8. (a) The regions with $x < a$ and $x > a$ do not contribute to the normalization. The normalization integral is

$$\int |\psi(x)|^2 \, dx = \int_{-a}^{+a} b^2 (a^2 - x^2)^2 \, dx = b^2 \int_{-a}^{+a} (a^4 - 2a^2 x^2 + x^4) \, dx = b^2 \left(a^4 x - 2a^2 \frac{x^3}{3} + \frac{x^5}{5} \right) \Bigg|_{-a/2}^{+a/2}$$

Evaluating the integral and setting it equal to 1, we find

$$b^2 \left(2a^5 - \frac{4a^5}{3} + \frac{2a^5}{5} \right) = 1 \qquad \text{or} \qquad b = \sqrt{\frac{15}{16a^5}}$$

(b) $P(x) \, dx = |\psi(x)|^2 \, dx = b^2 (a^2 - x^2)^2 \, dx$, and with $x = +a/2$ and $dx = 0.010a$ we obtain

$$P(x) \, dx = \frac{15}{16a^5} \left(\frac{a^2}{4} - a^2 \right)^2 (0.010a) = 0.0053$$

(c) $P(a/2 : a) = \int_{a/2}^{a} |\psi(x)|^2 \, dx = \int_{a/2}^{a} b^2 (a^2 - x^2)^2 \, dx = \frac{15}{16a^5} \left(a^4 x - 2a^2 \frac{x^3}{3} + \frac{x^5}{5} \right) \Bigg|_{a/2}^{a}$$

$$= \frac{15}{16a^5} \left[a^4 \left(a - \frac{a}{2} \right) - \frac{2a^2}{3} \left(a^3 - \frac{a^3}{8} \right) + \frac{1}{5} \left(a^5 - \frac{a^5}{32} \right) \right] = 0.104$$

9. With $\psi(x) = Cxe^{-bx}$, we have $d\psi/dx = Ce^{-bx} - bCxe^{-bx}$ and

$$\frac{d^2\psi}{dx^2} = -2bCe^{-bx} + b^2 Cxe^{-bx}$$

We now substitute $\psi(x)$ and $d^2\psi/dx^2$ into the Schrödinger equation:

$$-\frac{\hbar^2}{2m}(-2bCe^{-bx} + b^2 Cxe^{-bx}) + U(x)Cxe^{-bx} = ECxe^{-bx}$$

Canceling the common factor of Ce^{-bx} and solving for E,

$$E = \frac{\hbar^2 b}{mx} - \frac{\hbar^2 b^2}{2m} + U(x)$$

The energy E will be a constant only if the two terms that depend on x cancel each other:

$$\frac{\hbar^2 b}{mx} + U(x) = 0 \qquad \text{or} \qquad U(x) = -\frac{\hbar^2 b}{mx}$$

The cancellation of the two terms depending on x leaves only the remaining term for the energy:

$$E = -\frac{\hbar^2 b^2}{2m}$$

11. (a) The normalization integral is

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = \int_{-\infty}^{+\infty} A^2 x^2 e^{-2bx} dx = 2A^2 \int_0^{+\infty} x^2 e^{-2bx} dx = 2A^2 \frac{2}{(2b)^3} = 1$$

so $A = \sqrt{2b^3}$.

(b) The wave function can be written as $\psi(x) = Ae^{bx}$ for $x < 0$ and $\psi(x) = Ae^{-bx}$ for $x > 0$.

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = \int_{-\infty}^{0} A^2 e^{2bx} dx + \int_0^{+\infty} A^2 e^{-2bx} dx = \frac{2A^2}{2b}$$

and $A = \sqrt{b}$.

12. For the $x > 0$ wave function, we have

$$\frac{d\psi}{dx} = -Abe^{-bx} \qquad \text{and} \qquad \frac{d^2\psi}{dx^2} = Ab^2 e^{-bx}$$

Substituting into the Schrödinger equation then gives

$$-\frac{\hbar^2}{2m} Ab^2 e^{-bx} + UAe^{-bx} = EAe^{-bx} \qquad \text{or} \qquad -\frac{\hbar^2 b^2}{2m} + U = E$$

This is consistent with a constant value of U, which we can take to be 0, giving $E = -\hbar^2 b^2 / 2m$. Repeating the calculation for the $x < 0$ wave function gives an identical result.

So the potential energy is a constant (zero) for $x < 0$ and for $x > 0$. What happens at $x = 0$? Note that the wave function is continuous at $x = 0$ (both give $\psi = A$ at $x = 0$) but that the derivative $d\psi/dx$ is not continuous. This suggests an infinite discontinuity in $U(x)$ at $x = 0$, and because the wave functions approach 0 as $x \to \infty$ there must be a negative potential energy that produces the bound states. So the potential energy is

$$U(x) = 0 \qquad x < 0, \ x > 0$$
$$U(x) = -\infty \qquad x = 0$$

This type of function is known as a delta function.

13. (a) $\quad E_1 = \dfrac{h^2}{8mL^2} = \dfrac{(hc)^2}{8(mc^2)L^2} = \dfrac{(1240 \text{ eV} \cdot \text{nm})^2}{8(511,000 \text{ eV})(0.285 \text{ nm})^2} = 4.63 \text{ eV}$

$4 \to 1: \quad \Delta E = E_4 - E_1 = 16E_1 - E_1 = 15E_1 = 15(4.63 \text{ eV}) = 69.5 \text{ eV}$

(b) $\quad 4 \to 3: \quad \Delta E = E_4 - E_3 = 16E_1 - 9E_1 = 7E_1 = 7(4.63 \text{ eV}) = 32.4 \text{ eV}$

$\quad 4 \to 2: \quad \Delta E = E_4 - E_2 = 16E_1 - 4E_1 = 12E_1 = 12(4.63 \text{ eV}) = 55.6 \text{ eV}$

$\quad 3 \to 2: \quad \Delta E = E_3 - E_2 = 9E_1 - 4E_1 = 5E_1 = 5(4.63 \text{ eV}) = 23.2 \text{ eV}$

$\quad 3 \to 1: \quad \Delta E = E_3 - E_1 = 9E_1 - E_1 = 8E_1 = 8(4.63 \text{ eV}) = 37.0 \text{ eV}$

$\quad 2 \to 1: \quad \Delta E = E_2 - E_1 = 4E_1 - E_1 = 3E_1 = 3(4.63 \text{ eV}) = 13.9 \text{ eV}$

14. With $E_1 = 1.54 \text{ eV}$ and $E_n = n^2 E_1$ we have

$$\Delta E_3 = E_3 - E_1 = 9E_1 - E_1 = 8E_1 = 8(1.54 \text{ eV}) = 12.3 \text{ eV}$$

$$\Delta E_4 = E_4 - E_1 = 16E_1 - E_1 = 15E_1 = 15(1.54 \text{ eV}) = 23.1 \text{ eV}$$

16. (a) $P(0:L/3) = \int_0^{L/3} |\psi_1(x)|^2\, dx = \int_0^{L/3} \frac{2}{L}\sin^2\frac{\pi x}{L}\, dx = \frac{2}{\pi}\int_0^{\pi/3}\sin^2 u\, du$

$$= \frac{2}{\pi}\left(\frac{u}{4} - \frac{\sin 2u}{4}\right)\Big|_0^{\pi/3} = 0.1955$$

(b) $P(L/3:2L/3) = \int_{L/3}^{2L/3} |\psi_1(x)|^2\, dx = \int_{L/3}^{2L/3} \frac{2}{L}\sin^2\frac{\pi x}{L}\, dx = \frac{2}{\pi}\int_{\pi/3}^{2\pi/3}\sin^2 u\, du$

$$= \frac{2}{\pi}\left(\frac{u}{4} - \frac{\sin 2u}{4}\right)\Big|_{\pi/3}^{2\pi/3} = 0.6090$$

(c) $P(2L/3:L) = \int_{2L/3}^{L} |\psi_1(x)|^2\, dx = \int_{2L/3}^{L} \frac{2}{L}\sin^2\frac{\pi x}{L}\, dx = \frac{2}{\pi}\int_{2\pi/3}^{\pi}\sin^2 u\, du$

$$= \frac{2}{\pi}\left(\frac{u}{4} - \frac{\sin 2u}{4}\right)\Big|_{2\pi/3}^{\pi} = 0.1955$$

17. (a) $P(x)dx = |\psi_3(x)|^2\, dx = \frac{2}{L}\sin^2\frac{3\pi x}{L}\, dx = \frac{2}{0.189\text{ nm}}\sin^2\frac{3\pi(0.188\text{ nm})}{0.189\text{ nm}}0.001\text{ nm} = 2.63\times 10^{-5}$

(b) $P(x)dx = \frac{2}{L}\sin^2\frac{3\pi x}{L}\, dx = \frac{2}{0.189\text{ nm}}\sin^2\frac{3\pi(0.031\text{ nm})}{0.189\text{ nm}}0.001\text{ nm} = 0.0106$

(c) $P(x)dx = \frac{2}{L}\sin^2\frac{3\pi x}{L}\, dx = \frac{2}{0.189\text{ nm}}\sin^2\frac{3\pi(0.079\text{ nm})}{0.189\text{ nm}}0.001\text{ nm} = 5.42\times 10^{-3}$

(d) A classical particle has a uniform probability to be found anywhere within the region, so $P(x)\, dx = (0.001\text{ nm})/(0.189\text{ nm}) = 5.29\times 10^{-3}$.

18. With $E = E_0(n_x^2 + n_y^2)$ the levels above $50E_0$ are as follows:

n_x	n_y	E	n_x	n_y	E
6	4	$52E_0$	6	5	$61E_0$
4	6	$52E_0$	5	6	$61E_0$
7	2	$53E_0$	7	4	$65E_0$
2	7	$53E_0$	4	7	$65E_0$
7	3	$58E_0$	8	1	$65E_0$
3	7	$58E_0$	1	8	$65E_0$

The level at $E = 65E_0$ is 4-fold degenerate.

19. With $E = E_0(n_x^2 + n_y^2/4)$ the levels are as follows:

n_x	n_y	E	n_x	n_y	E
1	1	$1.25E_0$	2	3	$6.25E_0$
1	2	$2.00E_0$	1	5	$7.25E_0$
2	1	$2.25E_0$	2	4	$8.00E_0$
1	3	$3.25E_0$	3	1	$9.25E_0$
2	2	$5.00E_0$	1	6	$10.00E_0$
1	4	$5.00E_0$	3	2	$10.00E_0$

The levels at $E = 5.00E_0$ and $E = 10.00E_0$ are both 2-fold degenerate.

35. (a) $$E = n^2 E_1 = 10^2 \frac{h^2}{8mL^2} = \frac{100(hc)^2}{8mc^2L^2} = \frac{100(1240 \text{ eV} \cdot \text{nm})^2}{8(511,000 \text{ eV})(0.132 \text{ nm})^2} = 2160 \text{ eV}$$

(b) $$\Delta p = \sqrt{p^2} = \sqrt{2mE} = \frac{1}{c}\sqrt{2mc^2E} = \frac{1}{c}\sqrt{2(511,000 \text{ eV})(2160 \text{ eV})} = 4.70 \times 10^4 \text{ eV}/c$$

(c) $$\Delta x \sim \frac{\hbar}{\Delta p} = \frac{1}{2\pi}\frac{hc}{c\Delta p} = \frac{1}{2\pi}\frac{1240 \text{ eV} \cdot \text{nm}}{4.70 \times 10^4 \text{ eV}} = 4.2 \times 10^{-3} \text{ nm}$$

37. $(x^2)_{av} = \int_0^L |\psi(x)|^2 \, x^2 dx = \dfrac{2}{L}\int_0^L x^2 \sin^2 \dfrac{n\pi x}{L} dx = \dfrac{2}{L}\left(\dfrac{L}{n\pi}\right)^3 \int_0^{n\pi} u^2 \sin u \; du$ with $u = \dfrac{n\pi x}{L}$

The integral is a standard form that can be found in integral tables.

$(x^2)_{av} = \dfrac{2L^2}{(n\pi)^3}\left[\dfrac{u^3}{6} - \left(\dfrac{u^2}{4} - \dfrac{1}{8}\right)\sin 2u - \dfrac{u\cos 2u}{4}\right]_0^{n\pi} = \dfrac{2L^2}{(n\pi)^3}\left[\dfrac{(n\pi)^3}{6} - \dfrac{n\pi}{4}\right] = L^2\left(\dfrac{1}{3} - \dfrac{1}{2n^2\pi^2}\right)$

38. With $x_{av} = L/2$ from Example 5.5, we have

$$\Delta x = \sqrt{(x^2)_{av} - (x_{av})^2} = \sqrt{L^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) - \left(\frac{L}{2}\right)^2} = L\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}$$

39. (a) The particle has no preferred direction of motion, so it is equally likely to be moving in the positive and negative x directions. We therefore expect that $p_{av} = 0$.

   (b) Because the potential energy is zero inside the well, the kinetic energy is equal to the total energy:

   $$K = E_n \quad \text{or} \quad \frac{p^2}{2m} = \frac{h^2 n^2}{8mL^2} \quad \text{so} \quad p^2 = \frac{h^2 n^2}{4L^2}$$

   For a given level n, p^2 is constant so $(p^2)_{av}$ has that same value.

   (c) $\Delta p = \sqrt{(p^2)_{av} - (p_{av})^2} = \sqrt{\frac{h^2 n^2}{4L^2} - 0} = \frac{hn}{2L}$