rm T}^{+}(t)\,,\,\hat{H}_{\rm T}^{+}(t')\big]\Big\rangle\Big\}\ , \tag{13.48}$$

with $\Omega \equiv eV/\hbar$. $I_{\rm N}(t)$ is the usual *single particle tunneling current*, which is present both in normal metals as well as in superconductors. $I_{\rm J}(t)$ is the *Josephson pair tunneling current*, which is only present when the ensemble average is over states of indefinite particle number.

13.2.2 The single particle tunneling current $I_{\rm N}$

We now proceed to evaluate the so-called single-particle current $I_{\rm N}$ in Eqn. 13.47. This current is present, under voltage bias, between normal metal and normal metal, between normal metal

[4] We make use of the fact that $\hat{N}_{\rm L} + \hat{N}_{\rm R}$ commutes with $\hat{H}_{\rm T}^{\pm}$.

and superconductor, and between superconductor and superconductor. It is convenient to define the quantities

$$\begin{aligned}\mathcal{X}_{\mathsf{r}}(t-t') &\equiv -i\,\Theta(t-t')\left\langle\left[\hat{H}^{-}_{\mathrm{T}}(t)\,,\,\hat{H}^{+}_{\mathrm{T}}(t')\right]\right\rangle\\ \mathcal{X}_{\mathsf{a}}(t-t') &\equiv -i\,\Theta(t-t')\left\langle\left[\hat{H}^{-}_{\mathrm{T}}(t')\,,\,\hat{H}^{+}_{\mathrm{T}}(t)\right]\right\rangle \quad ,\end{aligned} \tag{13.49}$$

which differ by the order of the time values of the operators inside the commutator. We then have

$$\begin{aligned}I_{\mathrm{N}} &= \frac{ie}{\hbar^2}\int\limits_{-\infty}^{\infty} dt\,\left\{e^{+i\Omega t}\,\mathcal{X}_{\mathsf{r}}(t)+e^{-i\Omega t}\,\mathcal{X}_{\mathsf{a}}(t)\right\}\\ &= \frac{ie}{\hbar^2}\Big(\widetilde{\mathcal{X}}_{\mathsf{r}}(\Omega)+\widetilde{\mathcal{X}}_{\mathsf{a}}(-\Omega)\Big) \quad ,\end{aligned} \tag{13.50}$$

where $\widetilde{\mathcal{X}}_{\mathsf{a}}(\Omega)$ is the Fourier transform of $\mathcal{X}_{\mathsf{a}}(t)$ into the frequency domain. As we shall show presently, $\widetilde{\mathcal{X}}_{\mathsf{a}}(-\Omega)=-\widetilde{\mathcal{X}}^{*}_{\mathsf{r}}(\Omega)$, so we have

$$I_{\mathrm{N}}(V) = -\frac{2e}{\hbar^2}\,\mathrm{Im}\,\widetilde{\mathcal{X}}_{\mathsf{r}}(eV/\hbar) \quad . \tag{13.51}$$

Proof that $\widetilde{\mathcal{X}}_{\mathsf{a}}(\Omega)=-\widetilde{\mathcal{X}}^{*}_{\mathsf{r}}(-\Omega)$

Consider the general case

$$\begin{aligned}\mathcal{X}_{\mathsf{r}}(t) &= -i\,\Theta(t)\left\langle\left[\hat{A}(t)\,,\,\hat{A}^{\dagger}(0)\right]\right\rangle\\ \mathcal{X}_{\mathsf{a}}(t) &= -i\,\Theta(t)\left\langle\left[\hat{A}(0)\,,\,\hat{A}^{\dagger}(t)\right]\right\rangle \quad .\end{aligned} \tag{13.52}$$

We now spectrally decompose these expressions, inserting complete sets of states in between products of operators. One finds

$$\begin{aligned}\widetilde{\mathcal{X}}_{\mathsf{r}}(\omega) &= -i\int\limits_{-\infty}^{\infty} dt\,\Theta(t)\sum_{m,n}P_m\left\{\big|\langle\, m\,|\,\hat{A}\,|\,n\,\rangle\big|^2 e^{i(\omega_m-\omega_n)t}-\big|\langle\, m\,|\,\hat{A}^{\dagger}\,|\,n\,\rangle\big|^2 e^{-i(\omega_m-\omega_n)t}\right\}e^{i\omega t}\\ &= \sum_{m,n}P_m\left\{\frac{\big|\langle\, m\,|\,\hat{A}\,|\,n\,\rangle\big|^2}{\omega+\omega_m-\omega_n+i\epsilon}-\frac{\big|\langle\, m\,|\,\hat{A}^{\dagger}\,|\,n\,\rangle\big|^2}{\omega-\omega_m+\omega_n+i\epsilon}\right\} \quad ,\end{aligned} \tag{13.53}$$

where the eigenvalues of $\hat{K}$ are $\hbar\omega_m$, and $P_m=e^{-\hbar\omega_m/k_{\mathrm{B}}T}\big/\,\Xi$ is the thermal probability for state $|\,m\,\rangle$, where Ξ is the grand partition function. The corresponding expression for $\widetilde{\mathcal{X}}_{\mathsf{a}}(\omega)$ is

$$\widetilde{\mathcal{X}}_{\mathsf{a}}(\omega) = \sum_{m,n}P_m\left\{\frac{\big|\langle\, m\,|\,\hat{A}\,|\,n\,\rangle\big|^2}{\omega-\omega_m+\omega_n+i\epsilon}-\frac{\big|\langle\, m\,|\,\hat{A}^{\dagger}\,|\,n\,\rangle\big|^2}{\omega+\omega_m-\omega_n+i\epsilon}\right\} \quad , \tag{13.54}$$

whence follows $\widetilde{\mathcal{X}}_{\mathsf{a}}(-\omega) = -\widetilde{\mathcal{X}}^*_{\mathsf{r}}(\omega)$. QED. Note that in general

$$\begin{aligned}\mathcal{Z}(t) = -i\,\Theta(t)\,\big\langle \hat{A}(t)\,\hat{B}(0)\big\rangle &= -i\,\Theta(t)\sum_{m,n} P_m\,\langle\, m\,|\,e^{i\hat{K}t/\hbar}\,\hat{A}\,e^{-i\hat{K}t/\hbar}\,|\,n\,\rangle\langle\, n\,|\,\hat{B}\,|\,m\,\rangle \\ &= -i\,\Theta(t)\sum_{m,n} P_m\,\langle\, m\,|\,\hat{A}\,|\,n\,\rangle\langle\, n\,|\,\hat{B}\,|\,m\,\rangle\,e^{i(\omega_m-\omega_n)t} \quad ,\end{aligned} \tag{13.55}$$

the Fourier transform of which is

$$\widetilde{\mathcal{Z}}(\omega) = \int\limits_{-\infty}^{\infty} dt\; e^{i\omega t}\,\mathcal{Z}(t) = \sum_{m,n} P_m\,\frac{\langle\, m\,|\,\hat{A}\,|\,n\,\rangle\langle\, n\,|\,\hat{B}\,|\,m\,\rangle}{\omega+\omega_m-\omega_n+i\epsilon} \quad . \tag{13.56}$$

If we define the *spectral density* $\rho(\omega)$ as

$$\rho(\omega) = 2\pi\sum_{m,n} P_m\langle\, m\,|\,\hat{A}\,|\,n\,\rangle\langle\, n\,|\,\hat{B}\,|\,m\,\rangle\,\delta(\omega+\omega_m-\omega_n) \quad , \tag{13.57}$$

then we have

$$\widetilde{\mathcal{Z}}(\omega) = \int\limits_{-\infty}^{\infty} \frac{d\nu}{2\pi}\,\frac{\rho(\nu)}{\omega-\nu+i\epsilon} \quad . \tag{13.58}$$

Note that $\rho(\omega)$ is real if $B = A^\dagger$.

Evaluation of $\widetilde{\mathcal{X}}_{\mathsf{r}}(\omega)$

We must compute

$$\begin{aligned}\mathcal{X}_{\mathsf{r}}(t) &= -i\,\Theta(t)\sum_{i,j,\sigma}\sum_{k,l,\sigma'} T^*_{kl}\,T_{ij}\,\Big\langle\Big[c^\dagger_{\mathrm{R}\,j\,\sigma}(t)\,c_{\mathrm{L}\,i\,\sigma}(t)\,,\,c^\dagger_{\mathrm{L}\,k\,\sigma'}(0)\,c_{\mathrm{R}\,l\,\sigma'}(0)\Big]\Big\rangle \\ &= -i\,\Theta(t)\sum_{\boldsymbol{q},\boldsymbol{k},\sigma}|T_{\boldsymbol{q},\boldsymbol{k}}|^2\,\Big\{\big\langle c^\dagger_{\mathrm{R}\,\boldsymbol{k}\,\sigma}(t)\,c_{\mathrm{R}\,\boldsymbol{k}\,\sigma}(0)\big\rangle\,\big\langle c_{\mathrm{L}\,\boldsymbol{q}\,\sigma}(t)\,c^\dagger_{\mathrm{L}\,\boldsymbol{q}\,\sigma}(0)\big\rangle \\ &\qquad\qquad - \big\langle c^\dagger_{\mathrm{L}\,\boldsymbol{q}\,\sigma}(0)\,c_{\mathrm{L}\,\boldsymbol{q}\,\sigma}(t)\big\rangle\,\big\langle c_{\mathrm{R}\,\boldsymbol{k}\,\sigma}(0)\,c^\dagger_{\mathrm{R}\,\boldsymbol{k}\,\sigma}(t)\big\rangle\Big\}\end{aligned} \tag{13.59}$$

Note how we have taken $j = l \to \boldsymbol{k}$ and $i = k \to \boldsymbol{q}$, since *in each bank* wavevector is assumed to be a good quantum number. We now invoke the Bogoliubov transformation,

$$c_{\boldsymbol{k}\sigma} = u_{\boldsymbol{k}}\,\gamma_{\boldsymbol{k}\sigma} - \sigma\,v_{\boldsymbol{k}}\,e^{i\phi}\,\gamma^\dagger_{-\boldsymbol{k}\,-\sigma} \quad , \tag{13.60}$$

where we write $u_{\boldsymbol{k}} = \cos\vartheta_{\boldsymbol{k}}$ and $v_{\boldsymbol{k}} = \sin\vartheta_{\boldsymbol{k}}$. We then have

$$\begin{aligned}
\big\langle c^\dagger_{\mathrm{R}\,\boldsymbol{k}\,\sigma}(t)\, c_{\mathrm{R}\,\boldsymbol{k}\,\sigma}(0)\big\rangle &= u^2_{\boldsymbol{k}}\, e^{iE_{\boldsymbol{k}} t/\hbar}\, f(E_{\boldsymbol{k}}) + v^2_{\boldsymbol{k}}\, e^{-iE_{\boldsymbol{k}} t/\hbar}\, \big[1 - f(E_{\boldsymbol{k}})\big] \\
\big\langle c_{\mathrm{L}\,\boldsymbol{q}\,\sigma}(t)\, c^\dagger_{\mathrm{L}\,\boldsymbol{q}\,\sigma}(0)\big\rangle &= u^2_{\boldsymbol{q}}\, e^{-iE_{\boldsymbol{q}} t/\hbar}\, \big[1 - f(E_{\boldsymbol{q}})\big] + v^2_{\boldsymbol{q}}\, e^{iE_{\boldsymbol{q}} t/\hbar}\, f(E_{\boldsymbol{q}}) \\
\big\langle c^\dagger_{\mathrm{L}\,\boldsymbol{q}\,\sigma}(0)\, c_{\mathrm{L}\,\boldsymbol{q}\,\sigma}(t)\big\rangle &= u^2_{\boldsymbol{q}}\, e^{-iE_{\boldsymbol{q}} t/\hbar}\, f(E_{\boldsymbol{q}}) + v^2_{\boldsymbol{q}}\, e^{iE_{\boldsymbol{q}} t/\hbar}\, \big[1 - f(E_{\boldsymbol{q}})\big] \\
\big\langle c_{\mathrm{R}\,\boldsymbol{k}\,\sigma}(0)\, c^\dagger_{\mathrm{R}\,\boldsymbol{k}\,\sigma}(t)\big\rangle &= u^2_{\boldsymbol{k}}\, e^{iE_{\boldsymbol{k}} t/\hbar}\, \big[1 - f(E_{\boldsymbol{k}})\big] + v^2_{\boldsymbol{k}}\, e^{-iE_{\boldsymbol{k}} t/\hbar}\, f(E_{\boldsymbol{k}}) \quad .
\end{aligned} \tag{13.61}$$

We now appeal to Eqn. 13.35 and convert the $\boldsymbol{q}$ and $\boldsymbol{k}$ sums to integrals over $\xi_{\mathrm{L}\,\boldsymbol{q}}$ and $\xi_{\mathrm{R}\,\boldsymbol{k}}$. Pulling out the DOS factors $g_{\mathrm{L}} \equiv g_{\mathrm{L}}(\mu_{\mathrm{L}})$ and $g_{\mathrm{R}} \equiv g_{\mathrm{R}}(\mu_{\mathrm{R}})$, as well as the hopping integral $t \equiv t\big(\xi_{\mathrm{L}\,\boldsymbol{q}} = 0\,,\, \xi_{\mathrm{R}\,\boldsymbol{k}} = 0\big)$ from the integrand, we have

$$\begin{aligned}
\mathcal{X}_{\mathsf{r}}(t) = -i\,\Theta(t) \times \tfrac{1}{2}\, g_{\mathrm{L}}\, g_{\mathrm{R}}\, |t|^2 A \int\limits_{-\infty}^{\infty}\! d\xi \int\limits_{-\infty}^{\infty}\! d\xi' \times \qquad\qquad\qquad\qquad\qquad\qquad (13.62)\\
\bigg\{ \Big[u^2\, e^{-iEt/\hbar}\, (1-f) + v^2\, e^{iEt/\hbar}\, f\Big] \times \Big[u'^2\, e^{iE't/\hbar}\, f' + v'^2\, e^{-iE't/\hbar}\, (1-f')\Big] \\
- \Big[u^2\, e^{-iEt/\hbar}\, f + v^2\, e^{iEt/\hbar}\, (1-f)\Big] \times \Big[u'^2\, e^{iE't/\hbar}\, (1-f') + v'^2\, e^{-iE't/\hbar}\, f'\Big] \bigg\} \quad ,
\end{aligned}$$

where unprimed quantities correspond to the left bank (L) and primed quantities to the right bank (R). The ξ and ξ' integrals are simplified by the fact that in $u^2 = (E+\xi)/2E$ and $v^2 = (E-\xi)/2E$, *etc.* The terms proportional to ξ and ξ' and to $\xi\xi'$ drop out because everything else in the integrand is even in ξ and ξ' separately. Thus, we may replace u^2, v^2, u'^2, and v'^2 all by $\frac{1}{2}$. We now compute the Fourier transform, and we can read off the results keeping in mind the integral,

$$\int\limits_0^\infty\! dt\, e^{i\omega t}\, e^{i\Omega t}\, e^{-\epsilon t} = \frac{i}{\omega + \Omega + i\epsilon} \quad . \tag{13.63}$$

We then obtain

$$\begin{aligned}
\widetilde{\mathcal{X}}_{\mathsf{r}}(\omega) = \tfrac{1}{8}\, \hbar\, g_{\mathrm{L}}\, g_{\mathrm{R}}\, |t|^2 A \int\limits_{-\infty}^{\infty}\! d\xi \int\limits_{-\infty}^{\infty}\! d\xi' \, \bigg\{ \frac{2\,(f'-f)}{\hbar\omega + E' - E + i\epsilon} + \frac{1 - f - f'}{\hbar\omega - E - E' + i\epsilon} \qquad (13.64)\\
- \frac{1 - f - f'}{\hbar\omega + E + E' + i\epsilon} \bigg\} \quad .
\end{aligned}$$

Therefore,

$$\begin{aligned} I_{\rm N}(V,T) &= -\frac{2e}{\hbar^2}\,\mathsf{Im}\,\widetilde{\mathcal{X}}_{\mathsf{r}}(eV/\hbar) \\ &= \frac{\pi e}{\hbar}\,g_{\rm L}\,g_{\rm R}\,|t|^2 A \int\limits_0^\infty \! d\xi \int\limits_0^\infty \! d\xi' \, \Big\{ (1-f-f')\Big[\delta(E+E'-eV) - \delta(E+E'+eV)\Big] \\ &\qquad\qquad + 2\,(f'-f)\,\delta(E'-E+eV)\Big\} \quad . \end{aligned} \tag{13.65}$$

Single particle tunneling current in NIN junctions

We now evaluate $I_{\rm N}$ from Eqn. 13.65 for the case where both banks are normal metals. In this case, $E = \xi$ and $E' = \xi'$. (No absolute value symbol is needed since the ξ and ξ' integrals run over the positive real numbers.) At zero temperature, we have $f = 0$ and thus

$$\begin{aligned} I_{\rm N}(V,T=0) &= \frac{\pi e}{\hbar}\,g_{\rm L}\,g_{\rm R}\,|t|^2 A \int\limits_0^\infty \! d\xi \int\limits_0^\infty \! d\xi' \, \Big[\delta(\xi+\xi'-eV) - \delta(\xi+\xi'+eV)\Big] \\ &= \frac{\pi e}{\hbar}\,g_{\rm L}\,g_{\rm R}\,|t|^2 A \int\limits_0^{eV} \! d\xi = \frac{\pi e^2}{\hbar}\,g_{\rm L}\,g_{\rm R}\,|t|^2 A\,V \quad . \end{aligned} \tag{13.66}$$

We thus identify the normal state conductance of the junction as

$$G_{\rm N} \equiv \frac{\pi e^2}{\hbar}\,g_{\rm L}\,g_{\rm R}\,|t|^2 A \quad . \tag{13.67}$$

Single particle tunneling current in NIS junctions

Consider the case where one of the banks is a superconductor and the other a normal metal. We will assume $V > 0$ and work at $T = 0$. From Eqn. 13.65, we then have

$$\begin{aligned} I_{\rm N}(V,T=0) &= \frac{G_{\rm N}}{e} \int\limits_0^\infty \! d\xi \int\limits_0^\infty \! d\xi' \, \delta(\xi+E'-eV) = \frac{G_{\rm N}}{e} \int\limits_0^\infty \! d\xi \, \Theta(eV-E) \\ &= \frac{G_{\rm N}}{e} \int\limits_\Delta^{eV} \! dE \, \frac{E}{\sqrt{E^2-\Delta^2}} = G_n\,\sqrt{V^2-(\Delta/e)^2} \quad . \end{aligned} \tag{13.68}$$

The zero temperature conductance of the NIS junction is therefore

$$G_{\rm NIS}(V) = \frac{dI}{dV} = \frac{G_{\rm N}\,eV}{\sqrt{(eV)^2-\Delta^2}} \quad . \tag{13.69}$$

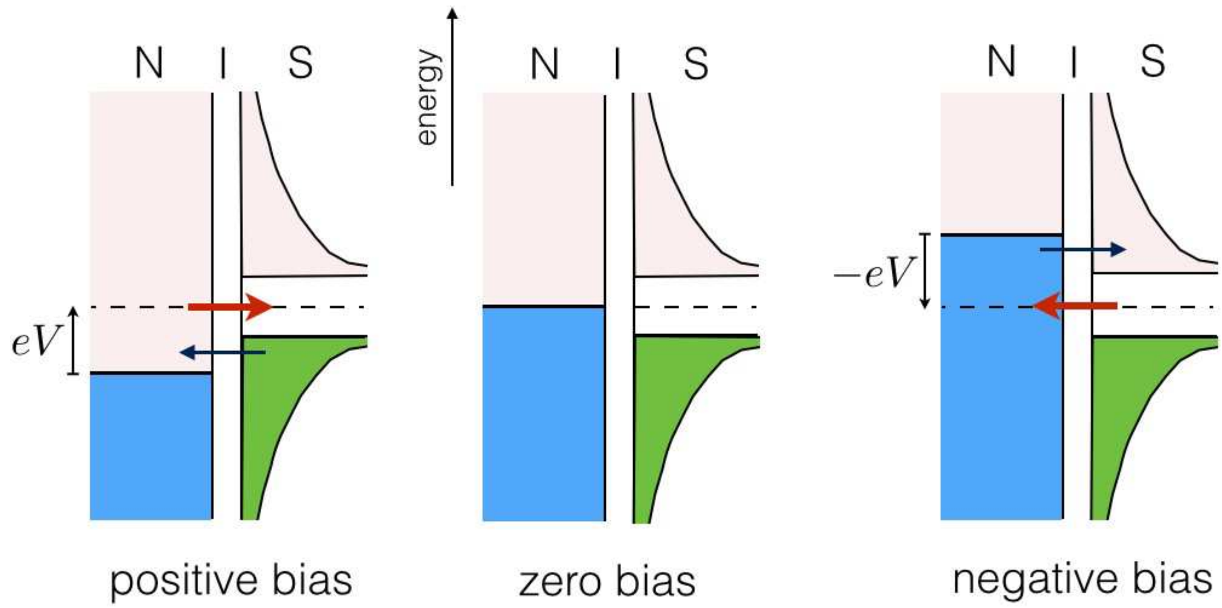

Figure 13.2: NIS tunneling for positive bias (left), zero bias (center), and negative bias (right). The left bank is maintained at an electrical potential V with respect to the right, hence $\mu_R = \mu_L + eV$. Blue regions indicate occupied fermionic states in the metal. Green regions indicate occupied electronic states in the superconductor. Light red regions indicate unoccupied states. Tunneling from or into the metal can only take place when its Fermi level lies outside the superconductor's gap region, meaning $|eV| > \Delta$, where V is the bias voltage. The arrow indicates the direction of electron number current. Black arrows indicate direction of electron current. Thick red arrows indicate direction of electrical current.

Hence the ratio G_{NIS}/G_{NIN} is

$$\frac{G_{NIS}(V)}{G_{NIN}(V)} = \frac{eV}{\sqrt{(eV)^2 - \Delta^2}} \quad . \tag{13.70}$$

It is to be understood that these expressions are to be multiplied by $\mathsf{sgn}(V)\,\Theta\big(e|V| - \Delta\big)$ to obtain the full result valid at all voltages.

Superconducting density of states

We define

$$\begin{aligned} n_S(E) &= 2\int \frac{d^3k}{(2\pi)^3}\,\delta(E - E_{\boldsymbol{k}}) \simeq g(\mu) \int\limits_{-\infty}^{\infty} d\xi\,\delta\Big(E - \sqrt{\xi^2 + \Delta^2}\Big) \\ &= g(\mu)\,\frac{2E}{\sqrt{E^2 - \Delta^2}}\,\Theta(E - \Delta) \quad . \end{aligned} \tag{13.71}$$

This is the density of energy states per unit volume for elementary excitations in the superconducting state. Note that there is an *energy gap* of size Δ, and that the missing states from this

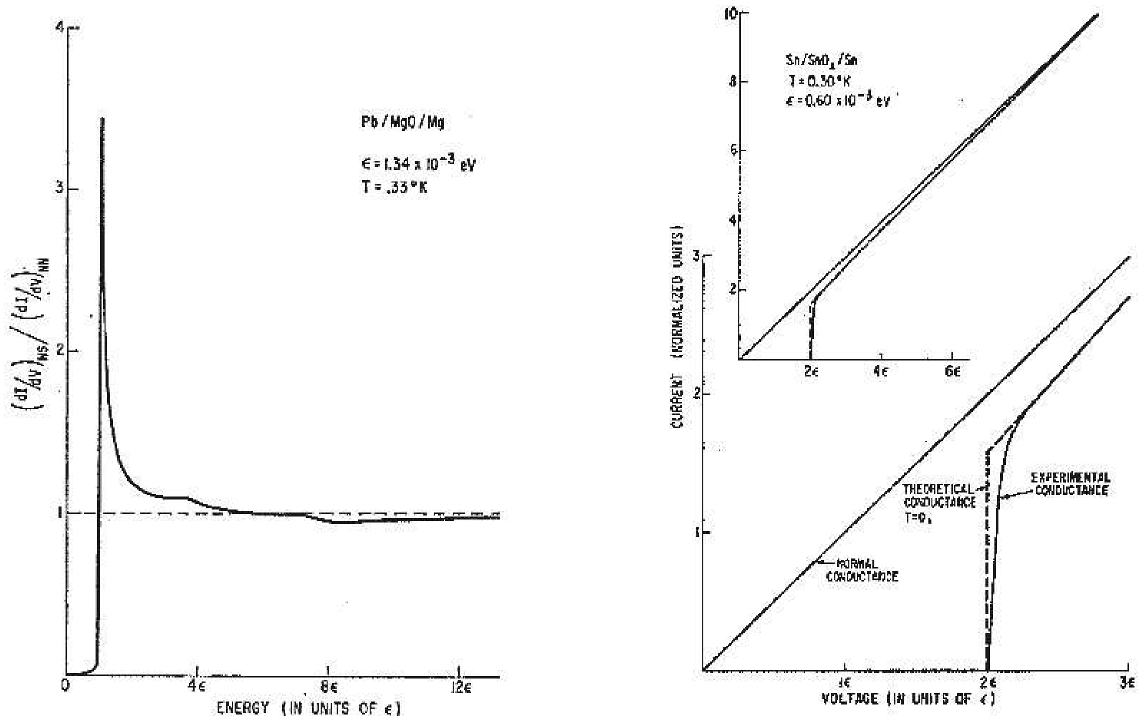

Figure 13.3: Tunneling data by Giaever *et al.* from *Phys. Rev.* **126**, 941 (1962). Left: normalized NIS tunneling conductance in a Pb/MgO/Mg sandwich junction. Pb is a superconductor for $T < T_c^{\rm Pb} = 7.19\,{\rm K}$, and Mg is a metal. A thin MgO layer provides a tunnel barrier. Right: I-V characteristic for a SIS junction Sn/SnO$_x$/Sn. Sn is a superconductor for $T < T_c^{\rm Sn} = 2.32\,{\rm K}$.

region pile up for $E \gtrsim \Delta$, resulting in a (integrable) divergence of $n_{\rm s}(E)$. In the limit $\Delta \to 0$, we have $n_{\rm s}(E) = 2\,g(\mu)\,\Theta(E)$. The factor of two arises because $n_{\rm s}(E)$ is the total density of states, which includes particle excitations above $k_{\rm F}$ as well as hole excitations below $k_{\rm F}$, both of which contribute $g(\mu)$. If $\Delta(\xi)$ is energy-dependent in the vicinity of $\xi = 0$, then we have

$$n(E) = g(\mu) \cdot \frac{E}{\xi} \cdot \left(1 + \frac{\Delta}{\xi}\frac{d\Delta}{d\xi}\right)^{-1}\Bigg|_{\xi=\sqrt{E^2-\Delta^2(\xi)}} \quad . \tag{13.72}$$

Here, $\xi = \sqrt{E^2 - \Delta^2(\xi)}$ is an implicit relation for $\xi(E)$.

The function $n_{\rm s}(E)$ vanishes for $E < 0$. We can, however, make a particle-hole transformation on the Bogoliubov operators, so that

$$\gamma_{\boldsymbol{k}\sigma} = \psi_{\boldsymbol{k}\sigma}\,\Theta(\xi_{\boldsymbol{k}}) + \psi^\dagger_{-\boldsymbol{k}\,-\sigma}\,\Theta(-\xi_{\boldsymbol{k}}) \quad . \tag{13.73}$$

We then have, up to constants,

$$\hat{K}_{\rm BCS} = \sum_{\boldsymbol{k}\sigma} \mathcal{E}_{\boldsymbol{k}\sigma}\,\psi^\dagger_{\boldsymbol{k}\sigma}\,\psi_{\boldsymbol{k}\sigma} \quad , \tag{13.74}$$

where

$$\mathcal{E}_{\boldsymbol{k}\sigma} = \begin{cases} +E_{\boldsymbol{k}\sigma} & \text{if } \xi_{\boldsymbol{k}} > 0 \\ -E_{\boldsymbol{k}\sigma} & \text{if } \xi_{\boldsymbol{k}} < 0 \end{cases} \quad . \tag{13.75}$$

The density of states for the ψ particles is then

$$\widetilde{n}_{\text{S}}(\mathcal{E}) = \frac{g_{\text{S}}\,|\mathcal{E}|}{\sqrt{\mathcal{E}^2 - \Delta^2}}\,\Theta\big(|\mathcal{E}| - \Delta\big) \quad , \tag{13.76}$$

were g_{S} is the metallic DOS at the Fermi level in the superconducting bank, *i.e.* above T_{c}. Note that $\widetilde{n}_{\text{S}}(-\mathcal{E}) = \widetilde{n}_{\text{S}}(\mathcal{E})$ is now an even function of $\mathcal{E}$, and that half of the weight from $n_{\text{S}}(E)$ has now been assigned to negative $\mathcal{E}$ states. The interpretation of Fig. 13.2 follows by writing

$$I_{\text{N}}(V, T=0) = \frac{G_{\text{N}}}{e g_{\text{S}}} \int\limits_0^{eV} \! d\mathcal{E}\; n_{\text{S}}(\mathcal{E}) \quad . \tag{13.77}$$

Note that this is properly odd under $V \to -V$. If $V > 0$, the tunneling current is proportional to the integral of the superconducting density of states from $\mathcal{E} = \Delta$ to $\mathcal{E} = eV$. Since $\widetilde{n}_{\text{S}}(\mathcal{E})$ vanishes for $|\mathcal{E}| < \Delta$, the tunnel current vanishes if $|eV| < \Delta$.

Single particle tunneling current in SIS junctions

We now come to the SIS case, where both banks are superconducting. From Eqn. 13.65, we have ($T = 0$)

$$\begin{aligned} I_{\text{N}}(V, T=0) &= \frac{G_{\text{N}}}{e} \int\limits_0^\infty \! d\xi \int\limits_0^\infty \! d\xi'\; \delta(E + E' - eV) \qquad (13.78) \\ &= \frac{G_{\text{N}}}{e} \int\limits_0^\infty \! dE \int\limits_0^\infty \! dE'\; \frac{E}{\sqrt{E^2 - \Delta_{\text{L}}^2}}\,\frac{E'}{\sqrt{E'^2 - \Delta_{\text{R}}^2}} \Big\{ \delta(E + E' - eV) - \delta(E + E' + eV) \Big\} \quad . \end{aligned}$$

While this integral has no general analytic form, we see that $I_{\text{N}}(V) = -I_{\text{N}}(-V)$, and that the threshold voltage V^* below which $I_{\text{N}}(V)$ vanishes is given by $eV^* = \Delta_{\text{L}} + \Delta_{\text{R}}$. For the special case $\Delta_{\text{L}} = \Delta_{\text{R}} \equiv \Delta$, one has

$$I_{\text{N}}(V) = \frac{G_{\text{N}}}{e} \left\{ \frac{(eV)^2}{eV + 2\Delta}\,\mathbb{K}(x) - (eV + 2\Delta)\Big(\mathbb{K}(x) - \mathbb{E}(x)\Big) \right\} \quad , \tag{13.79}$$

where $x = (eV - 2\Delta)/(eV + 2\Delta)$ and $\mathbb{K}(x)$ and $\mathbb{E}(x)$ are complete elliptic integrals of the first

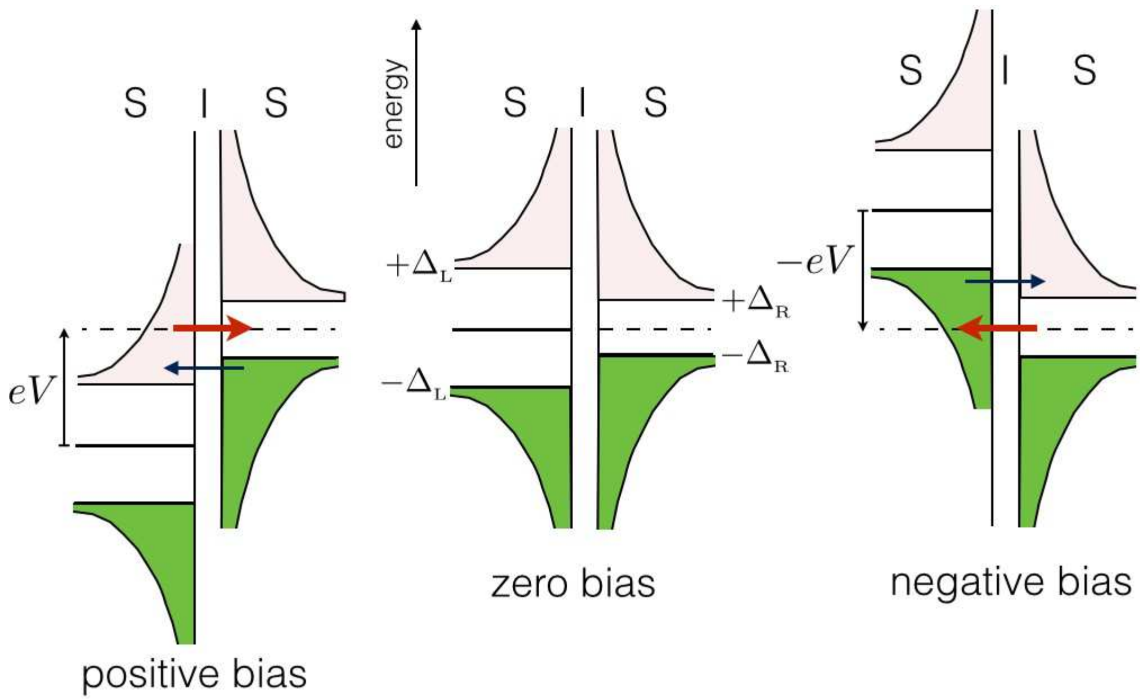

Figure 13.4: SIS tunneling for positive bias (left), zero bias (center), and negative bias (right). Green regions indicate occupied electronic states in each superconductor, where $\widetilde{n}_{\rm S}(\mathcal{E}) > 0$.

and second kinds, respectively:

$$\begin{aligned}\mathbb{K}(x) &= \int\limits_0^{\pi/2} \frac{d\theta}{\sqrt{1 - x^2 \sin^2\!\theta}} \\ \mathbb{E}(x) &= \int\limits_0^{\pi/2} \! d\theta \, \sqrt{1 - x^2 \sin^2\!\theta} \quad .\end{aligned} \tag{13.80}$$

We may also make progress by setting $eV = \Delta_{\rm L} + \Delta_{\rm R} + e\,\delta V$. One then has

$$I_{\rm N}(V^* + \delta V) = \frac{G_{\rm N}}{e} \int\limits_0^\infty \! d\xi_{\rm L} \int\limits_0^\infty \! d\xi_{\rm R} \; \delta\bigg(e\,\delta V - \frac{\xi_{\rm L}^2}{2\Delta_{\rm L}} - \frac{\xi_{\rm R}^2}{2\Delta_{\rm R}} \bigg) = \frac{\pi G_{\rm N}}{2e} \sqrt{\Delta_{\rm L}\,\Delta_{\rm R}} \quad . \tag{13.81}$$

Thus, the SIS tunnel current jumps discontinuously at $V = V^*$. At finite temperature, there is a smaller local maximum in $I_{\rm N}$ for $V = |\Delta_{\rm L} - \Delta_{\rm R}|/e$.

13.2.3 The Josephson pair tunneling current $I_{\rm J}$

Earlier we obtained the expression

$$I_{\rm J}(t) = \frac{e}{\hbar^2}\int\limits_{-\infty}^{\infty} dt'\, \Theta(t-t')\,\Big\{ e^{+i\Omega(t+t')} \Big\langle \big[\hat H^-_{\rm T}(t)\,,\,\hat H^-_{\rm T}(t')\big]\Big\rangle - e^{-i\Omega(t+t')} \Big\langle \big[\hat H^+_{\rm T}(t)\,,\,\hat H^+_{\rm T}(t')\big]\Big\rangle\Big\} \quad . \tag{13.82}$$

Proceeding in analogy to the case for $I_{\rm N}$, define now the anomalous response functions,

$$\begin{aligned} \mathcal{Y}_{\sf r}(t-t') &= -i\,\Theta(t-t')\,\Big\langle \big[\hat H^+_{\rm T}(t)\,,\,\hat H^+_{\rm T}(t')\big]\Big\rangle \\ \mathcal{Y}_{\sf a}(t-t') &= -i\,\Theta(t-t')\,\Big\langle \big[\hat H^-_{\rm T}(t')\,,\,\hat H^-_{\rm T}(t)\big]\Big\rangle \quad . \end{aligned} \tag{13.83}$$

The spectral representations of these response functions are

$$\begin{aligned} \widetilde{\mathcal{Y}}_{\sf r}(\omega) &= \sum_{m,n} P_m \left\{ \frac{\langle\, m\,|\,\hat H^+_{\rm T}\,|\,n\,\rangle\langle\, n\,|\,\hat H^+_{\rm T}\,|\,m\,\rangle}{\omega+\omega_m-\omega_n+i\epsilon} - \frac{\langle\, m\,|\,\hat H^+_{\rm T}\,|\,n\,\rangle\langle\, n\,|\,\hat H^+_{\rm T}\,|\,m\,\rangle}{\omega-\omega_m+\omega_n+i\epsilon}\right\} \\ \widetilde{\mathcal{Y}}_{\sf a}(\omega) &= \sum_{m,n} P_m \left\{ \frac{\langle\, m\,|\,\hat H^-_{\rm T}\,|\,n\,\rangle\langle\, n\,|\,\hat H^-_{\rm T}\,|\,m\,\rangle}{\omega-\omega_m+\omega_n+i\epsilon} - \frac{\langle\, m\,|\,\hat H^-_{\rm T}\,|\,n\,\rangle\langle\, n\,|\,\hat H^-_{\rm T}\,|\,m\,\rangle}{\omega+\omega_m-\omega_n+i\epsilon}\right\} \quad , \end{aligned} \tag{13.84}$$

from which we see $\widetilde{\mathcal{Y}}_{\sf a}(\omega) = -\widetilde{\mathcal{Y}}^*_{\sf r}(-\omega)$. The Josephson current is then given by

$$\begin{aligned} I_{\rm J}(t) &= -\frac{ie}{\hbar^2}\int\limits_{-\infty}^{\infty} dt'\,\Big\{ e^{-2i\Omega t}\,\mathcal{Y}_{\sf r}(t-t')\,e^{+i\Omega(t-t')} + e^{+2i\Omega t}\,\mathcal{Y}_{\sf a}(t-t')\,e^{-i\Omega(t-t')}\Big\} \\ &= \frac{2e}{\hbar^2}\,{\sf Im}\Big[e^{-2i\Omega t}\,\widetilde{\mathcal{Y}}_{\sf r}(\Omega)\Big] \quad , \end{aligned} \tag{13.85}$$

where $\Omega = eV/\hbar$.

Plugging in our expressions for $\hat H^{\pm}_{\rm T}$, we have

$$\begin{aligned} \mathcal{Y}_{\sf r}(t) &= -i\,\Theta(t)\sum_{\boldsymbol{k},\boldsymbol{q},\sigma} T_{\boldsymbol{k},\boldsymbol{q}}\,T_{-\boldsymbol{k},-\boldsymbol{q}}\,\Big\langle\Big[c^\dagger_{{\rm L}\,\boldsymbol{q}\,\sigma}(t)\,c_{{\rm R}\,\boldsymbol{k}\,\sigma}(t)\,,\,c^\dagger_{{\rm L}\,-\boldsymbol{q}\,-\sigma}(0)\,c_{{\rm R}\,-\boldsymbol{k}\,-\sigma}(0)\Big]\Big\rangle \\ &= 2i\,\Theta(t)\sum_{\boldsymbol{q},\boldsymbol{k}} T_{\boldsymbol{k},\boldsymbol{q}}\,T_{-\boldsymbol{k},-\boldsymbol{q}}\,\Big\{\big\langle c^\dagger_{{\rm L}\,\boldsymbol{q}\uparrow}(t)\,c^\dagger_{{\rm L}\,-\boldsymbol{q}\downarrow}(0)\big\rangle\,\big\langle c_{{\rm R}\,\boldsymbol{k}\uparrow}(t)\,c_{{\rm R}\,-\boldsymbol{k}\downarrow}(0)\big\rangle \\ &\qquad\qquad - \big\langle c^\dagger_{{\rm L}\,-\boldsymbol{q}\downarrow}(0)\,c^\dagger_{{\rm L}\,\boldsymbol{q}\uparrow}(t)\big\rangle\,\big\langle c_{{\rm R}\,-\boldsymbol{k}\downarrow}(0)\,c_{{\rm R}\,\boldsymbol{k}\uparrow}(t)\big\rangle\Big\} \quad . \end{aligned} \tag{13.86}$$

Again we invoke Bogoliubov,

$$c_{\boldsymbol{k}\uparrow} = u_{\boldsymbol{k}}\,\gamma_{\boldsymbol{k}\uparrow} - v_{\boldsymbol{k}}\,e^{i\phi}\,\gamma^{\dagger}_{-\boldsymbol{k}\downarrow} \qquad\qquad c^{\dagger}_{\boldsymbol{k}\uparrow} = u_{\boldsymbol{k}}\,\gamma^{\dagger}_{\boldsymbol{k}\uparrow} - v_{\boldsymbol{k}}\,e^{-i\phi}\,\gamma_{-\boldsymbol{k}\downarrow} \tag{13.87}$$

$$c_{-\boldsymbol{k}\downarrow} = u_{\boldsymbol{k}}\,\gamma_{-\boldsymbol{k}\downarrow} + v_{\boldsymbol{k}}\,e^{i\phi}\,\gamma^{\dagger}_{\boldsymbol{k}\uparrow} \qquad\qquad c^{\dagger}_{-\boldsymbol{k}\downarrow} = u_{\boldsymbol{k}}\,\gamma^{\dagger}_{-\boldsymbol{k}\downarrow} + v_{\boldsymbol{k}}\,e^{-i\phi}\,\gamma_{\boldsymbol{k}\uparrow} \tag{13.88}$$

to obtain

$$\begin{aligned}
\big\langle c^{\dagger}_{\mathrm{L}\,\boldsymbol{q}\uparrow}(t)\,c^{\dagger}_{\mathrm{L}\,-\boldsymbol{q}\downarrow}(0)\big\rangle &= u_{\boldsymbol{q}}\,v_{\boldsymbol{q}}\,e^{-i\phi_{\mathrm{L}}}\left\{e^{iE_{\boldsymbol{q}}t/\hbar}\,f(E_{\boldsymbol{q}}) - e^{-iE_{\boldsymbol{q}}t/\hbar}\big[1-f(E_{\boldsymbol{q}})\big]\right\}\\
\big\langle c_{\mathrm{R}\,\boldsymbol{k}\uparrow}(t)\,c_{\mathrm{R}\,-\boldsymbol{k}\downarrow}(0)\big\rangle &= u_{\boldsymbol{k}}\,v_{\boldsymbol{k}}\,e^{+i\phi_{\mathrm{R}}}\left\{e^{-iE_{\boldsymbol{k}}t/\hbar}\big[1-f(E_{\boldsymbol{k}})\big] - e^{iE_{\boldsymbol{k}}t/\hbar}\,f(E_{\boldsymbol{k}})\right\}\\
\big\langle c^{\dagger}_{\mathrm{L}\,-\boldsymbol{q}\downarrow}(0)\,c^{\dagger}_{\mathrm{L}\,\boldsymbol{q}\uparrow}(t)\big\rangle &= u_{\boldsymbol{q}}\,v_{\boldsymbol{q}}\,e^{-i\phi_{\mathrm{L}}}\left\{e^{iE_{\boldsymbol{q}}t/\hbar}\big[1-f(E_{\boldsymbol{q}})\big] - e^{-iE_{\boldsymbol{q}}t/\hbar}\,f(E_{\boldsymbol{q}})\right\}\\
\big\langle c_{\mathrm{R}\,-\boldsymbol{k}\downarrow}(0)\,c_{\mathrm{R}\,\boldsymbol{k}\uparrow}(t)\big\rangle &= u_{\boldsymbol{k}}\,v_{\boldsymbol{k}}\,e^{+i\phi_{\mathrm{R}}}\left\{e^{-iE_{\boldsymbol{k}}t/\hbar}\,f(E_{\boldsymbol{k}}) - e^{iE_{\boldsymbol{k}}t/\hbar}\big[1-f(E_{\boldsymbol{k}})\big]\right\}
\end{aligned}\tag{13.89}$$

We then have

$$\begin{aligned}
\mathcal{Y}_{\mathsf{r}}(t) = i\,\Theta(t)\times\tfrac{1}{2}\,g_{\mathrm{L}}\,g_{\mathrm{R}}\,|t|^2 A\,e^{i(\phi_{\mathrm{R}}-\phi_{\mathrm{L}})}\int\limits_{-\infty}^{\infty}\!d\xi\int\limits_{-\infty}^{\infty}\!d\xi'\;u\,v\,u'\,v'\times \qquad\qquad&\\
\Bigg\{\Big[e^{iEt/\hbar}\,f - e^{-iEt/\hbar}\,(1-f)\Big]\times\Big[e^{-iE't/\hbar}\,(1-f') - e^{iE't/\hbar}\,f'\Big]&\\
-\Big[e^{iEt/\hbar}\,(1-f) - e^{-iEt/\hbar}\,f\Big]\times\Big[e^{-iE't/\hbar}\,f' - e^{iE't/\hbar}\,(1-f')\Big]\Bigg\}&\quad,
\end{aligned}\tag{13.90}$$

where once again primed and unprimed symbols refer respectively to left (L) and right (R) banks. Recall that the BCS coherence factors give $uv = \frac{1}{2}\sin(2\vartheta) = \Delta/2E$. Taking the Fourier transform, we have

$$\begin{aligned}
\widetilde{\mathcal{Y}}_{\mathsf{r}}(\omega) = \tfrac{1}{2}\,\hbar\,g_{\mathrm{L}}\,g_{\mathrm{R}}\,|t|^2\,e^{i(\phi_{\mathrm{R}}-\phi_{\mathrm{L}})}A\int\limits_0^{\infty}\!d\xi\int\limits_0^{\infty}\!d\xi'\;\frac{\Delta}{E}\,\frac{\Delta'}{E'}\Bigg\{\frac{f-f'}{\hbar\omega+E-E'+i\epsilon} - \frac{f-f'}{\hbar\omega-E+E'+i\epsilon}&\\
+\frac{1-f-f'}{\hbar\omega+E+E'+i\epsilon} - \frac{1-f-f'}{\hbar\omega-E-E'+i\epsilon}\Bigg\}&\quad.
\end{aligned}\tag{13.91}$$

Setting $T=0$, we have

$$\begin{aligned}
\widetilde{\mathcal{Y}}_{\mathsf{r}}(\omega) &= \frac{\hbar^2 G_{\mathrm{N}}}{2\pi e^2}\,e^{i(\phi_{\mathrm{R}}-\phi_{\mathrm{L}})}\int\limits_0^{\infty}\!d\xi\int\limits_0^{\infty}\!d\xi'\;\frac{\Delta\Delta'}{EE'}\left\{\frac{1}{\hbar\omega+E+E'+i\epsilon} - \frac{1}{\hbar\omega-E-E'+i\epsilon}\right\}\\
&= \frac{\hbar^2 G_{\mathrm{N}}}{2\pi e^2}\,e^{i(\phi_{\mathrm{R}}-\phi_{\mathrm{L}})}\int\limits_{\Delta}^{\infty}\!dE\,\frac{\Delta}{\sqrt{E^2-\Delta^2}}\int\limits_{\Delta'}^{\infty}\!dE'\,\frac{\Delta'}{\sqrt{E'^2-\Delta'^2}}\times\frac{2\,(E+E')}{(\hbar\omega)^2-(E+E')^2}\quad.
\end{aligned}\tag{13.92}$$

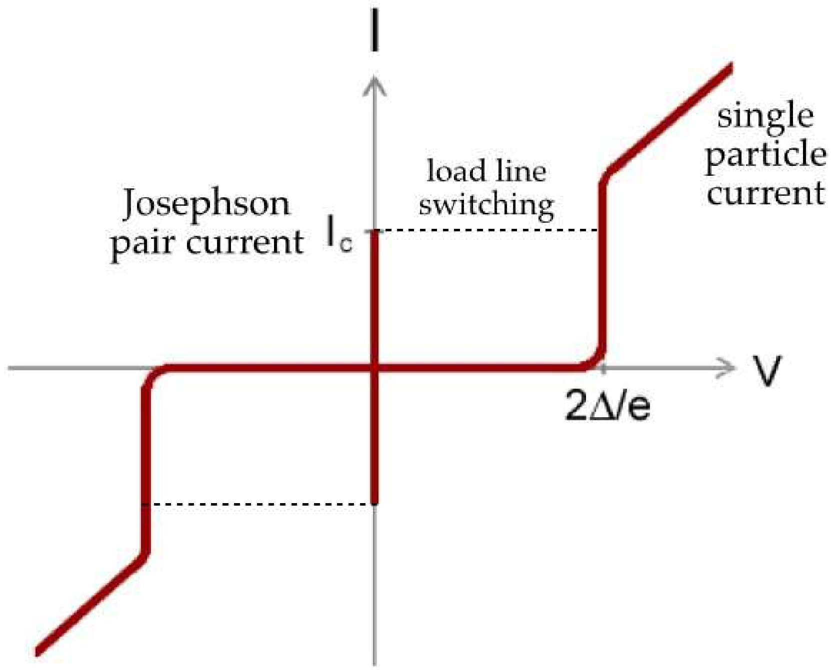

Figure 13.5: Current-voltage characteristics for a current-biased Josephson junction. Increasing current at zero bias voltage is possible up to $|I| = I_c$, beyond which the voltage jumps along the dotted line. Subsequent reduction in current leads to hysteresis.

There is no general analytic form for this integral. However, for the special case $\Delta = \Delta'$, we have

$$\widetilde{\mathcal{Y}}_{\mathsf{r}}(\omega) = \frac{G_{\rm N}\hbar^2}{2e^2}\,\Delta\,\mathbb{K}\bigg(\frac{\hbar|\omega|}{4\Delta}\bigg)\, e^{i(\phi_{\rm R}-\phi_{\rm L})} \quad , \tag{13.93}$$

where $\mathbb{K}(x)$ is the complete elliptic integral of the first kind. Thus,

$$I_{\rm J}(t) = G_{\rm N}\cdot\frac{\Delta}{e}\,\mathbb{K}\bigg(\frac{e|V|}{4\Delta}\bigg)\,\sin\!\bigg(\phi_{\rm R}-\phi_{\rm L}-\frac{2eVt}{\hbar}\bigg) \quad . \tag{13.94}$$

With $V = 0$, one finds (at finite T),

$$I_{\rm J} = G_{\rm N}\cdot\frac{\pi\Delta}{2e}\,\tanh\bigg(\frac{\Delta}{2k_{\rm B}T}\bigg)\,\sin(\phi_{\rm R}-\phi_{\rm L}) \quad . \tag{13.95}$$

Thus, there is a spontaneous current flow in the absence of any voltage bias, provided the phases remain fixed. The maximum current which flows under these conditions is called the *critical current* of the junction, I_c. Writing $R_{\rm N} = 1/G_{\rm N}$ for the normal state junction resistance, one has

$$I_c R_{\rm N} = \frac{\pi\Delta}{2e}\,\tanh\bigg(\frac{\Delta}{2k_{\rm B}T}\bigg) \quad , \tag{13.96}$$

which is known as the *Ambegaokar-Baratoff relation*. Note that I_c agrees with what we found in Eqn. 13.81 for V just above $V^* = 2\Delta$. I_c is also the current flowing in a normal junction at bias voltage $V = \pi\Delta/2e$. Setting $I_c = 2eJ/\hbar$ where J is the Josephson coupling, we find our $V = 0$ results here in complete agreement with those of Eqn. 13.29 when Coulomb charging energies of the grains are neglected.

Experimentally, one generally draws a current I across the junction and then measures the voltage difference. In other words, the junction is *current-biased*. Varying I then leads to a hysteretic voltage response, as shown in Fig. 13.5. The functional form of the oscillating current is then $I(t) = I_c \sin(\phi_R - \phi_L - \Omega t)$, which gives no DC average. With $R_N \approx 1.5\,\Omega$ and $\Delta = 1\,\text{meV}$, one obtains a critical current $I_c = 1\,\text{mA}$. For a junction of area $A \sim 1\,\text{mm}^2$, the critical current density is then $j_c = I_c/A \sim 10^3\,\text{A/m}^2$. Current densities in bulk type I and type II materials can approach $j \sim 10^{11}\,\text{A/m}^2$ and $10^9\,\text{A/m}^2$, respectively.

13.3 The Josephson Effect

13.3.1 Two grain junction

In §13.1 we discussed a model for superconducting grains. Consider now only a single pair of grains, and write

$$\hat{K} = -J\cos(\phi_L - \phi_R) + \frac{2e^2}{C_L}\,M_L^2 + \frac{2e^2}{C_R}\,M_R^2 - 2\mu_L M_L - 2\mu_R M_R \quad , \tag{13.97}$$

where $M_{L,R}$ is the number of Cooper pairs on each grain in excess of the background charge, which we assume here to be a multiple of $2e$. From the Heisenberg equations of motion, we have that

$$\dot{M}_L = \frac{i}{\hbar}\,\big[\hat{K}, M_L\big] = \frac{J}{\hbar}\,\sin(\phi_R - \phi_L) \quad , \tag{13.98}$$

which follows from the fact that $M_L = -i\,\partial/\partial\phi_L$. Similarly, we find $\dot{M}_R = -\frac{J}{\hbar}\sin(\phi_R - \phi_L)$. An electrical current $I = 2e\dot{M}_L = -2e\dot{M}_R$ then flows from left to right. The equations of motion for the phases are

$$\begin{aligned} \dot{\phi}_L &= \frac{i}{\hbar}\,\big[\hat{K}\,,\,\phi_L\big] = \frac{4e^2 M_L}{\hbar C_L} - \frac{2\mu_L}{\hbar} \\ \dot{\phi}_R &= \frac{i}{\hbar}\,\big[\hat{K}\,,\,\phi_R\big] = \frac{4e^2 M_R}{\hbar C_R} - \frac{2\mu_R}{\hbar} \quad . \end{aligned} \tag{13.99}$$

Let's assume the grains are large, so their self-capacitances are large too. In that case, we can neglect the Coulomb energy of each grain, and we obtain the *Josephson equations*

$$\frac{d\phi}{dt} = -\frac{2eV}{\hbar} \qquad , \qquad I(t) = I_c \sin\phi(t) \quad , \tag{13.100}$$

where $eV = \mu_{\mathrm{R}} - \mu_{\mathrm{L}}$, $I_{\mathrm{c}} = 2eJ/\hbar$, and $\phi \equiv \phi_{\mathrm{R}} - \phi_{\mathrm{L}}$. When quasiparticle tunneling is accounted for, the second of the Josephson equations is modified to

$$I = I_{\mathrm{c}} \sin\phi + \big(G_0 + G_1 \cos\phi\big) V \quad , \tag{13.101}$$

where $G_0 \equiv G_{\mathrm{N}}$ is the quasiparticle contribution to the current, and G_1 accounts for higher order effects.

13.3.2 Effect of in-plane magnetic field

Thus far we have assumed that the effective hopping amplitude t between the L and R banks is real. This is valid in the absence of an external magnetic field, which breaks time-reversal. In the presence of an external magnetic field, t is replaced by $t \to t\, e^{i\gamma}$, where $\gamma = \frac{e}{\hbar c} \int_{\mathrm{L}}^{\mathrm{R}} \boldsymbol{A} \cdot d\boldsymbol{l}$ is the Aharonov-Bohm phase. Without loss of generality, we consider the junction interface to lie in the (x, y) plane, and we take $\boldsymbol{H} = H\hat{\boldsymbol{y}}$. We are then free to choose the gauge $\boldsymbol{A} = -Hx\hat{\boldsymbol{z}}$. Then

$$\gamma(x) = \frac{e}{\hbar c} \int_{\mathrm{L}}^{\mathrm{R}} \boldsymbol{A} \cdot d\boldsymbol{l} = -\frac{e}{\hbar c}\, H\, (\lambda_{\mathrm{L}} + \lambda_{\mathrm{R}} + d)\, x \quad , \tag{13.102}$$

where $\lambda_{\mathrm{L,R}}$ are the penetration depths for the two superconducting banks, and d is the junction separation. Typically $\lambda_{\mathrm{L,R}} \sim 100\,\text{Å} - 1000\,\text{Å}$, while $d \sim 10\,\text{Å}$, so usually we may neglect the junction separation in comparison with the penetration depth.

In the case of the single particle current I_{N}, we needed the commutators $\big[\hat{H}^+_{\mathrm{T}}(t), \hat{H}^-_{\mathrm{T}}(0)\big]$ and $\big[\hat{H}^-_{\mathrm{T}}(t), \hat{H}^+_{\mathrm{T}}(0)\big]$. Since $\hat{H}^+_{\mathrm{T}} \propto t$ while $\hat{H}^-_{\mathrm{T}} \propto t^*$, the result depends on the product $|t|^2$, which has no phase. Thus, I_{N} is unaffected by an in-plane magnetic field. For the Josephson pair tunneling current I_{J}, however, we need $\big[\hat{H}^+_{\mathrm{T}}(t), \hat{H}^+_{\mathrm{T}}(0)\big]$ and $\big[\hat{H}^-_{\mathrm{T}}(t), \hat{H}^-_{\mathrm{T}}(0)\big]$. The former is proportional to t^2 and the latter to t^{*2}. Therefore the Josephson current density is

$$j_{\mathrm{J}}(x) = \frac{I_{\mathrm{c}}(T)}{A}\, \sin\!\bigg(\phi - \frac{2e}{\hbar c}\, H d_{\mathrm{eff}} x - \frac{2eVt}{\hbar}\bigg), \tag{13.103}$$

where $d_{\mathrm{eff}} \equiv \lambda_{\mathrm{L}} + \lambda_{\mathrm{R}} + d$ and $\phi = \phi_{\mathrm{R}} - \phi_{\mathrm{L}}$. Note that it is $2eHd_{\mathrm{eff}}/\hbar c = \mathsf{arg}(t^2)$ which appears in the argument of the sine. This may be interpreted as the Aharonov-Bohm phase accrued by a tunneling Cooper pair. We now assume our junction interface is a square of dimensions $L_x \times L_y$. At $V = 0$, the total Josephson current is then[5]

$$I_{\mathrm{J}} = \int_0^{L_x}\! dx \int_0^{L_y}\! dy\; j(x) = \frac{I_{\mathrm{c}} \phi_{\mathrm{L}}}{\pi\Phi}\, \sin(\pi\Phi/\phi_{\mathrm{L}}) \sin(\phi - \pi\Phi/\phi_{\mathrm{L}}) \quad , \tag{13.104}$$

[5]Take care not to confuse ϕ_{L}, the phase of the left superconducting bank, with ϕ_{L}, the London flux quantum $hc/2e$. To the untrained eye, these symbols look identical.

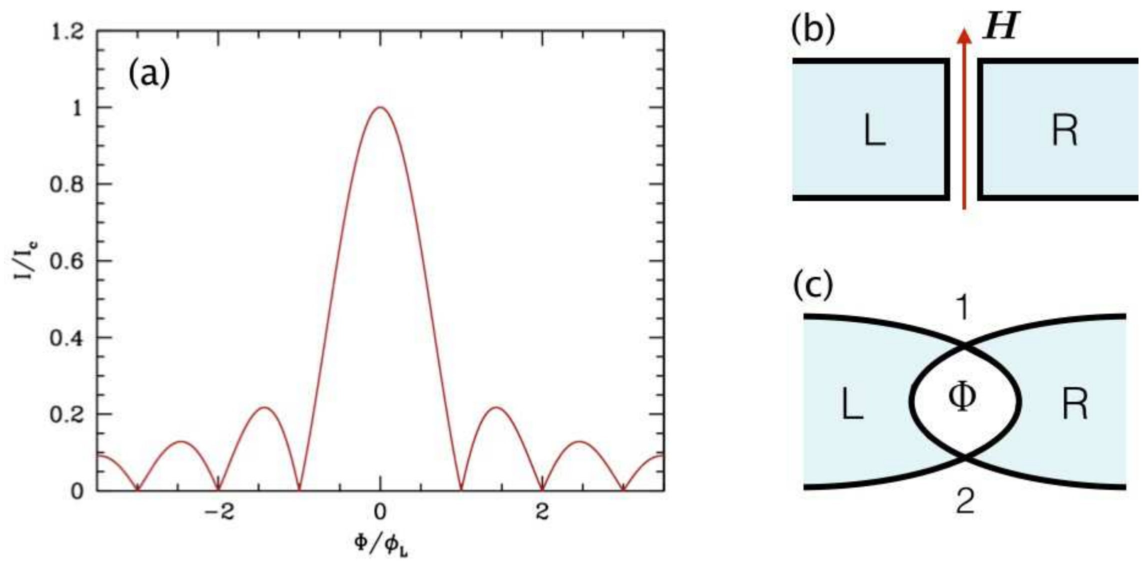

Figure 13.6: (a) Fraunhofer pattern of Josephson current *versus* flux due to in-plane magnetic field. (b) Sketch of Josephson junction experiment yielding (a). (c) Two-point superconducting quantum interferometer.

where $\Phi \equiv H L_x d_{\text{eff}}$ is the total magnetic flux through the junction. The maximum current occurs when $\phi - \pi\Phi/\phi_{\text{L}} = \pm\frac{1}{2}\pi$, where its magnitude is

$$I_{\text{max}}(\Phi) = I_{\text{c}} \left| \frac{\sin(\pi\Phi/\phi_{\text{L}})}{\pi\Phi/\phi_{\text{L}}} \right| \quad . \tag{13.105}$$

The shape $I_{\text{max}}(\Phi)$ is precisely that of the single slit Fraunhofer pattern from geometrical optics! (See Fig. 13.6.)

13.3.3 Two-point quantum interferometer

Consider next the device depicted in Fig. 13.6(c) consisting of two weak links between superconducting banks. The current flowing from L to R is

$$I = I_{\text{c},1} \sin\phi_1 + I_{\text{c},2} \sin\phi_2 \quad . \tag{13.106}$$

where $\phi_1 \equiv \phi_{\text{L},1} - \phi_{\text{R},1}$ and $\phi_2 \equiv \phi_{\text{L},2} - \phi_{\text{R},2}$ are the phase differences across the two Josephson junctions. The total flux Φ inside the enclosed loop is

$$\phi_2 - \phi_1 = \frac{2\pi\Phi}{\phi_{\text{L}}} \equiv 2\gamma \quad . \tag{13.107}$$

Writing $\phi_2 = \phi_1 + 2\gamma$, we extremize $I(\phi_1, \gamma)$ with respect to ϕ_1, and obtain

$$I_{\text{max}}(\gamma) = \sqrt{(I_{\text{c},1} + I_{\text{c},2})^2 \cos^2\!\gamma + (I_{\text{c},1} - I_{\text{c},2})^2 \sin^2\!\gamma} \quad . \tag{13.108}$$

If $I_{c,1} = I_{c,2}$, we have $I_{max}(\gamma) = 2I_c\,|\cos\gamma\,|$. This provides for an extremely sensitive measurement of magnetic fields, since $\gamma = \pi\Phi/\phi_L$ and $\phi_L = 2.07\times 10^{-7}\,\mathrm{G\,cm}^2$. Thus, a ring of area $1\,\mathrm{cm}^2$ allows for the detection of fields on the order of $10^{-7}\,\mathrm{G}$. This device is known as a Superconducting QUantum Interference Device, or SQUID. The limits of the SQUID's sensitivity are set by the noise in the SQUID or in the circuit amplifier.

13.3.4 RCSJ Model

In circuits, a Josephson junction, from a practical point of view, is always transporting current in parallel to some resistive channel. Josephson junctions also have electrostatic capacitance as well. Accordingly, consider the *resistively and capacitively shunted Josephson junction* (RCSJ), a sketch of which is provided in Fig. 13.8(c). The equations governing the RCSJ model are

$$\begin{aligned} I &= C\,\dot{V} + \frac{V}{R} + I_c \sin\phi \\ V &= \frac{\hbar}{2e}\,\dot{\phi} \quad , \end{aligned} \tag{13.109}$$

where we again take I to run from left to right. If the junction is *voltage-biased*, then integrating the second of these equations yields $\phi(t) = \phi_0 + \omega_J t$, where $\omega_J = 2eV/\hbar$ is the *Josephson frequency*. The current is then

$$I = \frac{V}{R} + I_c \sin(\phi_0 + \omega_J t) \quad . \tag{13.110}$$

If the junction is *current-biased*, then we substitute the second equation into the first, to obtain

$$\frac{\hbar C}{2e}\,\ddot{\phi} + \frac{\hbar}{2eR}\,\dot{\phi} + I_c \sin\phi = I \quad . \tag{13.111}$$

We adimensionalize by writing $s \equiv \omega_p t$, with $\omega_p = (2eI_c/\hbar C)^{1/2}$ is the *Josephson plasma frequency* (at zero current). We then have

$$\frac{d^2\phi}{ds^2} + \frac{1}{Q}\,\frac{d\phi}{ds} = j - \sin\phi \equiv -\frac{du}{d\phi} \quad , \tag{13.112}$$

where $Q = \omega_p \tau$ with $\tau = RC$, and $j = I/I_c$. The quantity Q^2 is called the *McCumber-Stewart parameter*. The resistance is $R(T \approx T_c) = R_N$, while $R(T \ll T_c) \approx R_N\,\exp(\Delta/k_B T)$. The dimensionless potential energy $u(\phi)$ is given by

$$u(\phi) = -j\phi - \cos\phi \tag{13.113}$$

and resembles a 'tilted washboard'; see Fig. 13.8(a,b). This is an $N = 2$ dynamical system on a cylinder. Writing $\omega \equiv \dot{\phi}$, we have

$$\frac{d}{ds}\begin{pmatrix}\phi \\ \omega\end{pmatrix} = \begin{pmatrix}\omega \\ j - \sin\phi - Q^{-1}\omega\end{pmatrix} \quad . \tag{13.114}$$

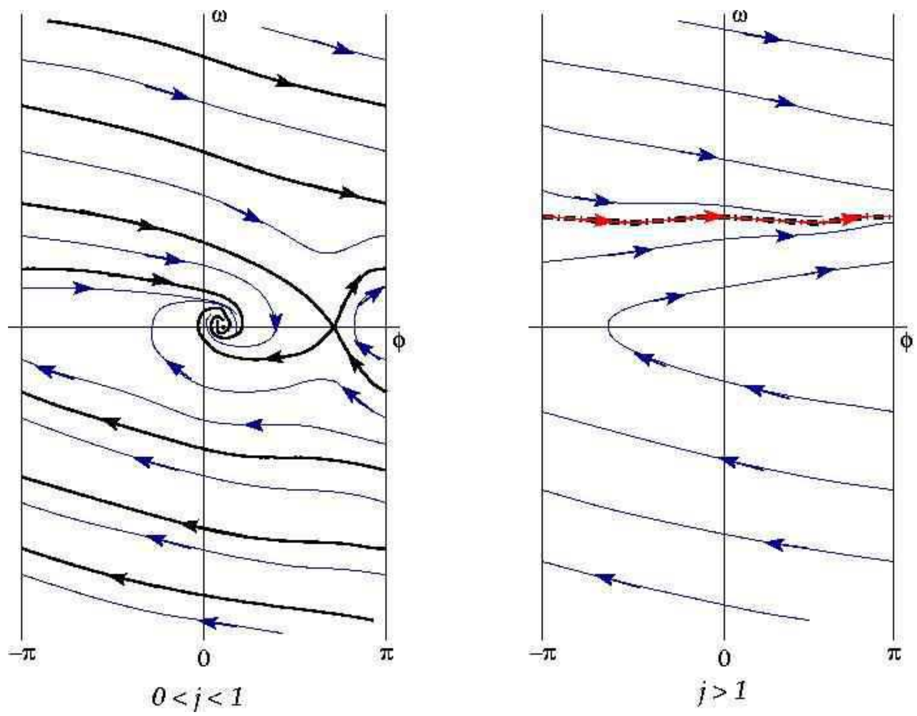

Figure 13.7: Phase flows for the equation $\ddot{\phi} + Q^{-1}\dot{\phi} + \sin\phi = j$. Left panel: $0 < j < 1$; note the separatrix (in black), which flows into the stable and unstable fixed points. Right panel: $j > 1$. The red curve overlying the thick black dot-dash curve is a *limit cycle*.

Note that $\phi \in [0, 2\pi]$ while $\omega \in (-\infty, \infty)$. Fixed points satisfy $\omega = 0$ and $j = \sin\phi$. Thus, for $|j| > 1$, there are no fixed points.

Strong damping

The RCSJ model dynamics are given by the second order ODE,

$$\partial_s^2 \phi + Q^{-1} \partial_s \phi = -u'(\phi) = j - \sin\phi \quad . \tag{13.115}$$

The parameter $Q = \omega_{\rm p}\tau$ determines the damping, with large Q corresponding to small damping. Consider the large damping limit $Q \ll 1$. In this case the inertial term proportional to $\ddot{\phi}$ may be ignored, and what remains is a first order ODE. Restoring dimensions,

$$\frac{d\phi}{dt} = \Omega\,(j - \sin\phi) \qquad , \tag{13.116}$$

where $\Omega = \omega_{\rm p}^2 RC = 2eI_{\rm c}R/\hbar$. We are effectively setting $C \equiv 0$, hence this is known as the RSJ model. The above equation describes a $N = 1$ dynamical system on the circle. When $|j| < 1$, *i.e.* $|I| < I_{\rm c}$, there are two fixed points, which are solutions to $\sin\phi^* = j$. The fixed point where $\cos\phi^* > 0$ is stable, while that with $\cos\phi^* < 0$ is unstable. The flow is toward the stable fixed

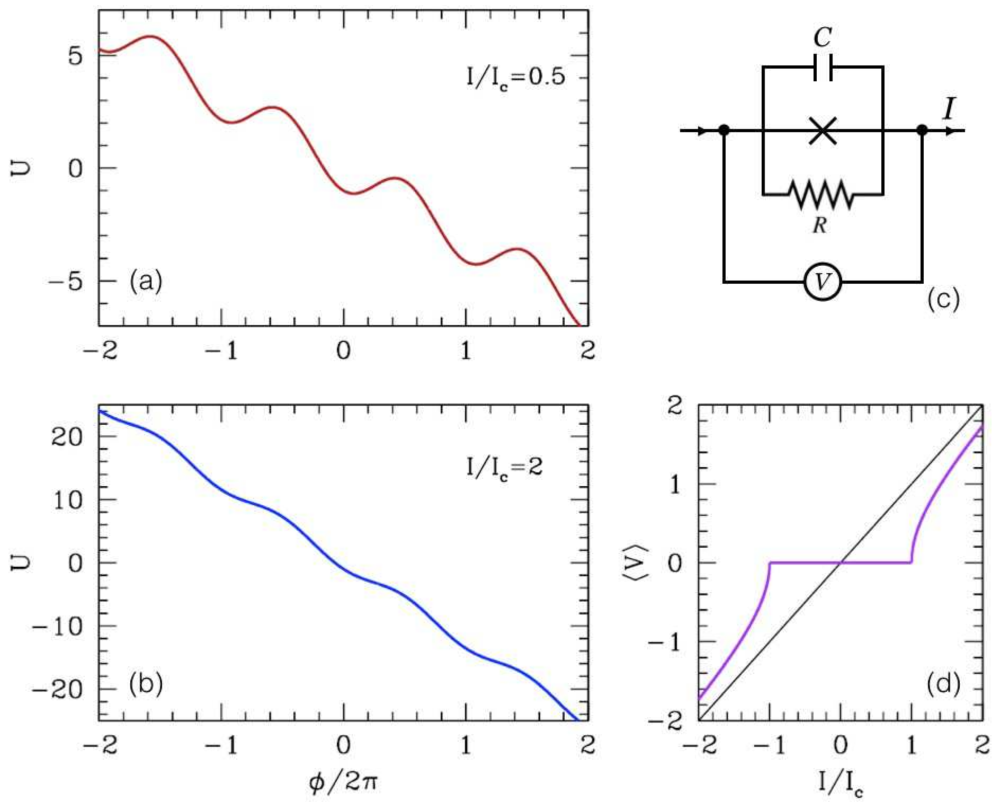

Figure 13.8: (a) Dimensionless washboard potential $u(\phi)$ for $I/I_{\rm c} = 0.5$. (b) $u(\phi)$ for $I/I_{\rm c} = 2.0$. (c) The resistively and capacitively shunted Josephson junction (RCSJ). (d) $\langle V \rangle$ *versus* I for the RSJ model.

point. At the fixed point, ϕ is constant, which means the voltage $V = \hbar\dot{\phi}/2e$ vanishes. There is current flow with no potential drop.

Consider the case $j > 1$. In this case there is a bottleneck in the ϕ evolution in the vicinity of $\phi = \frac{1}{2}\pi$, where $\dot{\phi}$ is smallest, but $\dot{\phi} > 0$ always. We compute the average voltage

$$\langle V \rangle = \frac{\hbar}{2e}\,\langle \dot{\phi} \rangle = \frac{\hbar}{2e} \cdot \frac{2\pi}{T} \quad , \tag{13.117}$$

where T is the rotational period for $\phi(t)$. We compute this using the equation of motion:

$$\varOmega T = \int\limits_0^{2\pi} \frac{d\phi}{j - \sin\phi} = \frac{2\pi}{\sqrt{j^2 - 1}} \quad . \tag{13.118}$$

Thus,

$$\langle V \rangle = \frac{\hbar}{2e}\sqrt{j^2 - 1} \cdot \frac{2eI_{\rm c}R}{\hbar} = R\sqrt{I^2 - I_{\rm c}^2} \quad . \tag{13.119}$$

This behavior is sketched in Fig. 13.8(d).

Josephson plasma oscillations

When $I < I_c$, the phase undergoes damped oscillations in the washboard minima. Expanding about the fixed point, we write $\phi = \sin^{-1} j + \delta\phi$, and obtain

$$\frac{d^2\delta\phi}{ds^2} + \frac{1}{Q}\,\frac{d\,\delta\phi}{ds} = -\sqrt{1-j^2}\,\delta\phi \quad . \tag{13.120}$$

This is the equation of a damped harmonic oscillator. With no damping ($Q = \infty$), the oscillation frequency is

$$\Omega(I) = \omega_p \left(1 - \frac{I^2}{I_c^2}\right)^{1/4} \quad . \tag{13.121}$$

When Q is finite, the frequency of the oscillations has an imaginary component, with solutions

$$\omega_\pm(I) = -\frac{i\,\omega_p}{2Q} \pm \omega_p \sqrt{\left(1 - \frac{I^2}{I_c^2}\right)^{1/2} - \frac{1}{4Q^2}} \quad . \tag{13.122}$$

Retrapping current in underdamped junctions

The energy of the junction is given by

$$E = \tfrac{1}{2} C V^2 + \frac{\hbar I_c}{2e}\,(1 - \cos\phi) \quad . \tag{13.123}$$

The first term may be thought of as a kinetic energy and the second as potential energy. Because the system is dissipative, energy is not conserved. Rather,

$$\dot{E} = C V \dot{V} + \frac{\hbar I_c}{2e}\,\dot{\phi}\,\sin\phi = V\big(C\dot{V} + I_c \sin\phi\big) = V\left(I - \frac{V}{R}\right) \quad . \tag{13.124}$$

Suppose the junction were completely undamped, *i.e.* $R = 0$. Then as the phase slides down the tilted washboard for $|I| < I_c$, it moves from peak to peak, picking up speed as it moves along. When $R > 0$, there is energy loss, and $\phi(t)$ might not make it from one peak to the next. Suppose we start at a local maximum $\phi = \pi$ with $V = 0$. What is the energy when ϕ reaches 3π? To answer that, we assume that energy is almost conserved, so

$$E = \tfrac{1}{2} C V^2 + \frac{\hbar I_c}{2e}\,(1 - \cos\phi) \approx \frac{\hbar I_c}{e} \quad \Rightarrow \quad V = \left(\frac{e\hbar I_c}{eC}\right)^{1/2} \big|\cos(\tfrac{1}{2}\phi)\big| \quad . \tag{13.125}$$

then

$$\begin{aligned}(\Delta E)_{\text{cycle}} &= \int\limits_{-\infty}^{\infty}\! dt\, V\left(I - \frac{V}{R}\right) = \frac{\hbar}{2e}\int\limits_{-\pi}^{\pi}\! d\phi \left\{ I - \frac{1}{R}\left(\frac{e\hbar I_c}{eC}\right)^{1/2} \cos(\tfrac{1}{2}\phi)\right\} \\ &= \frac{\hbar}{2e}\left\{2\pi I - \frac{4}{R}\left(\frac{e\hbar I_c}{eC}\right)^{1/2}\right\} = \frac{h}{2e}\left\{ I - \frac{4 I_c}{\pi Q}\right\} \quad .\end{aligned} \tag{13.126}$$

Thus, we identify $I_r \equiv 4I_c/\pi Q \ll I_c$ as the *retrapping current*. The idea here is to focus on the case where the phase evolution is on the cusp between trapped and free. If the system loses energy over the cycle, then subsequent motion will be attenuated, and the phase dynamics will flow to the zero voltage fixed point. Note that if the current I is reduced below I_c and then held fixed, eventually the junction will dissipate energy and enter the zero voltage state for any $|I| < I_c$. But if the current is swept and $\dot{I}/I$ is faster than the rate of energy dissipation, the retrapping occurs at $I = I_r$.

Thermal fluctuations

Restoring the proper units, the potential energy is $U(\phi) = (\hbar I_c/2e)\, u(\phi)$. Thus, thermal fluctuations may be ignored provided

$$k_B T \ll \frac{\hbar I_c}{2e} = \frac{\hbar}{2eR_N} \cdot \frac{\pi\Delta}{2e} \tanh\!\left(\frac{\Delta}{2k_B T}\right) \quad , \tag{13.127}$$

where we have invoked the Ambegaokar-Baratoff formula, Eqn. 13.96. BCS theory gives $\Delta = 1.764\, k_B T_c$, so we require

$$k_B T \ll \frac{h}{8R_N e^2} \cdot (1.764\, k_B T_c) \cdot \tanh\!\left(\frac{0.882\, T_c}{T}\right) \quad . \tag{13.128}$$

In other words,

$$\frac{R_N}{R_K} \ll \frac{0.22\, T_c}{T} \tanh\!\left(\frac{0.882\, T_c}{T}\right) \quad , \tag{13.129}$$

where $R_K = h/e^2 = 25812.8\,\Omega$ is the quantum unit of resistance[6].

We can model the effect of thermal fluctuations by adding a noise term to the RCSJ model, writing

$$C\dot{V} + \frac{V}{R} + I_c \sin\phi = I + \frac{V_f}{R} \quad , \tag{13.130}$$

where $V_f(t)$ is a stochastic term satisfying

$$\big\langle V_f(t)\, V_f(t')\big\rangle = 2k_B T R\, \delta(t-t') \quad . \tag{13.131}$$

Adimensionalizing, we now have

$$\frac{d^2\phi}{ds^2} + \gamma \frac{d\phi}{ds} = -\frac{\partial u}{\partial \phi} + \eta(s) \quad , \tag{13.132}$$

where $s = \omega_p t$, $\gamma = 1/\omega_p RC$, $u(\phi) = -j\phi - \cos\phi$, $j = I/I_c(T)$, and

$$\big\langle \eta(s)\, \eta(s')\big\rangle = \frac{2\omega_p k_B T}{I_c^2 R}\, \delta(s-s') \equiv 2\Theta\, \delta(s-s') \quad . \tag{13.133}$$

[6] R_K is called the *Klitzing* for Klaus von Klitzing, the discoverer of the integer quantum Hall effect.

Thus, $\Theta \equiv \omega_{\mathrm{p}} k_{\mathrm{B}} T / I_{\mathrm{c}}^2 R$ is a dimensionless measure of the temperature. Our problem is now that of a damped massive particle moving in the washboard potential and subjected to stochastic forcing due to thermal noise.

Writing $\omega = \partial_s \phi$, we have

$$\begin{aligned} \partial_s \phi &= \omega \\ \partial_s \omega &= -u'(\phi) - \gamma\omega + \sqrt{2\Theta}\, \eta(s) \quad . \end{aligned} \tag{13.134}$$

In this case, $W(s) = \int\limits_0^s ds'\, \eta(s')$ describes a Wiener process: $\big\langle W(s)\, W(s')\big\rangle = \mathsf{min}(s,s')$. The probability distribution $P(\phi,\omega,s)$ then satisfies the Fokker-Planck equation[7],

$$\frac{\partial P}{\partial s} = -\frac{\partial}{\partial \phi}\big(\omega P\big) + \frac{\partial}{\partial \omega}\Big\{\big[u'(\phi) + \gamma\omega\big] P\Big\} + \Theta\, \frac{\partial^2 P}{\partial \omega^2} \quad . \tag{13.135}$$

We cannot make much progress beyond numerical work starting from this equation. However, if the mean drift velocity of the 'particle' is everywhere small compared with the thermal velocity $v_{\mathrm{th}} \propto \sqrt{\Theta}$, and the mean free path $\ell \propto v_{\mathrm{th}}/\gamma$ is small compared with the scale of variation of ϕ in the potential $u(\phi)$, then, following the classic treatment by Kramers, we can convert the Fokker-Planck equation for the distribution $P(\phi,\omega,t)$ to the Smoluchowski equation for the distribution $P(\phi,t)$[8]. These conditions are satisfied when the damping γ is large. To proceed along these lines, simply assume that ω relaxes quickly, so that $\partial_s \omega \approx 0$ at all times. This says $\omega = -\gamma^{-1} u'(\phi) + \gamma^{-1} \sqrt{2\Theta}\, \eta(s)$. Plugging this into $\partial_s \phi = \omega$, we have

$$\partial_s \phi = -\gamma^{-1} u'(\phi) + \gamma^{-1} \sqrt{2\Theta}\, \eta(s) \quad , \tag{13.136}$$

the Fokker-Planck equation for which is[9]

$$\frac{\partial P(\phi,s)}{\partial s} = \frac{\partial}{\partial \phi}\Big[\gamma^{-1} u'(\phi)\, P(\phi,s)\Big] + \gamma^{-2} \Theta\, \frac{\partial^2 P(\phi,s)}{\partial \phi^2} \quad , \tag{13.137}$$

which is called the Smoluchowski equation. Note that $-\gamma^{-1} u'(\phi)$ plays the role of a local drift velocity, and $\gamma^{-2}\Theta$ that of a diffusion constant. This may be recast as

$$\frac{\partial P}{\partial s} = -\frac{\partial W}{\partial \phi} \quad , \qquad W(\phi,s) = -\gamma^{-1}\big(\partial_\phi u\big) P - \gamma^{-2}\, \Theta\, \partial_\phi P \quad . \tag{13.138}$$

[7] For the stochastic coupled ODEs $du_a = A_a\, dt + B_{ab}\, dW_b$ where each $W_a(t)$ is an independent Wiener process, *i.e.* $dW_a\, dW_b = \delta_{ab}\, dt$, then, using the Stratonovich stochastic calculus, one has the Fokker-Planck equation $\partial_t P = -\partial_a (A_a P) + \frac{1}{2} \partial_a \big[B_{ac}\, \partial_b (B_{bc} P)\big]$.

[8] See M. Ivanchenko and L. A. Zil'berman, *Sov. Phys. JETP* **28**, 1272 (1969) and, especially, V. Ambegaokar and B. I. Halperin, *Phys. Rev. Lett.* **22**, 1364 (1969).

[9] For the stochastic differential equation $dx = v_{\mathrm{d}}\, dt + \sqrt{2D}\, dW(t)$, where $W(t)$ is a Wiener process, the Fokker-Planck equation is $\partial_t P = -v_{\mathrm{d}}\, \partial_x P + D\, \partial_x^2 P$.

In steady state, we have that $\partial_s P = 0$, hence W must be a constant. We also demand $P(\phi, s) = P(\phi + 2\pi, s)$. To solve, define $F(\phi) \equiv e^{-\gamma\, u(\phi)/\Theta}$. In steady state, we then have

$$\frac{\partial}{\partial\phi}\bigg(\frac{P}{F}\bigg) = -\frac{\gamma^2 W}{\Theta} \cdot \frac{1}{F} \quad . \tag{13.139}$$

Integrating,

$$\begin{aligned} \frac{P(\phi)}{F(\phi)} - \frac{P(0)}{F(0)} &= -\frac{\gamma^2 W}{\Theta} \int\limits_0^\phi \frac{d\phi'}{F(\phi')} \\ \frac{P(2\pi)}{F(2\pi)} - \frac{P(\phi)}{F(\phi)} &= -\frac{\gamma^2 W}{\Theta} \int\limits_\phi^{2\pi} \frac{d\phi'}{F(\phi')} \quad . \end{aligned} \tag{13.140}$$

Multiply the first of these by $F(0)$ and the second by $F(2\pi)$, and then add, remembering that $P(2\pi) = P(0)$. One then obtains

$$P(\phi) = \frac{\gamma^2 W}{\Theta} \cdot \frac{F(\phi)}{F(2\pi) - F(0)} \cdot \left\{ \int\limits_0^\phi \! d\phi' \, \frac{F(0)}{F(\phi')} + \int\limits_\phi^{2\pi} \! d\phi' \, \frac{F(2\pi)}{F(\phi')} \right\} \quad . \tag{13.141}$$

We now are in a position to demand that $P(\phi)$ be normalized. Integrating over the circle, we obtain

$$W = \frac{G(j,\gamma)}{\gamma} \tag{13.142}$$

where

$$\frac{1}{G(j,\gamma/\Theta)} = \frac{\gamma/\Theta}{\exp(\pi\gamma/\Theta) - 1} \left[\int\limits_0^{2\pi} \! d\phi \, f(\phi) \right] \left[\int\limits_0^{2\pi} \frac{d\phi'}{f(\phi')} \right] + \frac{\gamma}{\Theta} \int\limits_0^{2\pi} \! d\phi \, f(\phi) \int\limits_\phi^{2\pi} \frac{d\phi'}{f(\phi')} \quad , \tag{13.143}$$

where $f(\phi) \equiv F(\phi)/F(0) = e^{-\gamma\, u(\phi)/\Theta} \, e^{\gamma\, u(0)/\Theta}$ is normalized such that $f(0) = 1$.

It remains to relate the constant W to the voltage. For any function $g(\phi)$, we have

$$\frac{d}{dt} \big\langle g\big(\phi(s)\big) \big\rangle = \int\limits_0^{2\pi} \! d\phi \, \frac{\partial P}{\partial s} \, g(\phi) = -\int\limits_0^{2\pi} \! d\phi \, \frac{\partial W}{\partial \phi} \, g(\phi) = \int\limits_0^{2\pi} \! d\phi \, W(\phi) \, g'(\phi) \quad . \tag{13.144}$$

Technically we should restrict $g(\phi)$ to be periodic, but we can still make sense of this for $g(\phi) = \phi$, with

$$\big\langle \partial_s \phi \big\rangle = \int\limits_0^{2\pi} \! d\phi \, W(\phi) = 2\pi W \quad , \tag{13.145}$$

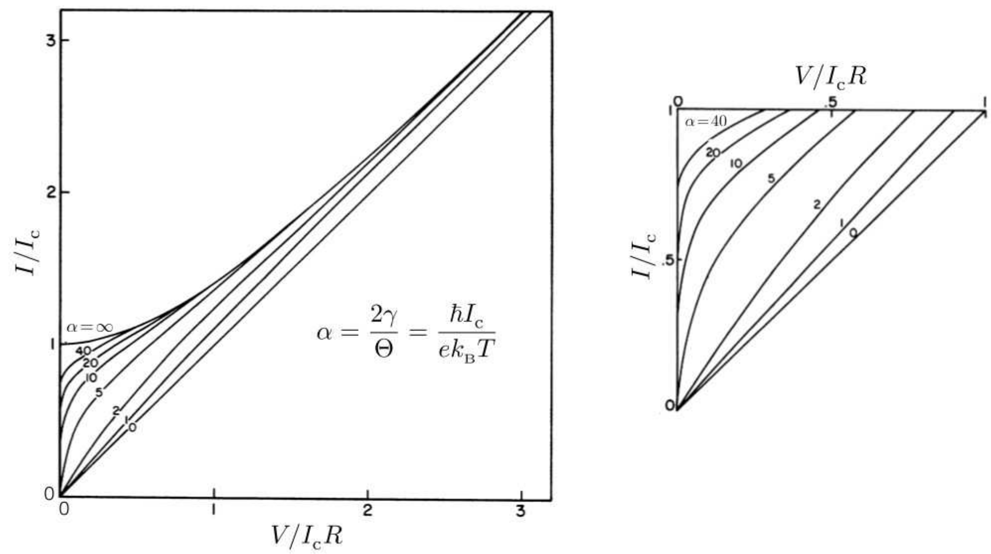

Figure 13.9: Left: scaled current bias $j = I/I_c$ *versus* scaled voltage $v = \langle V\rangle/I_c R$ for different values of the parameter γ/Θ, which is the ratio of damping to temperature. Right: detail of $j(v)$ plots. From Ambegaokar and Halperin (1969).

where the last expression on the RHS holds in steady state, where W is a constant. We could have chosen $g(\phi)$ to be a sawtooth type function, rising linearly on $\phi \in [0, 2\pi)$ then discontinuously dropping to zero, and only considered the parts where the integrands were smooth. Thus, after restoring physical units,

$$v \equiv \frac{\langle V\rangle}{I_c R} = \frac{\hbar\omega_p}{2eI_c R}\,\langle \partial_s \phi\rangle = 2\pi\, G(j, \gamma/\Theta) \quad . \tag{13.146}$$

AC Josephson effect

Suppose we add an AC bias to V, writing

$$V(t) = V_0 + V_1 \sin(\omega_1 t) \quad . \tag{13.147}$$

Integrating the Josephson relation $\dot\phi = 2eV/\hbar$, we have

$$\phi(t) = \omega_J\, t + \frac{V_1}{V_0}\,\frac{\omega_J}{\omega_1}\,\cos(\omega_1 t) + \phi_0 \quad . \tag{13.148}$$

where $\omega_J = 2eV_0/\hbar$. Thus,

$$I_J(t) = I_c \sin\!\bigg(\omega_J t + \frac{V_1\,\omega_J}{V_0\,\omega_1}\,\cos(\omega_1 t) + \phi_0\bigg) \quad . \tag{13.149}$$

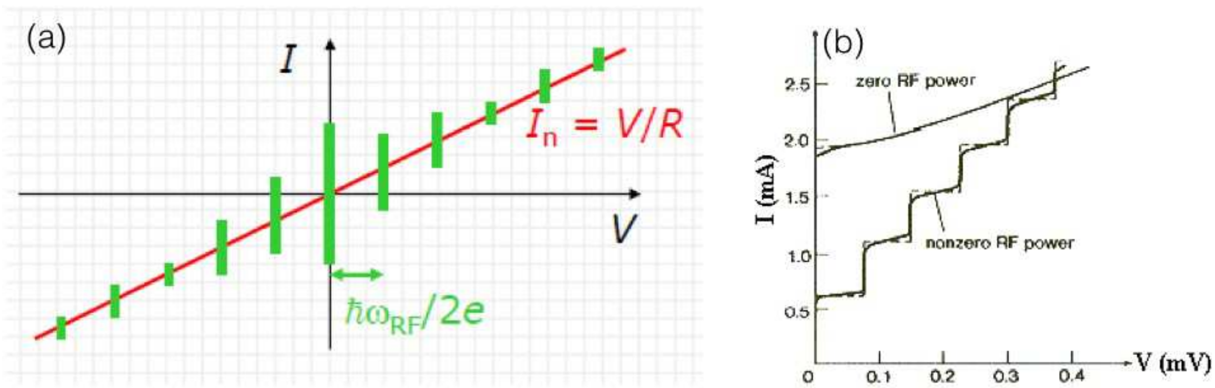

Figure 13.10: (a) Shapiro spikes in the voltage-biased AC Josephson effect. The Josephson current has a nonzero average only when $V_0 = n\hbar\omega_1/2e$, where ω_1 is the AC frequency. From *http://cmt.nbi.ku.dk/student_projects/bsc/heiselberg.pdf*. (b) Shapiro steps in the current-biased AC Josephson effect.

We now invoke the Bessel function generating relation,

$$e^{iz\cos\theta} = \sum_{n=-\infty}^{\infty} J_n(z)\, e^{-in\theta} \tag{13.150}$$

to write

$$I_{\rm J}(t) = I_{\rm c} \sum_{n=-\infty}^{\infty} J_n\!\left(\frac{V_1\,\omega_{\rm J}}{V_0\,\omega_1}\right) \sin\big[(\omega_{\rm J} - n\omega_1)\,t + \phi_0\big] \quad . \tag{13.151}$$

Thus, $I_{\rm J}(t)$ oscillates in time, except for terms for which

$$\omega_{\rm J} = n\omega_1 \quad \Rightarrow \quad V_0 = n\,\frac{\hbar\omega_1}{2e} \quad , \tag{13.152}$$

in which case

$$I_{\rm J}(t) = I_{\rm c}\, J_n\!\left(\frac{2eV_1}{\hbar\omega_1}\right) \sin\phi_0 \quad . \tag{13.153}$$

We now add back in the current through the resistor, to obtain

$$\begin{aligned} \big\langle I(t)\big\rangle &= \frac{V_0}{R} + I_{\rm c}\, J_n\!\left(\frac{2eV_1}{\hbar\omega_1}\right) \sin\phi_0 \\ &\in \left[\frac{V_0}{R} - I_{\rm c}\, J_n\!\left(\frac{2eV_1}{\hbar\omega_1}\right) , \, \frac{V_0}{R} + I_{\rm c}\, J_n\!\left(\frac{2eV_1}{\hbar\omega_1}\right)\right] \quad . \end{aligned} \tag{13.154}$$

This feature, depicted in Fig. 13.10(a), is known as *Shapiro spikes*.

Current-biased AC Josephson effect

When the junction is current-biased, we must solve

$$\frac{\hbar C}{2e}\,\ddot{\phi} + \frac{\hbar}{2eR}\,\dot{\phi} + I_c \sin\phi = I(t) \quad , \tag{13.155}$$

with $I(t) = I_0 + I_1 \cos(\omega_1 t)$. This results in the *Shapiro steps* shown in Fig. 13.10(b). To analyze this equation, we write our phase space coordinates on the cylinder as $(x_1, x_2) = (\phi, \omega)$, and add the forcing term to Eqn. 13.114, *viz.*

$$\begin{aligned} \frac{d}{dt}\begin{pmatrix} \phi \\ \omega \end{pmatrix} &= \begin{pmatrix} \omega \\ j - \sin\phi - Q^{-1}\omega \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ \cos(\nu s) \end{pmatrix} \\ \frac{d\boldsymbol{x}}{ds} &= \boldsymbol{V}(\boldsymbol{x}) + \varepsilon \boldsymbol{f}(\boldsymbol{x}, s) \quad , \end{aligned} \tag{13.156}$$

where $s = \omega_p t$, $\nu = \omega_1/\omega_p$, and $\varepsilon = I_1/I_c$. As before, we have $j = I_0/I_c$. When $\varepsilon = 0$, we have the RCSJ model, which for $|j| > 1$ has a stable limit cycle and no fixed points. The phase curves for the RCSJ model and the limit cycle for $|j| > 1$ are depicted in Fig. 13.7. In our case, the forcing term $\boldsymbol{f}(\boldsymbol{x}, s)$ has the simple form $f_1 = 0$, $f_2 = \cos(\nu s)$, but it could be more complicated and nonlinear in $\boldsymbol{x}$.

The phenomenon we are studying is called *synchronization*[10]. Linear oscillators perturbed by a harmonic force will oscillate with the forcing frequency once transients have damped out. Consider, for example, the equation $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\Omega t)$, where $\beta > 0$ is a damping coefficient. The solution is $x(t) = A(\Omega)\cos\big(\Omega t + \delta(\Omega)\big) + x_h(t)$, where $x_h(t)$ solves the homogeneous equation (*i.e.* with $f_0 = 0$) and decays to zero exponentially at large times. Nonlinear oscillators, such as the RCSJ model under study here, also can be synchronized to the external forcing, but not necessarily always. In the case of the Duffing oscillator, $\ddot{x} + 2\beta\dot{x} + x + \eta x^3$, with $\beta > 0$ and $\eta > 0$, the origin $(x = 0, \dot{x} = 0)$ is still a stable fixed point. In the presence of an external forcing $\varepsilon\, f_0 \cos(\Omega t)$, with β, η, and ε all small, varying the detuning $\delta\Omega = \Omega - 1$ (also assumed small) can lead to hysteresis in the amplitude of the oscillations, but the oscillator is always entrained, *i.e.* synchronized with the external forcing.

The situation changes considerably if the nonlinear oscillator has no stable fixed point but rather a stable limit cycle. This is the case, for example, for the van der Pol equation $\ddot{x} + 2\beta(x^2 - 1)\dot{x} + x = 0$, and it is also the case for the RCSJ model. The limit cycle $\boldsymbol{x}_0(s)$ has a period, which we call T_0, so $\boldsymbol{x}(s + T_0) = \boldsymbol{x}(s)$. All points on the limit cycle (LC) are fixed under the T_0-advance map g_{T_0}, where $g_\tau \boldsymbol{x}(s) = \boldsymbol{x}(s + \tau)$. We may parameterize points along the LC by an angle θ which increases uniformly in s, so that $\dot{\theta} = \nu_0 = 2\pi/T_0$. Furthermore, since each point $\boldsymbol{x}_0(\theta)$ is a fixed point under g_{T_0}, and the LC is presumed to be attractive, we may define the *θ-isochrone* as the set of points $\{\boldsymbol{x}\}$ in phase space which flow to $\boldsymbol{x}_0(\theta)$ under repeated application of g_{T_0}. For an N-dimensional phase space, the isochrones are $(N-1)$-dimensional hypersurfaces. For the

[10]See A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization* (Cambridge, 2001).

RCSJ model, which has $N = 2$, the isochrones are curves $\theta = \theta(\phi, \omega)$ on the (ϕ, ω) cylinder. In particular, the θ-isochrone is a curve which intersects the LC at the point $\boldsymbol{x}_0(\theta)$. We then have

$$\begin{aligned} \frac{d\theta}{ds} &= \sum_{j=1}^{N} \frac{\partial\theta}{\partial x_j} \frac{dx_j}{ds} \\ &= \nu_0 + \varepsilon \sum_{j=1}^{N} \frac{\partial\theta}{\partial x_j} f_j\big(\boldsymbol{x}(s), s\big) \quad . \end{aligned} \tag{13.157}$$

If we are close to the LC, we may replace $\boldsymbol{x}(s)$ on the RHS above with $\boldsymbol{x}_0(\theta)$, yielding

$$\frac{d\theta}{ds} = \nu_0 + \varepsilon F(\theta, s) \quad , \tag{13.158}$$

where

$$F(\theta, s) = \sum_{j=1}^{N} \frac{\partial\theta}{\partial x_j}\bigg|_{\boldsymbol{x}_0(\theta)} f_j\big(\boldsymbol{x}_0(\theta), s\big) \quad . \tag{13.159}$$

OK, so now here's the thing. The function $F(\theta, s)$ is separately periodic in both its arguments, so we may write

$$F(\theta, s) = \sum_{k,l} F_{k,l}\, e^{i(k\theta + l\nu s)} \quad , \tag{13.160}$$

where $\boldsymbol{f}\big(\boldsymbol{x}, s + \frac{2\pi}{\nu}\big) = \boldsymbol{f}(\boldsymbol{x}, s)$, *i.e.* ν is the forcing frequency. The unperturbed solution has $\dot\theta = \nu_0$, hence the forcing term in Eqn. 13.158 is resonant when $k\nu_0 + l\nu \approx 0$. This occurs when $\nu \approx \frac{p}{q}\,\nu_0$, where p and q are relatively prime integers. The resonance condition is satisfied when $k = rp$ and $l = -rq$ for any integer r

.

We now separate the resonant from nonresonant terms in the (k, l) sum, writing

$$\dot\theta = \nu_0 + \varepsilon \sum_{r=-\infty}^{\infty} F_{rp,-rq}\, e^{ir(p\theta - q\nu s)} + \text{NRT} \quad , \tag{13.161}$$

where NRT stands for "non-resonant terms". We next average over short time scales to eliminate these nonresonant terms, and focus on the dynamics of the average phase $\langle\theta\rangle$. Defining $\psi \equiv p\,\langle\theta\rangle - q\,\nu s$, we have

$$\begin{aligned} \dot\psi &= p\,\langle\dot\theta\rangle - q\nu \\ &= (p\nu_0 - q\nu) + \varepsilon p \sum_{r=-\infty}^{\infty} F_{rp,-rq}\, e^{ir\psi} \\ &= -\delta + \varepsilon\, G(\psi) \quad , \end{aligned} \tag{13.162}$$

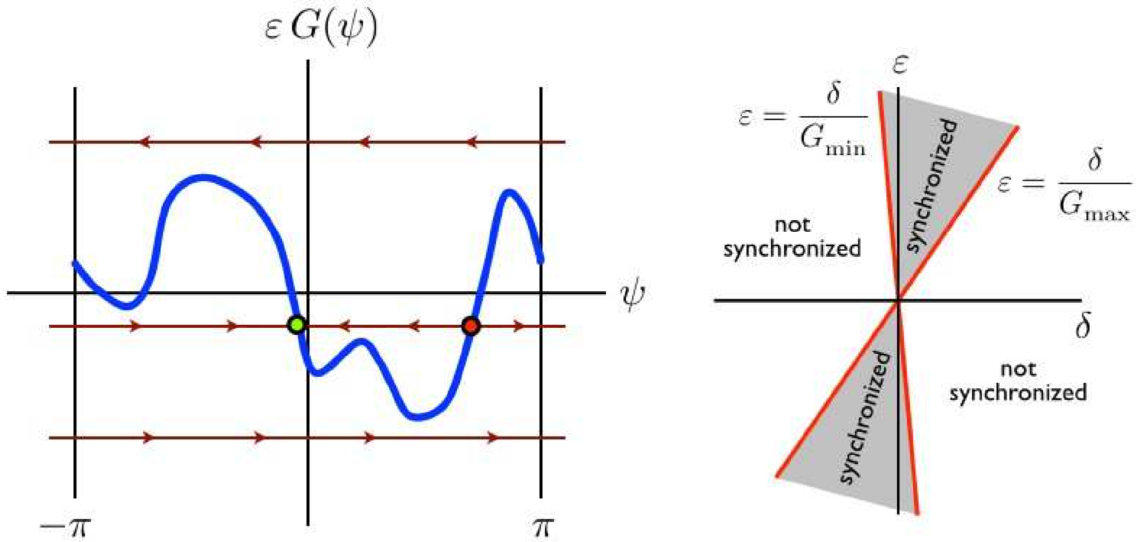

Figure 13.11: Left: graphical solution of $\dot{\psi} = -\delta + \varepsilon\, G(\psi)$. Fixed points are only possible if $-\varepsilon\, G_{\rm min} \leq \delta \leq G_{\rm max}$. Right: synchronization region, shown in grey, in the (δ, ε) plane.

where $\delta \equiv q\nu - p\nu_0$ is the detuning, and $G(\psi) \equiv p \sum_r F_{rp,-rq}\, e^{ir\psi}$ is the sum over resonant terms. This last equation is that of a simple $N = 1$ dynamical system on the circle! If the detuning δ falls within the range $\big[\varepsilon G_{\rm min}\,,\, \varepsilon G_{\rm max}\big]$, then ψ flows to a stable fixed point where $\delta = \varepsilon\, G(\psi^*)$. The oscillator is then synchronized with the forcing, because $\langle\dot{\theta}\rangle \to \frac{q}{p}\,\nu$. If the detuning is too large and lies outside this range, then there is no synchronization. Rather, $\psi(s)$ increases linearly with the time s, and $\langle\theta(t)\rangle = \theta_0 + \frac{q}{p}\,\nu s + \frac{1}{p}\,\psi(s)$, where

$$dt = \frac{d\psi}{\varepsilon\, G(\psi) - \delta} \qquad \Longrightarrow \qquad T_\psi = \int_0^{2\pi} \frac{d\psi}{\varepsilon\, G(\psi) - \delta} \quad . \tag{13.163}$$

For weakly forced, weakly nonlinear oscillators, resonance occurs only for $\nu = \pm\nu_0$, but in the case of weakly forced, strongly nonlinear oscillators, the general resonance condition is $\nu = \frac{p}{q}\,\nu_0$. The reason is that in the case of weakly nonlinear oscillators, the limit cycle is itself harmonic to zeroth order. There are then only two frequencies in its Fourier decomposition, *i.e.* $\pm\nu_0$. In the strongly nonlinear case, the limit cycle is decomposed into a fundamental frequency ν_0 plus all its harmonics. In addition, the forcing $\boldsymbol{f}(\boldsymbol{x}, s)$ can itself can be a general periodic function of s, involving multiples of the fundamental forcing frequency ν. For the case of the RCSJ, the forcing function is harmonic and independent of $\boldsymbol{x}$. This means that only the $l = \pm 1$ terms enter in the above analysis.

13.4 Ultrasonic Attenuation

Recall the electron-phonon Hamiltonian,

$$\begin{aligned}\hat{H}_{\text{el-ph}} &= \frac{1}{\sqrt{V}} \sum_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \sigma,\lambda}} g_{\boldsymbol{k}\boldsymbol{k}'\lambda} \left(a^\dagger_{\boldsymbol{k}'-\boldsymbol{k},\lambda} + a_{\boldsymbol{k}-\boldsymbol{k}',\lambda}\right) c^\dagger_{\boldsymbol{k}\sigma}\, c_{\boldsymbol{k}'\sigma} \qquad (13.164)\\ &= \frac{1}{\sqrt{V}} \sum_{\substack{\boldsymbol{k},\boldsymbol{k}\\ \sigma,\lambda}} g_{\boldsymbol{k}\boldsymbol{k}'\lambda} \big(a^\dagger_{\boldsymbol{k}'-\boldsymbol{k},\lambda} + a_{\boldsymbol{k}-\boldsymbol{k}',\lambda}\big)\big(u_{\boldsymbol{k}}\gamma^\dagger_{\boldsymbol{k}\sigma} - \sigma\, e^{-i\phi}\, v_{\boldsymbol{k}}\, \gamma_{-\boldsymbol{k}\,-\sigma}\big)\big(u_{\boldsymbol{k}'}\gamma_{\boldsymbol{k}'\sigma} - \sigma\, e^{i\phi}\, v_{\boldsymbol{k}'}\, \gamma^\dagger_{-\boldsymbol{k}'\,-\sigma}\big)\ .\end{aligned}$$

Let's now compute the phonon lifetime using Fermi's Golden Rule[11]. In the phonon absorption process, a phonon of wavevector $\boldsymbol{q}$ is absorbed by an electron of wavevector $\boldsymbol{k}$, converting it into an electron of wavevector $\boldsymbol{k}' = \boldsymbol{k} + \boldsymbol{q}$. The net absorption rate of $(\boldsymbol{q}, \lambda)$ phonons is then is given by the rate of

$$\Gamma^{\text{abs}}_{\boldsymbol{q}\lambda} = \frac{2\pi n_{\boldsymbol{q},\lambda}}{V} \sum_{\boldsymbol{k},\boldsymbol{k}',\sigma} \left|g_{\boldsymbol{k}\boldsymbol{k}'\lambda}\right|^2 \left(u_{\boldsymbol{k}}u_{\boldsymbol{k}'} - v_{\boldsymbol{k}}v_{\boldsymbol{k}'}\right)^2 f_{\boldsymbol{k}\sigma} \left(1 - f_{\boldsymbol{k}'\sigma}\right) \delta(E_{\boldsymbol{k}'} - E_{\boldsymbol{k}} - \hbar\omega_{\boldsymbol{q}\lambda})\, \delta_{\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{q} \bmod \boldsymbol{G}} \quad . \qquad (13.165)$$

Here $n_{\boldsymbol{q}\lambda}$ is the Bose function and $f_{\boldsymbol{k}\sigma}$ the Fermi function, and we have assumed that the phonon frequencies are all smaller than 2Δ, so we may ignore quasiparticle pair creation and pair annihilation processes. Note that the electron Fermi factors yield the probability that the state $|\boldsymbol{k}\sigma\rangle$ is occupied while $|\boldsymbol{k}'\sigma\rangle$ is vacant. *Mutatis mutandis*, the emission rate of these phonons is[12]

$$\Gamma^{\text{em}}_{\boldsymbol{q}\lambda} = \frac{2\pi(n_{\boldsymbol{q},\lambda}+1)}{V} \sum_{\boldsymbol{k},\boldsymbol{k}',\sigma} \left|g_{\boldsymbol{k}\boldsymbol{k}'\lambda}\right|^2 \left(u_{\boldsymbol{k}}u_{\boldsymbol{k}'} - v_{\boldsymbol{k}}v_{\boldsymbol{k}'}\right)^2 f_{\boldsymbol{k}'\sigma} \left(1 - f_{\boldsymbol{k}\sigma}\right) \delta(E_{\boldsymbol{k}'} - E_{\boldsymbol{k}} - \hbar\omega_{\boldsymbol{q}\lambda})\, \delta_{\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{q} \bmod \boldsymbol{G}} \quad . \qquad (13.166)$$

We then have

$$\frac{dn_{\boldsymbol{q}\lambda}}{dt} = -\alpha_{\boldsymbol{q}\lambda}\, n_{\boldsymbol{q}\lambda} + s_{\boldsymbol{q}\lambda} \quad , \qquad (13.167)$$

where

$$\alpha_{\boldsymbol{q}\lambda} = \frac{4\pi}{V} \sum_{\boldsymbol{k},\boldsymbol{k}'} \left|g_{\boldsymbol{k}\boldsymbol{k}'\lambda}\right|^2 \left(u_{\boldsymbol{k}}u_{\boldsymbol{k}'} - v_{\boldsymbol{k}}v_{\boldsymbol{k}'}\right)^2 \left(f_{\boldsymbol{k}} - f_{\boldsymbol{k}'}\right) \delta(E_{\boldsymbol{k}'} - E_{\boldsymbol{k}} - \hbar\omega_{\boldsymbol{q}\lambda})\, \delta_{\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{q} \bmod \boldsymbol{G}} \qquad (13.168)$$

is the attenuation rate, and $s_{\boldsymbol{q}\lambda}$ is due to spontaneous emission,

$$s_{\boldsymbol{q}\lambda} = \frac{4\pi}{V} \sum_{\boldsymbol{k},\boldsymbol{k}'} \left|g_{\boldsymbol{k}\boldsymbol{k}'\lambda}\right|^2 \left(u_{\boldsymbol{k}}u_{\boldsymbol{k}'} - v_{\boldsymbol{k}}v_{\boldsymbol{k}'}\right)^2 f_{\boldsymbol{k}'} \left(1 - f_{\boldsymbol{k}}\right) \delta(E_{\boldsymbol{k}'} - E_{\boldsymbol{k}} - \hbar\omega_{\boldsymbol{q}\lambda})\, \delta_{\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{q} \bmod \boldsymbol{G}} \quad . \qquad (13.169)$$

---

[11] Here we follow §3.4 of J. R. Schrieffer, *Theory of Superconductivity* (Benjamin-Cummings, 1964).

[12] Note the factor of $n + 1$ in the emission rate, where the additional 1 is due to spontaneous emission. The absorption rate includes only a factor of n.

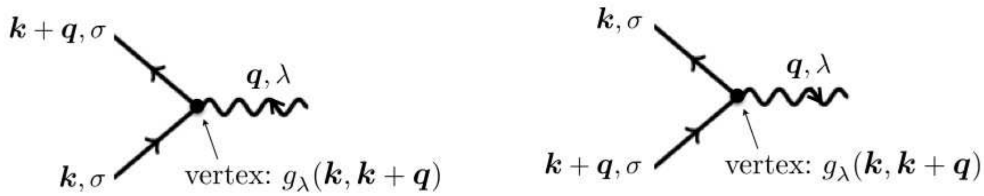

Figure 13.12: Phonon absorption and emission processes.

We now expand about the Fermi surface, writing

$$\frac{1}{V}\sum_{\boldsymbol{k},\boldsymbol{k}'} F(\xi_{\boldsymbol{k}},\xi_{\boldsymbol{k}'})\,\delta_{\boldsymbol{k}',\boldsymbol{k}+\boldsymbol{q}} = \tfrac{1}{4}\,g^2(\mu)\int\limits_{-\infty}^{\infty}\!d\xi\int\limits_{-\infty}^{\infty}\!d\xi'\,F(\xi,\xi')\int\!\frac{d\hat{\boldsymbol{k}}}{4\pi}\int\!\frac{d\hat{\boldsymbol{k}}'}{4\pi}\,\delta(k_{\rm F}\hat{\boldsymbol{k}}'-k_{\rm F}\hat{\boldsymbol{k}}-\boldsymbol{q})\ . \tag{13.170}$$

for any function $F(\xi,\xi')$. The integrals over $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{k}}'$ give

$$\int\!\frac{d\hat{\boldsymbol{k}}}{4\pi}\int\!\frac{d\hat{\boldsymbol{k}}'}{4\pi}\,\delta(k_{\rm F}\hat{\boldsymbol{k}}'-k_{\rm F}\hat{\boldsymbol{k}}-\boldsymbol{q}) = \frac{1}{4\pi k_{\rm F}^3}\cdot\frac{k_{\rm F}}{2q}\cdot\Theta(2k_{\rm F}-q)\quad . \tag{13.171}$$

The step function appears naturally because the constraint $k_{\rm F}\hat{\boldsymbol{k}}' = k_{\rm F}\hat{\boldsymbol{k}}+\boldsymbol{q}$ requires that $\boldsymbol{q}$ connect two points which lie on the metallic Fermi surface, so the largest $|\boldsymbol{q}|$ can be is $2k_{\rm F}$. We will drop the step function in the following expressions, assuming $q < 2k_{\rm F}$, but it is good to remember that it is implicitly present. Thus, ignoring *Umklapp* processes, we have

$$\alpha_{\boldsymbol{q}\lambda} = \frac{g^2(\mu)\,|g_{\boldsymbol{q}\lambda}|^2}{8\,k_{\rm F}^2\,q}\int\limits_{-\infty}^{\infty}\!d\xi\int\limits_{-\infty}^{\infty}\!d\xi'\,(uu'-vv')^2\,(f-f')\,\delta(E'-E-\hbar\omega_{\boldsymbol{q}\lambda})\quad . \tag{13.172}$$

We now use

$$(uu'\pm vv')^2 = \left(\sqrt{\frac{E+\xi}{2E}}\sqrt{\frac{E'+\xi'}{2E'}}\pm\sqrt{\frac{E-\xi}{2E}}\sqrt{\frac{E'-\xi'}{2E'}}\right)^2 = \frac{EE'+\xi\xi'\pm\Delta^2}{EE'} \tag{13.173}$$

and change variables $\big(\,\xi = E\,dE/\sqrt{E^2-\Delta^2}\,\big)$ to write

$$\alpha_{\boldsymbol{q}\lambda} = \frac{g^2(\mu)\,|g_{\boldsymbol{q}\lambda}|^2}{2\,k_{\rm F}^2\,q}\int\limits_{\Delta}^{\infty}\!dE\int\limits_{\Delta}^{\infty}\!dE'\,\frac{(EE'-\Delta^2)(f-f')}{\sqrt{E^2-\Delta^2}\,\sqrt{E'^2-\Delta^2}}\,\delta(E'-E-\hbar\omega_{\boldsymbol{q}\lambda})\quad . \tag{13.174}$$

We now satisfy the Dirac delta function, which means we eliminate the E' integral and set $E' = E+\hbar\omega_{\boldsymbol{q}\lambda}$ everywhere else in the integrand. Clearly the $f-f'$ term will be first order in the smallness of $\hbar\omega_{\boldsymbol{q}}$, so in all other places we may set $E'=E$ to lowest order. This simplifies the above expression considerably, and we are left with

$$\alpha_{\boldsymbol{q}\lambda} = \frac{g^2(\mu)\,|g_{\boldsymbol{q}\lambda}|^2\,\hbar\omega_{\boldsymbol{q}\lambda}}{2\,k_{\rm F}^2\,q}\int\limits_{\Delta}^{\infty}\!dE\left(-\frac{\partial f}{\partial E}\right) = \frac{g^2(\mu)\,|g_{\boldsymbol{q}\lambda}|^2\,\hbar\omega_{\boldsymbol{q}\lambda}}{2\,k_{\rm F}^2\,q}\,f(\Delta)\quad , \tag{13.175}$$

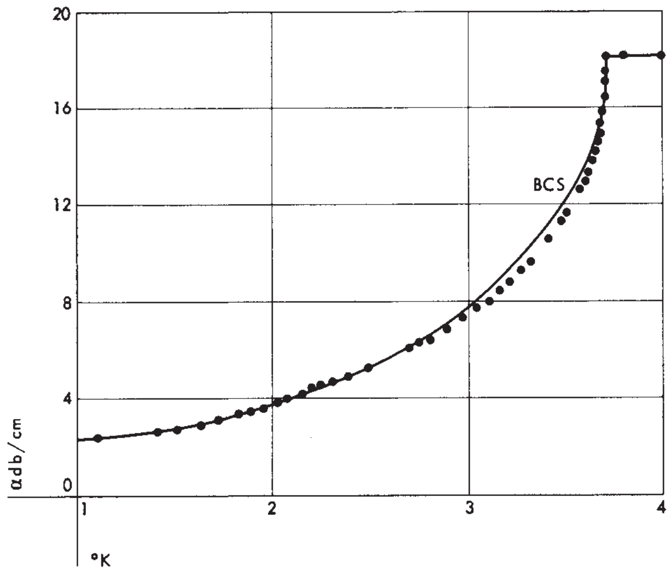

Figure 13.13: Ultrasonic attenuation in tin, compared with predictions of the BCS theory. From R. W. Morse, *IBM Jour. Res. Dev.* **6**, 58 (1963).

where $q < 2k_{\mathrm{F}}$ is assumed. For $q \to 0$, we have $\omega_{\boldsymbol{q}\lambda}/q \to c_\lambda(\hat{\boldsymbol{q}})$, the phonon velocity.

We may now write the ratio of the phonon attenuation rate in the superconducting and normal states as

$$\frac{\alpha_{\mathsf{S}}(T)}{\alpha_{\mathsf{N}}(T)} = \frac{f(\Delta)}{f(0)} = \frac{2}{\exp\!\Big(\frac{\Delta(T)}{k_{\mathrm{B}}T}\Big) + 1} \quad . \tag{13.176}$$

The ratio naturally goes to unity at $T = T_{\mathrm{c}}$, where Δ vanishes. Results from early experiments on superconducting Sn are shown in Fig. 13.13.

13.5 Nuclear Magnetic Relaxation

We start with the hyperfine Hamiltonian,

$$\hat{H}_{\mathrm{HF}} = A \sum_{\boldsymbol{k},\boldsymbol{k}'} \sum_{\boldsymbol{R}} \varphi^*_{\boldsymbol{k}}(\boldsymbol{R})\, \varphi_{\boldsymbol{k}'}(\boldsymbol{R}) \Big[J^+_{\boldsymbol{R}}\, c^\dagger_{\boldsymbol{k}\downarrow}\, c_{\boldsymbol{k}'\uparrow} + J^-_{\boldsymbol{R}}\, c^\dagger_{\boldsymbol{k}\uparrow}\, c_{\boldsymbol{k}'\downarrow} + J^z_{\boldsymbol{R}} \left(c^\dagger_{\boldsymbol{k}\uparrow}\, c_{\boldsymbol{k}'\uparrow} - c^\dagger_{\boldsymbol{k}\downarrow}\, c_{\boldsymbol{k}'\downarrow} \right) \Big] \tag{13.177}$$

where $\boldsymbol{J}_{\boldsymbol{R}}$ is the nuclear spin operator on nuclear site $\boldsymbol{R}$, satisfying

$$\left[J^\mu_{\boldsymbol{R}} \,,\, J^\nu_{\boldsymbol{R}'} \right] = i\, \epsilon_{\mu\nu\lambda}\, J^\lambda_{\boldsymbol{R}}\, \delta_{\boldsymbol{R},\boldsymbol{R}'} \quad , \tag{13.178}$$

and where $\varphi_{\boldsymbol{k}}(\boldsymbol{R})$ is the amplitude of the electronic Bloch wavefunction (with band index suppressed) on the nuclear site $\boldsymbol{R}$. Using

$$c_{\boldsymbol{k}\sigma} = u_{\boldsymbol{k}}\, \gamma_{\boldsymbol{k}\sigma} - \sigma\, v_{\boldsymbol{k}}\, e^{i\phi}\, \gamma^\dagger_{-\boldsymbol{k}\,-\sigma} \tag{13.179}$$

we have for $\boldsymbol{S}_{\boldsymbol{k}\boldsymbol{k}'} = \frac{1}{2}\, c^\dagger_{\boldsymbol{k}\mu}\, \boldsymbol{\sigma}_{\mu\nu}\, c_{\boldsymbol{k}'\nu}$,

$$\begin{aligned}
S^+_{\boldsymbol{k}\boldsymbol{k}'} &= u_{\boldsymbol{k}} u_{\boldsymbol{k}'} \gamma^\dagger_{\boldsymbol{k}\uparrow}\gamma_{\boldsymbol{k}'\downarrow} - v_{\boldsymbol{k}} v_{\boldsymbol{k}'}\, \gamma_{-\boldsymbol{k}\downarrow}\gamma^\dagger_{-\boldsymbol{k}'\uparrow} + u_{\boldsymbol{k}} v_{\boldsymbol{k}'}\, e^{i\phi}\, \gamma^\dagger_{\boldsymbol{k}\uparrow}\gamma^\dagger_{-\boldsymbol{k}'\uparrow} - u_{\boldsymbol{k}} v_{\boldsymbol{k}'}\, e^{-i\phi}\, \gamma_{-\boldsymbol{k}\downarrow}\gamma_{\boldsymbol{k}'\downarrow} \\
S^-_{\boldsymbol{k}\boldsymbol{k}'} &= u_{\boldsymbol{k}} u_{\boldsymbol{k}'} \gamma^\dagger_{\boldsymbol{k}\downarrow}\gamma_{\boldsymbol{k}'\uparrow} - v_{\boldsymbol{k}} v_{\boldsymbol{k}'}\, \gamma_{-\boldsymbol{k}\uparrow}\gamma^\dagger_{-\boldsymbol{k}'\downarrow} - u_{\boldsymbol{k}} v_{\boldsymbol{k}'}\, e^{i\phi}\, \gamma^\dagger_{\boldsymbol{k}\downarrow}\gamma^\dagger_{-\boldsymbol{k}'\downarrow} + u_{\boldsymbol{k}} v_{\boldsymbol{k}'}\, e^{-i\phi}\, \gamma_{-\boldsymbol{k}\uparrow}\gamma_{\boldsymbol{k}'\uparrow} \qquad (13.180) \\
S^z_{\boldsymbol{k}\boldsymbol{k}'} &= \frac{1}{2}\sum_\sigma \Big(u_{\boldsymbol{k}} u_{\boldsymbol{k}'}\, \gamma^\dagger_{\boldsymbol{k}\sigma}\gamma_{\boldsymbol{k}'\sigma} + v_{\boldsymbol{k}} v_{\boldsymbol{k}'}\, \gamma_{-\boldsymbol{k}\,-\sigma}\gamma^\dagger_{-\boldsymbol{k}'\,-\sigma} - \sigma\, u_{\boldsymbol{k}} v_{\boldsymbol{k}'}\; e^{i\phi}\, \gamma^\dagger_{\boldsymbol{k}\sigma}\gamma^\dagger_{-\boldsymbol{k}'\,-\sigma} - \sigma\, v_{\boldsymbol{k}} u_{\boldsymbol{k}'}\; e^{-i\phi}\, \gamma_{-\boldsymbol{k}\,-\sigma}\gamma_{\boldsymbol{k}'\sigma} \Big) \ .
\end{aligned}$$

Let's assume our nuclei are initially spin polarized, and let us calculate the rate $1/T_1$ at which the J^z component of the nuclear spin relaxes. Again appealing to the Golden Rule,

$$\frac{1}{T_1} = 2\pi\, |A|^2 \sum_{\boldsymbol{k},\boldsymbol{k}'} |\varphi_{\boldsymbol{k}}(0)|^2\, |\varphi_{\boldsymbol{k}'}(0)|^2\, \big(u_{\boldsymbol{k}} u_{\boldsymbol{k}'} + v_{\boldsymbol{k}} v_{\boldsymbol{k}'}\big)^2 f_{\boldsymbol{k}}\big(1 - f_{\boldsymbol{k}'}\big)\, \delta(E_{\boldsymbol{k}'} - E_{\boldsymbol{k}} - \hbar\omega) \qquad (13.181)$$

where ω is the nuclear spin precession frequency in the presence of internal or external magnetic fields. Assuming $\varphi_{\boldsymbol{k}}(\boldsymbol{R}) = C/\sqrt{V}$, we write $V^{-1}\sum_{\boldsymbol{k}} \to \frac{1}{2} g(\mu) \int d\xi$ and we appeal to Eqn. 13.173. Note that the coherence factors in this case give $(uu' + vv')^2$, as opposed to $(uu' - vv')^2$ as we found in the case of ultrasonic attenuation (more on this below). What we then obtain is

$$\frac{1}{T_1} = 2\pi\, |A|^2\, |C|^4\, g^2(\mu) \int\limits_\Delta^\infty dE\; \frac{E(E+\hbar\omega) + \Delta^2}{\sqrt{E^2-\Delta^2}\,\sqrt{(E+\hbar\omega)^2 - \Delta^2}}\; f(E)\,\big[1 - f(E+\hbar\omega)\big] \quad . \qquad (13.182)$$

Let's first evaluate this expression for normal metals, where $\Delta = 0$. We have

$$\frac{1}{T_{1,\text{N}}} = 2\pi\, |A|^2\, |C|^4\, g^2(\mu) \int\limits_0^\infty d\xi\; f(\xi)\big[1 - f(\xi + \hbar\omega)\big] = \pi\, |A|^2\, |C|^4\, g^2(\mu)\, k_{\text{B}} T \quad , \qquad (13.183)$$

where we have assumed $\hbar\omega \ll k_{\text{B}}T$, and used $f(\xi)\big[1 - f(\xi)\big] = -k_{\text{B}}T\, f'(\xi)$. The assumption $\omega \to 0$ is appropriate because the nuclear magneton is so tiny: $\mu_{\text{N}}/k_{\text{B}} = 3.66 \times 10^{-4}\text{K/T}$, so the nuclear splitting is on the order of mK even at fields as high as 10 T. The NMR relaxation rate is thus proportional to temperature, a result known as the *Korringa law*.

Now let's evaluate the ratio of NMR relaxation rates in the superconducting and normal states. Assuming $\hbar\omega \ll \Delta$, we have

$$\frac{T^{-1}_{1,\text{S}}}{T^{-1}_{1,\text{N}}} = 2 \int\limits_\Delta^\infty dE\; \frac{E(E+\hbar\omega) + \Delta^2}{\sqrt{E^2-\Delta^2}\,\sqrt{(E+\hbar\omega)^2 - \Delta^2}} \left(-\frac{\partial f}{\partial E}\right) \quad . \qquad (13.184)$$

We dare not send $\omega \to 0$ in the integrand, because this would lead to a logarithmic divergence. Numerical integration shows that for $\hbar\omega \lesssim \frac{1}{2} k_{\text{B}} T_{\text{c}}$, the above expression has a peak just below $T = T_{\text{c}}$. This is the famous *Hebel-Slichter peak*.

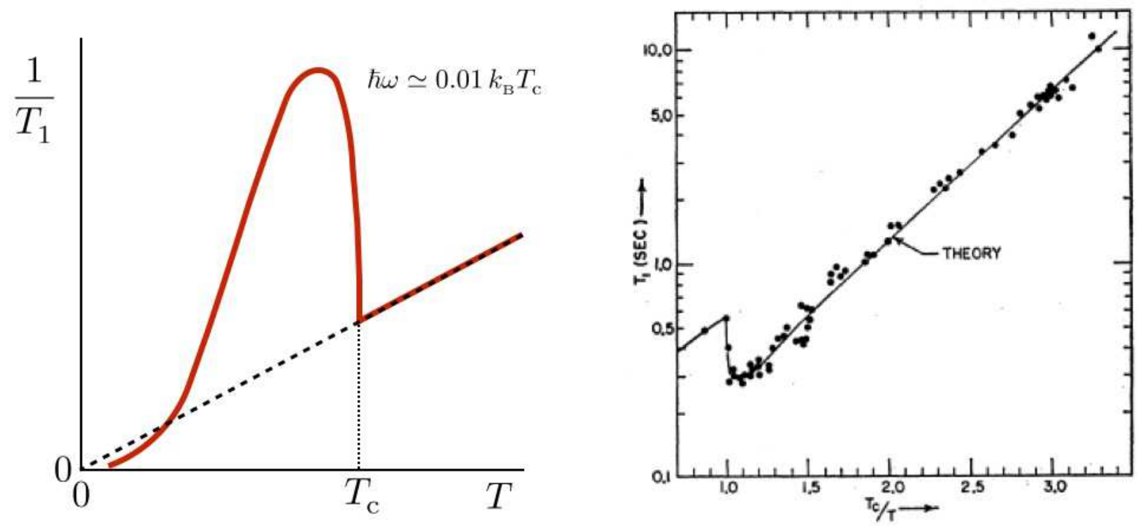

Figure 13.14: Left: Sketch of NMR relaxation rate $1/T_1$ *versus* temperature as predicted by BCS theory, with $\hbar\omega \approx 0.01\, k_B T_c$, showing the Hebel-Slichter peak. Right: T_1 *versus* T_c/T in a powdered aluminum sample, from Y. Masuda and A. G. Redfield, *Phys. Rev.* **125**, 159 (1962). The Hebel-Slichter peak is seen here as a dip.

These results for acoustic attenuation and spin relaxation exemplify so-called *case I* and *case II* responses of the superconductor, respectively. In case I, the transition matrix element is proportional to $uu' - vv'$, which vanishes at $\xi = 0$. In case II, the transition matrix element is proportional to $uu' + vv'$.

13.6 General Theory of BCS Linear Response

Consider a general probe of the superconducting state described by the perturbation Hamiltonian

$$\hat{V}(t) = \sum_{\boldsymbol{k},\sigma} \sum_{\boldsymbol{k}',\sigma'} \Big[B\big(\boldsymbol{k}\sigma \,|\, \boldsymbol{k}'\sigma'\big)\, e^{-i\omega t} + B^*\big(\boldsymbol{k}'\sigma' \,|\, \boldsymbol{k}\sigma\big)\, e^{+i\omega t} \Big] c^\dagger_{\boldsymbol{k}\sigma}\, c_{\boldsymbol{k}'\sigma'} \quad . \tag{13.185}$$

An example would be ultrasonic attenuation, where

$$\hat{V}_{\text{ultra}}(t) = U \sum_{\boldsymbol{k},\boldsymbol{k}',\sigma} \phi_{\boldsymbol{k}'-\boldsymbol{k}}(t)\, c^\dagger_{\boldsymbol{k}\sigma}\, c_{\boldsymbol{k}'\sigma'} \quad . \tag{13.186}$$

Here $\phi(\boldsymbol{r}) = \nabla \cdot \boldsymbol{u}$ is the deformation of the lattice and U is the deformation potential, with the interaction of the local deformation with the electrons given by $U\phi(\boldsymbol{r})\, n(\boldsymbol{r})$, where $n(\boldsymbol{r})$ is the total electron number density at $\boldsymbol{r}$. Another example is interaction with microwaves. In this case, the bare dispersion is corrected by $\boldsymbol{p} \to \boldsymbol{p} + \frac{e}{c}\boldsymbol{A}$, hence

$$\hat{V}_{\mu\text{wave}}(t) = \frac{e\hbar}{2m^* c} \sum_{\boldsymbol{k},\boldsymbol{k}',\sigma} (\boldsymbol{k} + \boldsymbol{k}') \cdot \boldsymbol{A}_{\boldsymbol{k}'-\boldsymbol{k}}(t)\, c^\dagger_{\boldsymbol{k}\sigma}\, c_{\boldsymbol{k}'\sigma'} \quad , \tag{13.187}$$

where m^* is the band mass.

Consider now a general perturbation Hamiltonian of the form

$$\hat{V} = -\sum_i \big(\phi_i(t)\, C_i^\dagger + \phi_i^*(t)\, C_i\big) \tag{13.188}$$

where C_i are operators labeled by i. We write

$$\phi_i(t) = \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \hat{\phi}_i(\omega)\, e^{-i\omega t} \quad . \tag{13.189}$$

According to the general theory of linear response formulated in chapter 9, the power dissipation due to this perturbation is given by

$$\begin{aligned} P(\omega) = &-i\omega\, \hat{\phi}_i^*(\omega)\, \hat{\phi}_j(\omega)\, \hat{\chi}_{C_i C_j^\dagger}(\omega) + i\omega\, \hat{\phi}_i(\omega)\, \hat{\phi}_j^*(\omega)\, \hat{\chi}_{C_i^\dagger C_j}(-\omega) \\ &- i\omega\, \hat{\phi}_i^*(\omega)\, \hat{\phi}_j^*(-\omega)\, \hat{\chi}_{C_i C_j}(\omega) + i\omega\, \hat{\phi}_i(\omega)\, \hat{\phi}_j(-\omega)\, \hat{\chi}_{C_i^\dagger C_j^\dagger}(-\omega) \quad . \end{aligned} \tag{13.190}$$

where $\hat{H} = \hat{H}_0 + \hat{V}$ and $C_i(t) = e^{i\hat{H}_0 t/\hbar}\, C_i\, e^{-i\hat{H}_0 t/\hbar}$ is the operator C_i in the interaction representation.

$$\hat{\chi}_{AB}(\omega) = \frac{i}{\hbar} \int\limits_0^{\infty} dt\, e^{-i\omega t} \left\langle \big[A(t)\,,\, B(0)\big] \right\rangle \tag{13.191}$$

For our application, we have $i \equiv (\boldsymbol{k}\sigma \,|\, \boldsymbol{k}'\sigma')$ and $j \equiv (\boldsymbol{p}\mu \,|\, \boldsymbol{p}'\mu')$, with $C_i^\dagger = c^\dagger_{\boldsymbol{k}\sigma}\, c_{\boldsymbol{k}'\sigma'}$ and $C_j = c^\dagger_{\boldsymbol{p}'\mu'} c_{\boldsymbol{p}\mu}$, *etc.* So we need to compute the response function,

$$\hat{\chi}_{C_i C_j^\dagger}(\omega) = \frac{i}{\hbar} \int\limits_0^{\infty} dt \left\langle \Big[c^\dagger_{\boldsymbol{k}'\sigma'}(t)\, c_{\boldsymbol{k}\sigma}(t)\,,\, c^\dagger_{\boldsymbol{p}\mu}(0)\, c_{\boldsymbol{p}'\mu'}(0) \Big] \right\rangle e^{i\omega t} \quad . \tag{13.192}$$

OK, so strap in, because this is going to be a bit of a bumpy ride.

We evaluate the commutator in real time and then Fourier transform to the frequency domain. Using Wick's theorem for fermions[13],

$$\langle c_1^\dagger\, c_2\, c_3^\dagger\, c_4 \rangle = \langle c_1^\dagger\, c_2 \rangle\, \langle c_3^\dagger\, c_4 \rangle - \langle c_1^\dagger\, c_3^\dagger \rangle\, \langle c_2\, c_4 \rangle + \langle c_1^\dagger\, c_4 \rangle\, \langle c_2\, c_3^\dagger \rangle \quad , \tag{13.193}$$

[13] Wick's theorem is valid when taking expectation values in Slater determinant states.

we have

$$\begin{aligned}\chi_{C_i C_j^\dagger}(t) &= \frac{i}{\hbar}\Big\langle\Big[c^\dagger_{\boldsymbol{k}'\sigma'}(t)\,c_{\boldsymbol{k}\sigma}(t)\,,\,c^\dagger_{\boldsymbol{p}\mu}(0)\,c_{\boldsymbol{p}'\mu'}(0)\Big]\Big\rangle\,\Theta(t) \\ &= -\frac{i}{\hbar}\Big[F^a_{\boldsymbol{k}'\sigma'}(t)\,F^b_{\boldsymbol{k}\sigma}(t) - F^c_{\boldsymbol{k}\sigma}(t)\,F^d_{\boldsymbol{k}'\sigma'}(t)\Big]\,\delta_{\boldsymbol{p},\boldsymbol{k}}\,\delta_{\boldsymbol{p}',\boldsymbol{k}'}\,\delta_{\mu,\sigma}\,\delta_{\mu',\sigma'} \\ &\qquad + \frac{i}{\hbar}\Big[G^a_{\boldsymbol{k}'\sigma'}(t)\,G^b_{\boldsymbol{k}\sigma}(t) - G^c_{\boldsymbol{k}\sigma}(t)\,G^d_{\boldsymbol{k}'\sigma'}(t)\Big]\,\sigma\sigma'\,\delta_{\boldsymbol{p},-\boldsymbol{k}'}\,\delta_{\boldsymbol{p}',-\boldsymbol{k}}\,\delta_{\mu,-\sigma'}\,\delta_{\mu',-\sigma}\quad,\end{aligned} \tag{13.194}$$

where, using the Bogoliubov transformation,

$$\begin{aligned} c_{\boldsymbol{k}\sigma} &= u_{\boldsymbol{k}}\,\gamma_{\boldsymbol{k}\sigma} - \sigma\,v_{\boldsymbol{k}}\,e^{+i\phi}\,\gamma^\dagger_{-\boldsymbol{k}\,-\sigma} \\ c^\dagger_{-\boldsymbol{k}\,-\sigma} &= u_{\boldsymbol{k}}\,\gamma^\dagger_{-\boldsymbol{k}\,-\sigma} + \sigma\,v_{\boldsymbol{k}}\,e^{-i\phi}\,\gamma_{\boldsymbol{k}\sigma}\quad,\end{aligned} \tag{13.195}$$

we find

$$\begin{aligned} F^a_{\boldsymbol{q}\nu}(t) &= -i\,\Theta(t)\,\big\langle c^\dagger_{\boldsymbol{q}\nu}(t)\,c_{\boldsymbol{q}\nu}(0)\big\rangle = -i\,\Theta(t)\,\Big\{u^2_{\boldsymbol{q}}\,e^{iE_{\boldsymbol{q}}t/\hbar}\,f(E_{\boldsymbol{q}}) + v^2_{\boldsymbol{q}}\,e^{-iE_{\boldsymbol{q}}t/\hbar}\big[1-f(E_{\boldsymbol{q}})\big]\Big\} \\ F^b_{\boldsymbol{q}\nu}(t) &= -i\,\Theta(t)\,\big\langle c_{\boldsymbol{q}\nu}(t)\,c^\dagger_{\boldsymbol{q}\nu}(0)\big\rangle = -i\,\Theta(t)\,\Big\{u^2_{\boldsymbol{q}}\,e^{-iE_{\boldsymbol{q}}t/\hbar}\big[1-f(E_{\boldsymbol{q}})\big] + v^2_{\boldsymbol{q}}\,e^{iE_{\boldsymbol{q}}t/\hbar}\,f(E_{\boldsymbol{q}})\Big\} \\ F^c_{\boldsymbol{q}\nu}(t) &= -i\,\Theta(t)\,\big\langle c^\dagger_{\boldsymbol{q}\nu}(0)\,c_{\boldsymbol{q}\nu}(t)\big\rangle = -i\,\Theta(t)\,\Big\{u^2_{\boldsymbol{q}}\,e^{-iE_{\boldsymbol{q}}t/\hbar}\,f(E_{\boldsymbol{q}}) + v^2_{\boldsymbol{q}}\,e^{iE_{\boldsymbol{q}}t/\hbar}\big[1-f(E_{\boldsymbol{q}})\big]\Big\} \\ F^d_{\boldsymbol{q}\nu}(t) &= -i\,\Theta(t)\,\big\langle c_{\boldsymbol{q}\nu}(0)\,c^\dagger_{\boldsymbol{q}\nu}(t)\big\rangle = -i\,\Theta(t)\,\Big\{u^2_{\boldsymbol{q}}\,e^{iE_{\boldsymbol{q}}t/\hbar}\big[1-f(E_{\boldsymbol{q}})\big] + v^2_{\boldsymbol{q}}\,e^{-iE_{\boldsymbol{q}}t/\hbar}\,f(E_{\boldsymbol{q}})\Big\}\end{aligned} \tag{13.196}$$

and

$$\begin{aligned} G^a_{\boldsymbol{q}\nu}(t) &= -i\,\Theta(t)\,\big\langle c^\dagger_{\boldsymbol{q}\nu}(t)\,c^\dagger_{-\boldsymbol{q}\,-\nu}(0)\big\rangle = -i\,\Theta(t)\,u_{\boldsymbol{q}}\,v_{\boldsymbol{q}}\,e^{-i\phi}\Big\{e^{iE_{\boldsymbol{q}}t/\hbar}\,f(E_{\boldsymbol{q}}) - e^{-iE_{\boldsymbol{q}}t/\hbar}\big[1-f(E_{\boldsymbol{q}})\big]\Big\} \\ G^b_{\boldsymbol{q}\nu}(t) &= -i\,\Theta(t)\,\big\langle c_{\boldsymbol{q}\nu}(t)\,c_{-\boldsymbol{q}\,-\nu}(0)\big\rangle = -i\,\Theta(t)\,u_{\boldsymbol{q}}\,v_{\boldsymbol{q}}\,e^{+i\phi}\Big\{e^{-E_{\boldsymbol{q}}t/\hbar}\big[1-f(E_{\boldsymbol{q}})\big] - e^{-iE_{\boldsymbol{q}}t/\hbar}\,f(E_{\boldsymbol{q}})\Big\} \\ G^c_{\boldsymbol{q}\nu}(t) &= -i\,\Theta(t)\,\big\langle c^\dagger_{\boldsymbol{q}\nu}(0)\,c^\dagger_{-\boldsymbol{q}\,-\nu}(t)\big\rangle = -i\,\Theta(t)\,u_{\boldsymbol{q}}\,v_{\boldsymbol{q}}\,e^{-i\phi}\Big\{e^{iE_{\boldsymbol{q}}t/\hbar}\big[1-f(E_{\boldsymbol{q}})\big] - e^{-iE_{\boldsymbol{q}}t/\hbar}\,f(E_{\boldsymbol{q}})\Big\} \\ G^d_{\boldsymbol{q}\nu}(t) &= -i\,\Theta(t)\,\big\langle c^\dagger_{\boldsymbol{q}\nu}(0)\,c^\dagger_{-\boldsymbol{q}\,-\nu}(t)\big\rangle = -i\,\Theta(t)\,u_{\boldsymbol{q}}\,v_{\boldsymbol{q}}\,e^{+i\phi}\Big\{e^{-iE_{\boldsymbol{q}}t/\hbar}\,f(E_{\boldsymbol{q}}) - e^{iE_{\boldsymbol{q}}t/\hbar}\big[1-f(E_{\boldsymbol{q}})\big]\Big\}\quad.\end{aligned} \tag{13.197}$$

Taking the Fourier transforms, we have[14]

$$\hat{F}^a(\omega) = \frac{u^2 f}{\omega + E + i\epsilon} + \frac{v^2\,(1-f)}{\omega - E + i\epsilon}\quad,\qquad \hat{F}^c(\omega) = \frac{u^2 f}{\omega - E + i\epsilon} + \frac{v^2\,(1-f)}{\omega + E + i\epsilon} \tag{13.198}$$

$$\hat{F}^b(\omega) = \frac{u^2\,(1-f)}{\omega - E + i\epsilon} + \frac{v^2 f}{\omega + E + i\epsilon}\quad,\qquad \hat{F}^d(\omega) = \frac{u^2\,(1-f)}{\omega + E + i\epsilon} + \frac{v^2 f}{\omega - E + i\epsilon} \tag{13.199}$$

[14]Here we are being somewhat loose and have set $\hbar = 1$ to avoid needless notational complication. We shall restore the proper units at the end of our calculation.

and

$$\hat{G}^a(\omega) = u\,v\,e^{-i\phi}\bigg(\frac{f}{\omega+E+i\epsilon} - \frac{1-f}{\omega-E+i\epsilon}\bigg) \quad , \quad \hat{G}^c(\omega) = u\,v\,e^{+i\phi}\bigg(\frac{1-f}{\omega-E+i\epsilon} - \frac{f}{\omega+E+i\epsilon}\bigg) \tag{13.200}$$

$$\hat{G}^b(\omega) = u\,v\,e^{+i\phi}\bigg(\frac{1-f}{\omega+E+i\epsilon} - \frac{f}{\omega-E+i\epsilon}\bigg) \quad , \quad \hat{G}^d(\omega) = u\,v\,e^{+i\phi}\bigg(\frac{f}{\omega+E+i\epsilon} - \frac{1-f}{\omega-E+i\epsilon}\bigg) \quad . \tag{13.201}$$

Using the result that the Fourier transform of a product is a convolution of Fourier transforms, we have from Eqn. 13.194,

$$\begin{aligned}\hat{\chi}_{C_i C_j^\dagger}(\omega) &= \frac{i}{\hbar}\,\delta_{\boldsymbol{p},\boldsymbol{k}}\,\delta_{\boldsymbol{p}',\boldsymbol{k}'}\,\delta_{\mu,\sigma}\,\delta_{\mu',\sigma'}\int\limits_{-\infty}^{\infty}\frac{d\nu}{2\pi}\,\Big[\hat{F}^c_{\boldsymbol{k}\sigma}(\nu)\,\hat{F}^d_{\boldsymbol{k}'\sigma'}(\omega-\nu) - \hat{F}^a_{\boldsymbol{k}'\sigma'}(\nu)\,\hat{F}^b_{\boldsymbol{k}\sigma}(\omega-\nu)\Big] \\ &\quad + \frac{i}{\hbar}\,\delta_{\boldsymbol{p},-\boldsymbol{k}'}\,\delta_{\boldsymbol{p}',-\boldsymbol{k}}\,\delta_{\mu,-\sigma'}\,\delta_{\mu',-\sigma}\int\limits_{-\infty}^{\infty}\frac{d\nu}{2\pi}\,\Big[\hat{G}^a_{\boldsymbol{k}\sigma}(\nu)\,\hat{G}^b_{\boldsymbol{k}'\sigma'}(\omega-\nu) - \hat{G}^c_{\boldsymbol{k}'\sigma'}(\nu)\,\hat{G}^d_{\boldsymbol{k}\sigma}(\omega-\nu)\Big] \quad .\end{aligned} \tag{13.202}$$

The integrals are easily done via the contour method. For example, one has

$$\begin{aligned} i\int\limits_{-\infty}^{\infty}\frac{d\nu}{2\pi}\,\hat{F}^c_{\boldsymbol{k}\sigma}(\nu)\,\hat{F}^d_{\boldsymbol{k}'\sigma'}(\omega-\nu) &= -\int\limits_{-\infty}^{\infty}\frac{d\nu}{2\pi i}\,\bigg(\frac{u^2\,f}{\nu-E+i\epsilon} + \frac{v^2\,(1-f)}{\nu+E+i\epsilon}\bigg)\bigg(\frac{u'^2\,(1-f')}{\omega-\nu+E'+i\epsilon} + \frac{v'^2\,f'}{\omega-\nu-E'+i\epsilon}\bigg) \\ &= \frac{u^2\,u'^2\,(1-f)\,f'}{\omega+E-E'+i\epsilon} + \frac{v^2\,u'^2\,f\,f'}{\omega-E-E'+i\epsilon} + \frac{u^2\,v'^2\,(1-f)(1-f')}{\omega+E+E'+i\epsilon} + \frac{v^2\,v'^2\,f(1-f')}{\omega-E+E'+i\epsilon} \ .\end{aligned} \tag{13.203}$$

One then finds (with proper units restored),

$$\begin{aligned}\hat{\chi}_{C_i C_j^\dagger}(\omega) &= \delta_{\boldsymbol{p},\boldsymbol{k}}\,\delta_{\boldsymbol{p}',\boldsymbol{k}'}\,\delta_{\mu,\sigma}\,\delta_{\mu',\sigma'}\Bigg(\frac{u^2u'^2\,(f-f')}{\hbar\omega-E+E'+i\epsilon} - \frac{v^2v'^2\,(f-f')}{\hbar\omega+E-E'+i\epsilon} \\ &\qquad\qquad + \frac{u^2v'^2\,(1-f-f')}{\hbar\omega+E+E'+i\epsilon} - \frac{v^2u'^2\,(1-f-f')}{\hbar\omega-E-E'+i\epsilon}\Bigg) \\ &\quad + \delta_{\boldsymbol{p},-\boldsymbol{k}'}\,\delta_{\boldsymbol{p}',-\boldsymbol{k}}\,\delta_{\mu,-\sigma'}\,\delta_{\mu',-\sigma}\Bigg(\frac{f'-f}{\hbar\omega-E+E'+i\epsilon} - \frac{f'-f}{\hbar\omega+E-E'+i\epsilon} \\ &\qquad\qquad + \frac{1-f-f'}{\hbar\omega+E+E'+i\epsilon} - \frac{1-f-f'}{\hbar\omega-E-E'+i\epsilon}\Bigg)\,uvu'v'\sigma\sigma' \quad .\end{aligned} \tag{13.204}$$

We are almost done. Note that $C_i = c^\dagger_{\boldsymbol{k}'\sigma'} c_{\boldsymbol{k}\sigma}$ means $C_i^\dagger = c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}'\sigma'}$, hence once we have $\hat{\chi}_{C_i C_j^\dagger}(\omega)$ we can easily obtain from it $\hat{\chi}_{C_i^\dagger C_j^\dagger}(\omega)$ and the other response functions in Eqn. 13.190, simply by permuting the wavevector and spin labels.

13.6.1 Case I and case II probes

The last remaining piece in the derivation is to note that, for virtually all cases of interest,

$$\sigma\sigma' B(-\boldsymbol{k}' \; -\sigma' \,|\, -\boldsymbol{k} \; -\sigma) = \eta \, B(\boldsymbol{k}\sigma \,|\, \boldsymbol{k}'\sigma') \quad , \tag{13.205}$$

where $B(\boldsymbol{k}\sigma \,|\, \boldsymbol{k}'\sigma')$ is the transition matrix element in the original fermionic (*i.e.* 'pre-Bogoliubov') representation, from Eqn. 13.185, and where $\eta = +1$ (case I) or $\eta = -1$ (case II). The eigenvalue η tells us how the perturbation Hamiltonian transforms under the combined operations of time reversal and particle-hole transformation. The action of time reversal is

$$\mathcal{T} \,|\, \boldsymbol{k}\,\sigma \,\rangle = \sigma \,|\, -\boldsymbol{k} \; -\sigma \,\rangle \quad \Rightarrow \quad c^\dagger_{\boldsymbol{k}\sigma} \to \sigma \, c^\dagger_{-\boldsymbol{k}\,-\sigma} \tag{13.206}$$

The particle-hole transformation sends $c^\dagger_{\boldsymbol{k}\sigma} \to c_{\boldsymbol{k}\sigma}$. Thus, under the combined operation,

$$\begin{aligned}\sum_{\boldsymbol{k},\sigma} \sum_{\boldsymbol{k}',\sigma'} B(\boldsymbol{k}\sigma \,|\, \boldsymbol{k}'\sigma') \, c^\dagger_{\boldsymbol{k}\sigma} \, c_{\boldsymbol{k}'\sigma'} &\to -\sum_{\boldsymbol{k},\sigma} \sum_{\boldsymbol{k}',\sigma'} \sigma\sigma' \, B(-\boldsymbol{k}' \; -\sigma' \,|\, -\boldsymbol{k} \; -\sigma) \, c^\dagger_{\boldsymbol{k}\sigma} \, c_{\boldsymbol{k}'\sigma'} + \text{const.} \\ &\to -\eta \sum_{\boldsymbol{k},\sigma} \sum_{\boldsymbol{k}',\sigma'} B(\boldsymbol{k}\sigma \,|\, \boldsymbol{k}'\sigma') \, c^\dagger_{\boldsymbol{k}\sigma} \, c_{\boldsymbol{k}'\sigma'} + \text{const.} \quad .\end{aligned} \tag{13.207}$$

If we can write $B(\boldsymbol{k}\sigma \,|\, \boldsymbol{k}'\sigma') = B_{\sigma\sigma'}(\xi_{\boldsymbol{k}}, \xi_{\boldsymbol{k}'})$, then, further assuming that our perturbation corresponds to a definite η , we have that the power dissipated is

$$\begin{aligned}P = \tfrac{1}{2}\, g^2(\mu) \sum_{\sigma,\sigma'} \int\limits_{-\infty}^{\infty}\! d\omega \, \omega \int\limits_{-\infty}^{\infty}\! d\xi \int\limits_{-\infty}^{\infty}\! d\xi' \, \big|B_{\sigma\sigma'}(\xi,\xi';\omega)\big|^2 \times \\ \bigg\{ \big(uu' - \eta vv'\big)^2 \, (f - f') \, \Big[\delta(\hbar\omega + E - E') + \delta(\hbar\omega + E' - E)\Big] \\ + \tfrac{1}{2}(uv' + \eta vu')^2 \, (1 - f - f') \, \Big[\delta(\hbar\omega - E - E') - \delta(\hbar\omega + E + E')\Big] \bigg\} \quad .\end{aligned} \tag{13.208}$$

The coherence factors entering the above expression are

$$\begin{aligned}\tfrac{1}{2}(uu' - \eta vv')^2 &= \frac{1}{2}\left(\sqrt{\frac{E+\xi}{2E}}\sqrt{\frac{E'+\xi'}{2E'}} - \eta\sqrt{\frac{E-\xi}{2E}}\sqrt{\frac{E'-\xi'}{2E'}}\right)^2 = \frac{EE' + \xi\xi' - \eta\Delta^2}{2EE'} \\ \tfrac{1}{2}(uv' + \eta vu')^2 &= \frac{1}{2}\left(\sqrt{\frac{E+\xi}{2E}}\sqrt{\frac{E'-\xi'}{2E'}} + \eta\sqrt{\frac{E-\xi}{2E}}\sqrt{\frac{E'+\xi'}{2E'}}\right)^2 = \frac{EE' - \xi\xi' + \eta\Delta^2}{2EE'} \quad .\end{aligned} \tag{13.209}$$

Integrating over ξ and ξ' kills the $\xi\xi'$ terms, and we define the coherence factors

$$F(E,E',\Delta) \equiv \frac{EE' - \eta\Delta^2}{2EE'} \qquad , \qquad \widetilde{F}(E,E',\Delta) \equiv \frac{EE' + \eta\Delta^2}{2EE'} = 1 - F \quad . \tag{13.210}$$

case	$\hbar\omega \ll 2\Delta$	$\hbar\omega \gg 2\Delta$	$\hbar\omega \approx 2\Delta$	$\hbar\omega \gg 2\Delta$
I ($\eta = +1$)	$F \approx 0$	$F \approx \frac{1}{2}$	$\widetilde{F} \approx 1$	$\widetilde{F} \approx \frac{1}{2}$
II ($\eta = -1$)	$F \approx 1$	$F \approx \frac{1}{2}$	$\widetilde{F} \approx 0$	$\widetilde{F} \approx \frac{1}{2}$

Table 13.1: Frequency dependence of the BCS coherence factors $F(E, E+\hbar\omega, \Delta)$ and $\widetilde{F}(E, \hbar\omega - E, \Delta)$ for $E \approx \Delta$.

The behavior of $F(E, E', \Delta)$ is summarized in Tab. 13.1. If we approximate $B_{\sigma\sigma'}(\xi, \xi'; \omega) \approx B_{\sigma\sigma'}(0, 0\,; \omega)$, and we define $|\mathcal{B}(\omega)|^2 = \sum_{\sigma,\sigma'} \big|B_{\sigma\sigma'}(0,0\,;\omega)\big|^2$, then we have

$$P = \int\limits_{-\infty}^{\infty} \! d\omega \, |\mathcal{B}(\omega)|^2 \, \mathcal{P}(\omega) \quad , \tag{13.211}$$

where

$$\begin{aligned}\mathcal{P}(\omega) \equiv \omega \int\limits_{\Delta}^{\infty} \! dE \int\limits_{\Delta}^{\infty} \! dE' \, \widetilde{n}_{\mathsf{s}}(E) \, \widetilde{n}_{\mathsf{s}}(E') \, \Big\{ & F(E, E', \Delta) \, (f - f') \, \Big[\delta(\hbar\omega + E - E') + \delta(\hbar\omega + E' - E)\Big] \\ & + \widetilde{F}(E, E', \Delta) \, (1 - f - f') \, \Big[\delta(\hbar\omega - E - E') - \delta(\hbar\omega + E + E')\Big] \Big\} \quad ,\end{aligned} \tag{13.212}$$

with

$$\widetilde{n}_{\mathsf{s}}(E) = \frac{g(\mu)\,|E|}{\sqrt{E^2 - \Delta^2}} \, \Theta(E^2 - \Delta^2) \quad , \tag{13.213}$$

which is the superconducting density of states from Eqn. 13.76. Note that the coherence factor for quasiparticle scattering is F, while that for quasiparticle pair creation or annihilation is $\widetilde{F} = 1 - F$.

13.6.2 Electromagnetic absorption

The interaction of light and matter is given in Eqn. 13.187. We have

$$B(\boldsymbol{k}\sigma \,|\, \boldsymbol{k}'\sigma') = \frac{e\hbar}{2mc} \, (\boldsymbol{k} + \boldsymbol{k}') \cdot \boldsymbol{A}_{\boldsymbol{k}-\boldsymbol{k}'} \, \delta_{\sigma\sigma'} \quad , \tag{13.214}$$

from which we see

$$\sigma\sigma' \, B(-\boldsymbol{k}' \, -\sigma' \,|\, -\boldsymbol{k} \, -\sigma) = -B(\boldsymbol{k}\sigma \,|\, \boldsymbol{k}'\sigma') \quad , \tag{13.215}$$

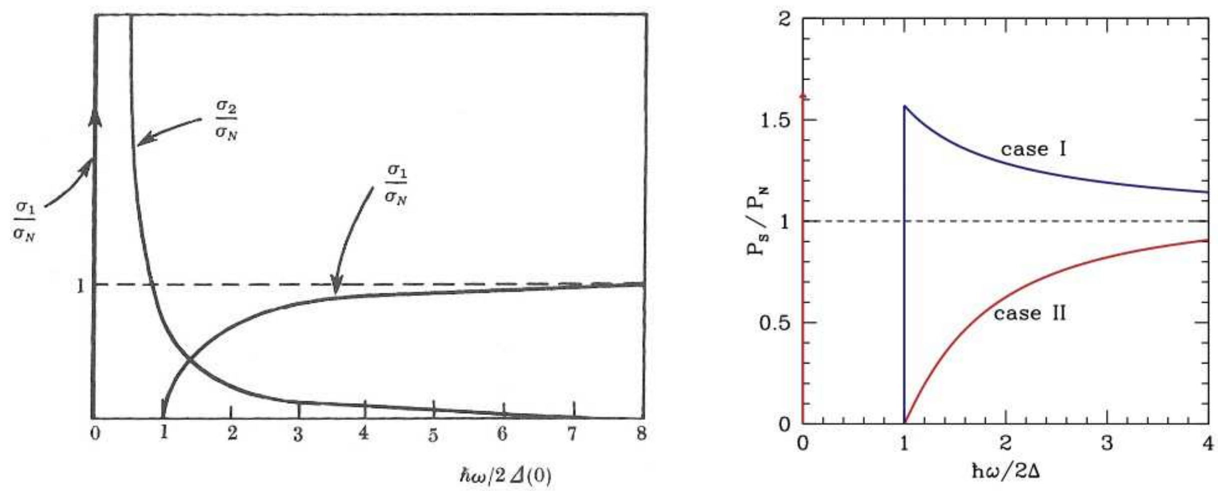

Figure 13.15: Left: real (σ_1) and imaginary (σ_2) parts of the conductivity of a superconductor, normalized by the metallic value of σ_1 just above T_c. From J. R. Schrieffer, *Theory of Superconductivity*. Right: ratio of $\mathcal{P}_s(\omega)/\mathcal{P}_N(\omega)$ for case I (blue) and case II (red) probes.

hence we have $\eta = -1$, *i.e.* case II. Let's set $T = 0$, so $f = f' = 0$. We see from Eqn. 13.212 that $\mathcal{P}(\omega) = 0$ for $\omega < 2\Delta$. We then have

$$\mathcal{P}(\omega) = \tfrac{1}{2}\, g^2(\mu) \int\limits_{\Delta}^{\hbar\omega-\Delta} dE\, \frac{E(\hbar\omega - E) - \Delta^2}{\sqrt{(E^2 - \Delta^2)\,\big((\hbar\omega - E)^2 - \Delta^2\big)}} \quad . \tag{13.216}$$

If we set $\Delta = 0$, we obtain $\mathcal{P}_N(\omega) = \frac{1}{2}\omega^2$. The ratio between superconducting and normal values is

$$\frac{\sigma_{1,S}(\omega)}{\sigma_{1,N}(\omega)} = \frac{\mathcal{P}_S(\omega)}{\mathcal{P}_N(\omega)} = \frac{1}{\omega} \int\limits_{\Delta}^{\hbar\omega-\Delta} dE\, \frac{E(\hbar\omega - E) - \Delta^2}{\sqrt{(E^2 - \Delta^2)\,\big((\hbar\omega - E)^2 - \Delta^2\big)}} \quad , \tag{13.217}$$

where $\sigma_1(\omega)$ is the real (dissipative) part of the conductivity. The result can be obtained in closed form in terms of elliptic integrals[15], and is

$$\frac{\sigma_{1,S}(\omega)}{\sigma_{1,N}(\omega)} = \left(1 + \frac{1}{x}\right) \mathbb{E}\!\left(\frac{1-x}{1+x}\right) - \frac{2}{x}\, \mathbb{K}\!\left(\frac{1-x}{1+x}\right) \quad , \tag{13.218}$$

where $x = \hbar\omega/2\Delta$. The imaginary part $\sigma_{2,S}(\omega)$ may then be obtained by Kramers-Kronig transform, and is

$$\frac{\sigma_{2,S}(\omega)}{\sigma_{1,N}(\omega)} = \frac{1}{2}\left(1 + \frac{1}{x}\right) \mathbb{E}\!\left(\frac{2\sqrt{x}}{1+x}\right) - \frac{1}{2}\left(1 - \frac{1}{x}\right) \mathbb{K}\!\left(\frac{2\sqrt{x}}{1+x}\right) \quad . \tag{13.219}$$

[15] See D. C. Mattis and J. Bardeen, *Phys. Rev.* **111**, 412 (1958).

The conductivity sum rule,

$$\int_0^\infty d\omega\, \sigma_1(\omega) = \frac{\pi n e^2}{2m} \quad , \tag{13.220}$$

is satisfied in translation-invariant systems[16]. In a superconductor, when the gap opens, the spectral weight in the region $\omega \in (0, 2\Delta)$ for case I probes shifts to the $\omega > 2\Delta$ region. One finds $\lim_{\omega\to 2\Delta^+} \mathcal{P}_{\mathsf{S}}(\omega)/\mathcal{P}_{\mathsf{N}}(\omega) = \frac{1}{2}\pi$. Case II probes, however, lose spectral weight in the $\omega > 2\Delta$ region in addition to developing a spectral gap. The missing spectral weight emerges as a delta function peak at zero frequency. The London equation $\boldsymbol{j} = -(c/4\pi\lambda_{\mathrm{L}})\, \boldsymbol{A}$ gives

$$-i\omega\, \sigma(\omega)\, \boldsymbol{E}(\omega) = -i\omega\, \boldsymbol{j}(\omega) = -\frac{c^2}{4\pi\lambda_{\mathrm{L}}^2}\, \boldsymbol{E}(\omega) \quad , \tag{13.221}$$

which says

$$\sigma(\omega) = \frac{c^2}{4\pi\lambda_{\mathrm{L}}^2}\, \frac{i}{\omega} + Q\, \delta(\omega) \quad , \tag{13.222}$$

where Q is as yet unknown[17]. We can determine the value of Q via Kramers-Kronig, *viz.*

$$\sigma_2(\omega) = -\,\mathsf{P}\!\int_{-\infty}^{\infty} \frac{d\nu}{\pi}\, \frac{\sigma_1(\nu)}{\nu - \omega} \quad , \tag{13.223}$$

where P denotes principal part. Thus,

$$\frac{c^2}{4\pi\lambda_{\mathrm{L}}^2\omega} = -Q \int_{-\infty}^{\infty} \frac{d\nu}{\pi}\, \frac{\delta(\nu)}{\nu - \omega} = \frac{Q}{\pi} \qquad \Rightarrow \qquad Q = \frac{c^2}{4\lambda_{\mathrm{L}}} \quad . \tag{13.224}$$

Thus, the full London $\sigma(\omega) = \sigma_1(\omega) + i\sigma_2(\omega)$ may be written as

$$\sigma(\omega) = \lim_{\epsilon\to 0^+} \frac{c^2}{4\lambda_{\mathrm{L}}}\, \frac{1}{\epsilon - i\pi\omega} = \frac{c^2}{4\lambda_{\mathrm{L}}} \left\{ \delta(\omega) + \frac{i}{\pi\omega} \right\} \quad . \tag{13.225}$$

Note that the London form for $\sigma_1(\omega)$ includes only the delta-function and none of the structure due to thermally excited quasiparticles ($\omega < 2\Delta$) or pair-breaking ($\omega > 2\Delta$). *Nota bene: while the real part of the conductivity $\sigma_1(\omega)$ includes a $\delta(\omega)$ piece which is finite below 2Δ, because it lies at zero frequency, it does not result in any energy dissipation.* It is also important to note that the electrodynamic response in London theory is purely local. The actual electromagnetic response kernel $K_{\mu\nu}(\boldsymbol{q}, \omega)$ computed using BCS theory is $\boldsymbol{q}$-dependent, even at $\omega = 0$. This says that a magnetic field $\boldsymbol{B}(\boldsymbol{x})$ will induce screening currents at positions $\boldsymbol{x}'$ which are not too

[16] Neglecting interband transitions, the conductivity sum rule is satisfied under replacement of the electron mass m by the band mass m^*.

[17] Note that $\omega\, \delta(\omega) = 0$ when multiplied by any nonsingular function in an integrand.

distant from $\boldsymbol{x}$. The relevant length scale here turns out to be the *coherence length* $\xi_0 = \hbar v_{\rm F}/\pi\Delta_0$ (at zero temperature).

At finite temperature, $\sigma_1(\omega, T)$ exhibits a Hebel-Slichter peak, also known as the *coherence peak*. Examples from two presumably non-s-wave superconductors are shown in Fig. 13.16.

Impurities and translational invariance

Observant students may notice that our derivation of $\sigma(\omega)$ makes no sense. The reason is that $B(\boldsymbol{k}\sigma \,|\, \boldsymbol{k}'\sigma') \propto (\boldsymbol{k} + \boldsymbol{k}') \cdot \boldsymbol{A}_{\boldsymbol{k}-\boldsymbol{k}'}$, which is not of the form $B_{\sigma\sigma'}(\xi_{\boldsymbol{k}}, \xi_{\boldsymbol{k}'})$. For an electromagnetic field of frequency ω, the wavevector $q = \omega/c$ may be taken to be $q \to 0$, since the wavelength of light in the relevant range (optical frequencies and below) is enormous on the scale of the Fermi wavelength of electrons in the metallic phase. We then have that $\boldsymbol{k} = \boldsymbol{k}' + \boldsymbol{q}$, in which case the coherence factor $u_{\boldsymbol{k}} v_{\boldsymbol{k}'} - v_{\boldsymbol{k}} u_{\boldsymbol{k}'}$ vanishes as $\boldsymbol{q} \to 0$ and $\sigma_1(\omega)$ vanishes as well! This is because in the absence of translational symmetry breaking due to impurities, the current operator $\boldsymbol{j}$ commutes with the Hamiltonian, hence matrix elements of the perturbation $\boldsymbol{j} \cdot \boldsymbol{A}$ cannot cause any electronic transitions, and therefore there can be no dissipation. But this is not quite right, because the crystalline potential itself breaks translational invariance. What is true is this: *with no disorder, the dissipative conductivity $\sigma_1(\omega)$ vanishes on frequency scales below those corresponding to interband transitions*. Of course, this is also true in the metallic phase as well.

As shown by Mattis and Bardeen, if we relax the condition of momentum conservation, which is appropriate in the presence of impurities which break translational invariance, then we basically arrive back at the condition $B(\boldsymbol{k}\sigma \,|\, \boldsymbol{k}'\sigma') \approx B_{\sigma\sigma'}(\xi_{\boldsymbol{k}}, \xi_{\boldsymbol{k}'})$. One might well wonder whether we should be classifying perturbation operators by the η parity in the presence of impurities, but provided $\Delta\tau \ll \hbar$, the Mattis-Bardeen result, which we have derived above, is correct.

13.7 Electromagnetic Response of Superconductors

Here we follow chapter 8 of Schrieffer, *Theory of Superconductivity*. In chapter 2 the lecture notes, we derived the linear response result,

$$\big\langle j_\mu(\boldsymbol{x},t)\big\rangle = -\frac{c}{4\pi}\int\! d^3x' \!\int\! dt'\; K_{\mu\nu}(\boldsymbol{x},t \;\; \boldsymbol{x}',t')\, A^\nu(\boldsymbol{x}',t') \quad , \tag{13.226}$$

where $\boldsymbol{j}(\boldsymbol{x},t)$ is the electrical current density, which is a sum of paramagnetic and diamagnetic contributions, *viz.*

$$\begin{aligned}\big\langle j^{\rm p}_\mu(\boldsymbol{x},t)\big\rangle &= \frac{i}{\hbar c}\int\! d^3x' \!\int\! dt'\; \big\langle \big[j^{\rm p}_\mu(\boldsymbol{x},t), j^{\rm p}_\nu(\boldsymbol{x}',t')\big]\big\rangle\, \Theta(t-t')\, A^\nu(\boldsymbol{x}',t') \\ \big\langle j^{\rm d}_\mu(\boldsymbol{x},t)\big\rangle &= -\frac{e}{mc^2}\,\big\langle j^{\rm p}_0(\boldsymbol{x},t)\big\rangle\, A^\mu(\boldsymbol{x},t)\,(1-\delta_{\mu 0}) \quad ,\end{aligned} \tag{13.227}$$

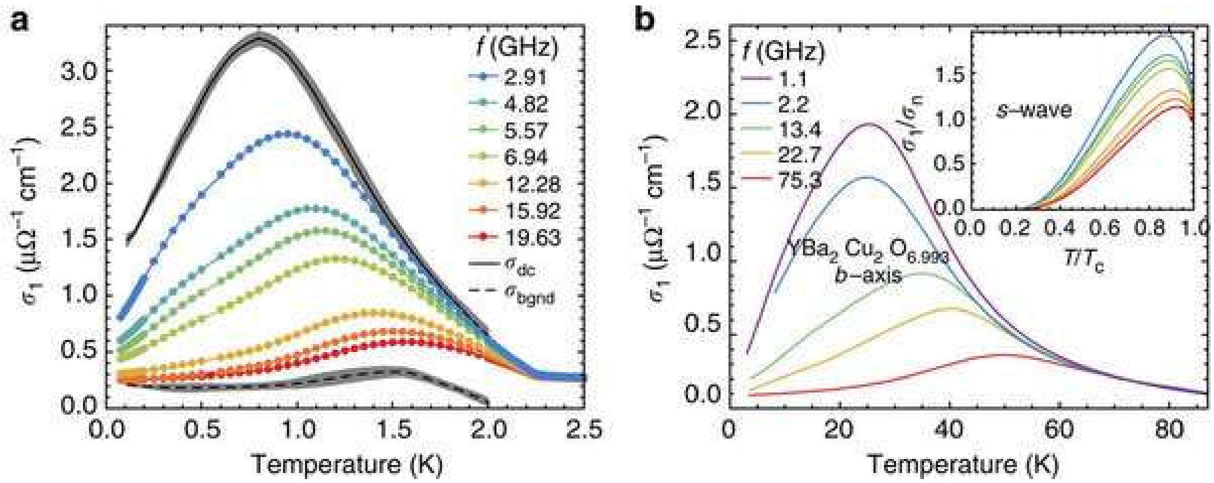

Figure 13.16: Real part of the conductivity $\sigma_1(\omega, T)$ in $\mathrm{CeCoIn_5}$ (left; $T_{\mathrm{c}} = 2.25\,\mathrm{K}$) and in $\mathrm{YBa_2Cu_3O_{6.993}}$ (right; $T_{\mathrm{c}} = 89\,\mathrm{K}$), each showing a coherence peak *versus* temperature over a range of low frequencies. Inset at right shows predictions for s-wave BCS superconductors. Both these materials are believed to involve a more exotic pairing structure. From C. J. S. Truncik *et al.*, *Nature Comm.* **4**, 2477 (2013).

with $j^{\mathrm{p}}_0(\boldsymbol{x}) = ce\, n(\boldsymbol{x})$. We then conclude[18]

$$\begin{aligned} K_{\mu\nu}(\boldsymbol{x}t; \boldsymbol{x}'t') = \frac{4\pi}{i\hbar c^2} \Big\langle \big[j^{\mathrm{p}}_\mu(\boldsymbol{x}, t), j^{\mathrm{p}}_\nu(\boldsymbol{x}', t') \big] \Big\rangle \, \Theta(t - t') \\ + \frac{4\pi e}{mc^3} \big\langle j^{\mathrm{p}}_0(\boldsymbol{x}, t) \big\rangle \, \delta(\boldsymbol{x} - \boldsymbol{x}') \, \delta(t - t') \, \delta_{\mu\nu} \, (1 - \delta_{\mu 0}) \quad . \end{aligned} \tag{13.228}$$

In Fourier space, we may write

$$K_{\mu\nu}(\boldsymbol{q}, t) = \overbrace{\frac{4\pi}{i\hbar c^2} \Big\langle \big[j^{\mathrm{p}}_\mu(\boldsymbol{q}, t), j^{\mathrm{p}}_\nu(-\boldsymbol{q}, 0) \big] \Big\rangle \, \Theta(t)}^{K^{\mathrm{p}}_{\mu\nu}(\boldsymbol{q},t)} + \overbrace{\frac{4\pi n e^2}{mc^2} \, \delta(t) \, \delta_{\mu\nu} \, (1 - \delta_{\mu 0})}^{K^{\mathrm{d}}_{\mu\nu}(\boldsymbol{q},t)} \quad , \tag{13.229}$$

where the paramagnetic current operator is

$$\boldsymbol{j}^{\mathrm{p}}(\boldsymbol{q}) = -\frac{e\hbar}{m} \sum_{\boldsymbol{k},\sigma} \big(\boldsymbol{k} + \tfrac{1}{2}\boldsymbol{q} \big) \, c^\dagger_{\boldsymbol{k}\sigma} \, c_{\boldsymbol{k}+\boldsymbol{q}\,\sigma} \quad . \tag{13.230}$$

The calculation of the electromagnetic response kernel $K_{\mu\nu}(\boldsymbol{q}, \omega)$ is tedious, but it yields all we need to know about the electromagnetic response of superconductors. For example, if we work in a gauge where $A^0 = 0$, we have $\boldsymbol{E}(\omega) = i\omega \boldsymbol{A}(\omega)/c$ and hence the conductivity tensor is

$$\sigma_{ij}(\boldsymbol{q}, \omega) = \frac{ic^2}{4\pi\omega} \, K_{ij}(\boldsymbol{q}, \omega) \quad , \tag{13.231}$$

[18]We use a Minkowski metric $g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(-,+,+,+)$ to raise and lower indices.

where i and j are spatial indices. Using the results of §13.6, the diamagnetic response kernel at $\omega = 0$ is

$$K^{\rm p}_{ij}(\boldsymbol{q},\omega=0) = -\frac{8\pi\hbar e^2}{mc^2}\int\!\frac{d^3\!k}{(2\pi)^3}\,\big(k_i+\tfrac{1}{2}q_i\big)\big(k_j+\tfrac{1}{2}q_j\big)\,L(\boldsymbol{k},\boldsymbol{q}) \quad , \tag{13.232}$$

where

$$\begin{aligned} L(\boldsymbol{k},\boldsymbol{q}) = \left(\frac{E_{\boldsymbol{k}}E_{\boldsymbol{k}+\boldsymbol{q}} - \xi_{\boldsymbol{k}}\xi_{\boldsymbol{k}+\boldsymbol{q}} - \Delta_{\boldsymbol{k}}\Delta_{\boldsymbol{k}+\boldsymbol{q}}}{2E_{\boldsymbol{k}}E_{\boldsymbol{k}+\boldsymbol{q}}}\right)\left(\frac{1-f(E_{\boldsymbol{k}})-f(E_{\boldsymbol{k}+\boldsymbol{q}})}{E_{\boldsymbol{k}}+E_{\boldsymbol{k}+\boldsymbol{q}}+i\epsilon}\right)& \\ + \left(\frac{E_{\boldsymbol{k}}E_{\boldsymbol{k}+\boldsymbol{q}} + \xi_{\boldsymbol{k}}\xi_{\boldsymbol{k}+\boldsymbol{q}} + \Delta_{\boldsymbol{k}}\Delta_{\boldsymbol{k}+\boldsymbol{q}}}{2E_{\boldsymbol{k}}E_{\boldsymbol{k}+\boldsymbol{q}}}\right)\left(\frac{f(E_{\boldsymbol{k}+\boldsymbol{q}})-f(E_{\boldsymbol{k}})}{E_{\boldsymbol{k}}-E_{\boldsymbol{k}+\boldsymbol{q}}+i\epsilon}\right)& \quad . \end{aligned} \tag{13.233}$$

At $T=0$, we have $f(E_{\boldsymbol{k}}) = f(E_{\boldsymbol{k}+\boldsymbol{q}}) = 0$, and only the first term contributes. As $\boldsymbol{q}\to 0$, we have $L(\boldsymbol{k},\boldsymbol{q}\to 0) = 0$ because the coherence factor vanishes while the energy denominator remains finite. Thus, only the diamagnetic response remains, and at $T=0$ we therefore have

$$\lim_{\boldsymbol{q}\to 0} K_{ij}(\boldsymbol{q},0) = \frac{\delta_{ij}}{\lambda^2_{\rm L}} \quad . \tag{13.234}$$

This should be purely transverse, but it is not – a defect of our mean field calculation. This can be repaired, but for our purposes it suffices to take the transverse part, *i.e.*

$$\lim_{\boldsymbol{q}\to 0} K_{ij}(\boldsymbol{q},0) = \frac{\delta_{ij} - \hat{q}_i\,\hat{q}_j}{\lambda^2_{\rm L}} \quad . \tag{13.235}$$

Thus, as long as $\lambda_{\rm L}$ is finite, the $\omega\to 0$ conductivity diverges.

At finite temperature, we have

$$\lim_{\boldsymbol{q}\to 0} L(\boldsymbol{k},\boldsymbol{q}) = -\frac{\partial f}{\partial E}\bigg|_{E=E_{\boldsymbol{k}}} = \frac{1}{k_{\rm B}T}\,f(E_{\boldsymbol{k}})\big[1-f(E_{\boldsymbol{k}})\big] \quad , \tag{13.236}$$

hence

$$\begin{aligned} \lim_{\boldsymbol{q}\to 0} K^{\rm p}_{ij}(\boldsymbol{q},\omega=0) &= -\frac{8\pi\hbar e^2}{mc^2 k_{\rm B}T}\int\!\frac{d^3\!k}{(2\pi)^3}\,k_i\,k_j\,\frac{e^{E_{\boldsymbol{k}}/k_{\rm B}T}}{\big(e^{E_{\boldsymbol{k}}/k_{\rm B}T}+1\big)^2} \\ &= -\frac{4\pi n e^2}{mc^2\hbar}\left[1-\frac{n_{\rm s}(T)}{n}\right]\delta_{ij} \quad , \end{aligned} \tag{13.237}$$

where $n = k^3_{\rm F}/3\pi^2$ is the total electron number density, and

$$\frac{n_{\rm s}(T)}{n} = 1 - \frac{\hbar^2\beta}{mk^3_{\rm F}}\int\limits_0^\infty\! dk\,k^4\,\frac{e^{\beta E_{\boldsymbol{k}}}}{\big(e^{\beta E_{\boldsymbol{k}}}+1\big)^2} \equiv 1 - \frac{n_{\rm n}(t)}{n} \quad , \tag{13.238}$$

where

$$n_{\rm n}(T) = \frac{\hbar^2}{3\pi^2 m} \int\limits_0^\infty\! dk\, k^4 \left(-\frac{\partial f}{\partial E}\right)_{E=E_{\boldsymbol{k}}} \tag{13.239}$$

is the normal fluid density. Expanding about $k = k_{\rm F}$, where $-\frac{\partial f}{\partial E}$ is sharply peaked at low temperatures, we find

$$\begin{aligned} n_{\rm n}(T) &= \frac{\hbar^2}{3m} \cdot 2 \int \frac{d^3\!k}{(2\pi)^3}\, k^2 \left(-\frac{\partial f}{\partial E}\right) \\ &= \frac{\hbar^2 k_{\rm F}^2}{3m}\, g(\varepsilon_{\rm F}) \cdot 2 \int\limits_0^\infty\! d\xi \left(-\frac{\partial f}{\partial E}\right) = 2n \int\limits_0^\infty\! d\xi \left(-\frac{\partial f}{\partial E}\right) \quad , \end{aligned} \tag{13.240}$$

which agrees precisely with what we found in Eqn. 3.136. Note that when the gap vanishes at $T_{\rm c}$, the integral yields $\frac{1}{2}$, and thus $n_{\rm n}(T_{\rm c}) = n$, as expected.

There is a slick argument, due to Landau, which yields this result. Suppose a superflow is established at some velocity $\boldsymbol{v}$. In steady state, any normal current will be damped out, and the electrical current will be $\boldsymbol{j} = -en_{\rm s}\boldsymbol{v}$. Now hop on a frame moving with the supercurrent. The superflow in the moving frame is stationary, so the current is due to normal electrons (quasiparticles), and $\boldsymbol{j}' = -en_{\rm n}(-\boldsymbol{v}) = +en_{\rm n}\boldsymbol{v}$. That is, the normal particles which were at rest in the lab frame move with velocity $-\boldsymbol{v}$ in the frame of the superflow, which we denote with a prime. The quasiparticle distribution in this primed frame is

$$f'_{\boldsymbol{k}\sigma} = \frac{1}{e^{\beta(E_{\boldsymbol{k}} + \hbar\boldsymbol{v}\cdot\boldsymbol{k})} + 1} \quad , \tag{13.241}$$

since, for a Galilean-invariant system, which we are assuming, the energy is

$$\begin{aligned} E' &= E + \boldsymbol{v}\cdot\boldsymbol{P} + \tfrac{1}{2} M\boldsymbol{v}^2 \\ &= \sum_{\boldsymbol{k},\sigma} \big(E_{\boldsymbol{k}} + \hbar\boldsymbol{k}\cdot\boldsymbol{v}\big)\, n_{\boldsymbol{k}\sigma} + \tfrac{1}{2} M\boldsymbol{v}^2 \quad . \end{aligned} \tag{13.242}$$

Expanding now in $\boldsymbol{v}$,

$$\begin{aligned} \boldsymbol{j}' &= -\frac{e\hbar}{mV} \sum_{\boldsymbol{k},\sigma} f'_{\boldsymbol{k}\sigma}\, \boldsymbol{k} = -\frac{e\hbar}{mV} \sum_{\boldsymbol{k},\sigma} \boldsymbol{k} \left\{ f(E_{\boldsymbol{k}}) + \hbar\boldsymbol{k}\cdot\boldsymbol{v}\, \frac{\partial f(E)}{\partial E}\bigg|_{E=E_{\boldsymbol{k}}} + \ldots \right\} \\ &= \frac{2\hbar^2 e\boldsymbol{v}}{3m} \int \frac{d^3\!k}{(2\pi)^3}\, k^2 \left(-\frac{\partial f}{\partial E}\right)_{E=E_{\boldsymbol{k}}} = \frac{\hbar^2 e\boldsymbol{v}}{3\pi^2 m} \int\limits_0^\infty\! dk\, k^4 \left(-\frac{\partial f}{\partial E}\right)_{E=E_{\boldsymbol{k}}} = en_{\rm n}\boldsymbol{v} \quad , \end{aligned} \tag{13.243}$$

yielding the exact same expression for $n_{\rm n}(T)$. So we conclude that $\lambda_{\rm L}^2 = mc^2/4\pi n_{\rm s}(T)e^2$, with $n_{\rm s}(T=0) = n$ and $n_{\rm s}(T \geq T_{\rm c}) = 0$. The difference $n_{\rm s}(0) - n_{\rm s}(T)$ is exponentially small in $\Delta_0/k_{\rm B}T$ for small T.

Microwave absorption measurements usually focus on the quantity $\lambda_{\rm L}(T) - \lambda_{\rm L}(0)$. A piece of superconductor effectively changes the volume – and hence the resonant frequency – of the cavity in which it is placed. Measuring the cavity resonance frequency shift $\Delta\omega_{\rm res}$ as a function of temperature allows for a determination of the difference $\Delta\lambda_{\rm L}(T) \propto \Delta\omega_{\rm res}(T)$.

Note that anything but an exponential dependence of $\Delta \ln \lambda_{\rm L}$ on $1/T$ indicates that there are low-lying quasiparticle excitations. The superconducting density of states is then replaced by

$$g_{\rm s}(E) = g_{\rm n} \int \frac{d\hat{\boldsymbol{k}}}{4\pi} \, \frac{E}{\sqrt{E^2 - \Delta^2(\hat{\boldsymbol{k}})}} \, \Theta\big(E^2 - \Delta^2(\hat{\boldsymbol{k}})\big) \quad , \tag{13.244}$$

where the gap $\Delta(\hat{\boldsymbol{k}})$ depends on direction in $\boldsymbol{k}$-space. If $g(E) \propto E^\alpha$ as $E \to 0$, then

$$n_{\rm n}(T) \propto \int\limits_0^\infty \! dE \; g_{\rm s}(E) \left(- \frac{\partial f}{\partial E} \right) \propto T^\alpha \quad , \tag{13.245}$$

in contrast to the exponential $\exp(-\Delta_0/k_{\rm B}T)$ dependence for the s-wave (full gap) case. For example, if

$$\Delta(\hat{\boldsymbol{k}}) = \Delta_0 \, \sin^n\!\theta \; e^{in\varphi} \propto \Delta_0 \, Y_{nn}(\theta,\varphi) \quad , \tag{13.246}$$

then we find $g_{\rm s}(E) \propto E^{2/n}$. For $n=2$ we would then predict a linear dependence of $\Delta \ln \lambda_{\rm L}(T)$ on T at low temperatures. Of course it is also possible to have *line nodes* of the gap function, *e.g.* $\Delta(\hat{\boldsymbol{k}}) = \Delta_0 \, (3\cos^2\theta - 1) \propto \Delta_0 \, Y_{20}(\theta,\varphi)$.

EXERCISE: Compute the energy dependence of $g_{\rm s}(E)$ when the gap function has line nodes.