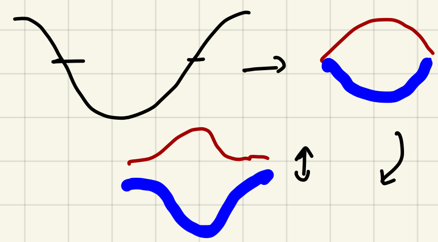
$$E^0_{var}(\zeta) = 4N\hbar\sqrt{\frac{K}{M}} + 2NK\zeta^2 - \frac{N}{4\pi}\int_{-\pi}^{\pi} d\theta\, \sqrt{t_1^2 + t_2^2 + 2t_1 t_2 \cos\theta}$$

where the subscript "var" reminds us this is a variational energy, i.e. $E^0_{var} = \langle \Psi_{var} | H_{SSH} | \Psi_{var} \rangle$. In the limit where $\alpha^2 t \ll K$, we have

$$\frac{E_0^{var}(\zeta)}{N} = 4\hbar\sqrt{\frac{K}{M}} + 2K\zeta^2 - \frac{4t}{\pi} - \frac{8t}{\pi}\alpha^2\zeta^2 \ln\left(\frac{2}{\sqrt{e}\,\alpha\zeta}\right) + \dots$$

and minimizing wrt ζ gives



$$\zeta^* = \frac{2}{\sqrt{e}\,\alpha}\, e^{-\pi K/4\alpha^2 t}$$

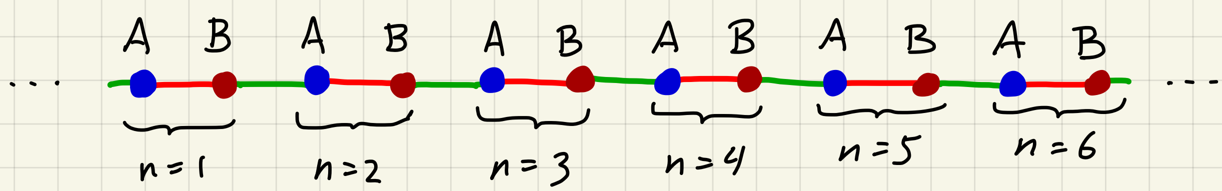
Thus, the system prefers to spontaneously dimerize!

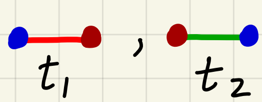
Lecture 3 (Jan. 12) : **Edge states in the SSH model**

The effective Hamiltonian for the fermionic sector of the SSH model is

$$H = -\sum_{n=1}^{N_c} \left(t_1 a_n^\dagger b_n + t_2 b_n^\dagger a_{n+1} + H.c. \right)$$

where $N_c = \frac{1}{2}N$ is the number of unit cells, each of which contains one A site and one B site :



hopping amplitudes : 

$N_c = \frac{1}{2}N = \#\text{cells}$

If we write $H|\psi\rangle = E|\psi\rangle$ with

$$|\psi\rangle = \sum_n \left(A_n a_n^\dagger + B_n b_n^\dagger\right)|0\rangle$$

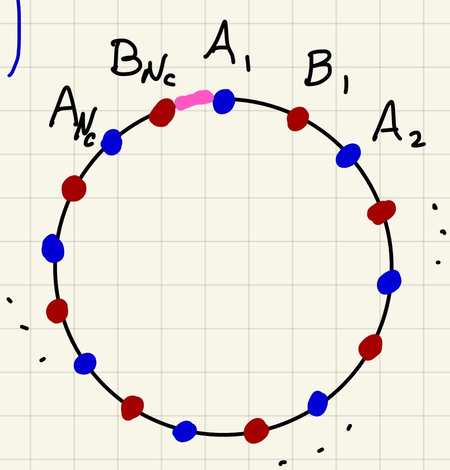
then

$$EA_n = -t_2 B_{n-1} - t_1 B_n$$
$$EB_n = -t_1 A_n - t_2 A_{n+1}$$

On a ring with periodic boundary conditions, $A_{n+N_c} = A_n$ and $B_{n+N_c} = B_n$. Thus,

$$\begin{pmatrix} 0 & t_1 \\ t_2 & E \end{pmatrix}\begin{pmatrix} A_{n+1} \\ B_n \end{pmatrix} = -\begin{pmatrix} E & t_2 \\ t_1 & 0 \end{pmatrix}\begin{pmatrix} A_n \\ B_{n-1} \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} A_{n+1} \\ B_n \end{pmatrix} = \frac{1}{t_1 t_2}\begin{pmatrix} E^2 - t_1^2 & Et_2 \\ -Et_2 & -t_2^2 \end{pmatrix}\begin{pmatrix} A_n \\ B_{n-1} \end{pmatrix}$$



With translational invariance, we have

$$\begin{pmatrix} A_{n+1} \\ B_n \end{pmatrix} = z\begin{pmatrix} A_n \\ B_{n-1} \end{pmatrix}$$

with $z = e^{ik\tilde{a}}$ with $\tilde{a} = 2a$, and so

$$\begin{pmatrix} E^2 - t_1^2 - zt_1t_2 & Et_2 \\ -Et_2 & -t_2^2 - zt_1t_2 \end{pmatrix}\begin{pmatrix} A_n \\ B_{n-1} \end{pmatrix} = 0$$

which requires that the determinant vanish, which says

$$zE^2 = (t_1 + zt_2)(t_2 + zt_1) \Rightarrow E = \pm|t_1 + zt_2|,$$

as we found previously.

Now let's cut the link between B_{N_c} and A_1. We then have

$$\text{①} \quad \begin{pmatrix} A_2 \\ B_1 \end{pmatrix} = \frac{1}{t_1 t_2} \begin{pmatrix} E^2 - t_1^2 & 0 \\ -Et_2 & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ B_{N_c} \end{pmatrix} \equiv L \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

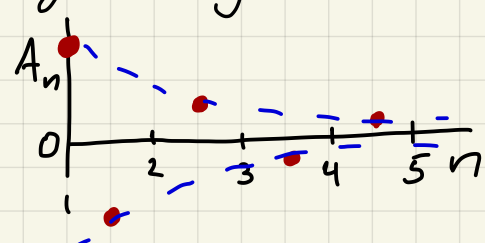
$$\text{②} \quad \begin{pmatrix} A_{N_c} \\ B_{N_c-1} \end{pmatrix} = \frac{1}{t_1 t_2} \begin{pmatrix} 0 & -Et_2 \\ 0 & E^2 - t_1^2 \end{pmatrix} \begin{pmatrix} A_1 \\ B_{N_c} \end{pmatrix} \equiv R \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

For $n \in \{2, \ldots, N_c - 1\}$ we still have

$$\text{③} \quad \begin{pmatrix} A_{n+1} \\ B_n \end{pmatrix} = \frac{1}{t_1 t_2} \begin{pmatrix} E^2 - t_1^2 & Et_2 \\ -Et_2 & -t_2^2 \end{pmatrix} \begin{pmatrix} A_n \\ B_{n-1} \end{pmatrix} \equiv M \begin{pmatrix} A_n \\ B_{n-1} \end{pmatrix}$$

We now show that in the thermodynamic limit that when $|r| < 1$, where $r \equiv t_1/t_2$, there are two $E=0$ edge states. Setting $E=0$, ① says $A_2 = -rA_1$ and $B_1 = 0$. Now iterate ③, which says $A_n = (-r)^{n-1} A_1$ and $B_n = 0$. In the $N_c \to \infty$ limit, the normalized $E=0$ wavefunction localized at the left $(n=1)$ edge is given by

$$A_n = \sqrt{1-r^2}\,(-r)^{n-1} e^{i\alpha}$$
$$B_n = 0$$



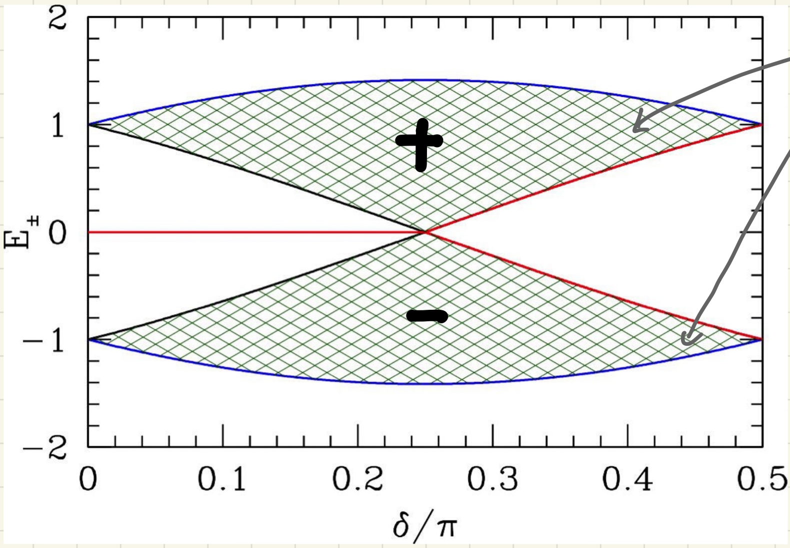
To find the second zero mode, start at the right edge $n = N_c$ with ②, which says $B_{N_c-1} = -rB_{N_c}$ and $A_{N_c} = 0$. Then iterate the inverse of ③, i.e.

$$\begin{pmatrix} A_n \\ B_{n-1} \end{pmatrix} = \frac{1}{t_1 t_2} \begin{pmatrix} -t_2^2 & -Et_2 \\ Et_2 & E^2 - t_1^2 \end{pmatrix} \begin{pmatrix} A_{n+1} \\ B_n \end{pmatrix}$$

to obtain $B_n = (-r)^{N_c - n} B_{N_c}$ and $A_n = 0$, hence

$$A_n = 0$$
$$B_n = \sqrt{1-r^2}\, e^{i\beta} (-r)^{N_c - n}$$

is the normalized wavefunction. For $|r|<1$ there are thus two $E=0$ edge modes. For $|r|>1$ these modes are unnormalizable. The spectrum for $t_1 = t \sin\delta$ and $t_2 = t\cos\delta$ is as follows:



bulk spectrum
$E(k) = \pm \left| t_1 + e^{ik\tilde{a}} t_2 \right|$

NB: If we cut the link between A_1 and B_1, then the $E=0$ edge modes appear for $|r|>1$ instead. For N_c finite,

$$\begin{pmatrix} A_{N_c} \\ B_{N_c - 1} \end{pmatrix} = M^{N_c - 2} \begin{pmatrix} A_2 \\ B_1 \end{pmatrix} = M^{N_c - 2} L \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = R \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \det\left(M^{N_c - 2} L - R\right) = 0$$

- **Topology and SSH**

The cell functions of the SSH model are spinors

$$\vec{u}_\pm(k) = \begin{pmatrix} u_{A\pm}(k) \\ u_{B\pm}(k) \end{pmatrix}$$

which satisfy

$$\underbrace{-\begin{pmatrix} 0 & t(k) \\ t^*(k) & 0 \end{pmatrix}}^{H(k)} \overbrace{\begin{pmatrix} u_{A\pm}(k) \\ u_{B\pm}(k) \end{pmatrix}}^{\vec{u}_\pm(k)} = \overbrace{E_\pm(k)}^{E_\pm(k)} \overbrace{\begin{pmatrix} u_{A\pm}(k) \\ u_{B\pm}(k) \end{pmatrix}}^{\vec{u}_\pm(k)}$$

where $t(k) \equiv t_1 + t_2 e^{-ik\tilde{a}}$ and $E_\pm(k) = \pm|t(k)|$. We define the **polarization** $P_\pm$ of each band as

$$P_\pm = i\int_{-\pi/\tilde{a}}^{\pi/\tilde{a}} \frac{dk}{2\pi} \langle \vec{u}_\pm(k) | \frac{\partial}{\partial k} | \vec{u}_\pm(k) \rangle = \int_{-\pi/\tilde{a}}^{\pi/\tilde{a}} \frac{dk}{2\pi} A_\pm(k)$$

where

$$A_\pm(k) = i \langle \vec{u}_\pm(k) | \frac{\partial}{\partial k} | \vec{u}_\pm(k) \rangle$$

is the Berry connection (or geometric connection). Note that $P_\pm$ is defined only modulo an integer, because under a gauge transformation $|\vec{u}_\pm(k)\rangle \to e^{-i\varphi(k)} |\vec{u}_\pm(k)\rangle$ which is single-valued, we have

$$A_\pm(k) \to A_\pm(k) + \frac{\partial \varphi}{\partial k}$$

$e^{2\pi i P_\pm}$ gauge-invariant

$$P_\pm \to P_\pm + \underbrace{\frac{1}{2\pi}\left[\varphi(\pi/\tilde{a}) - \varphi(-\pi/\tilde{a})\right]}_{\text{must be an integer, } n \in \mathbb{Z}} = \overbrace{P_\pm + n}$$

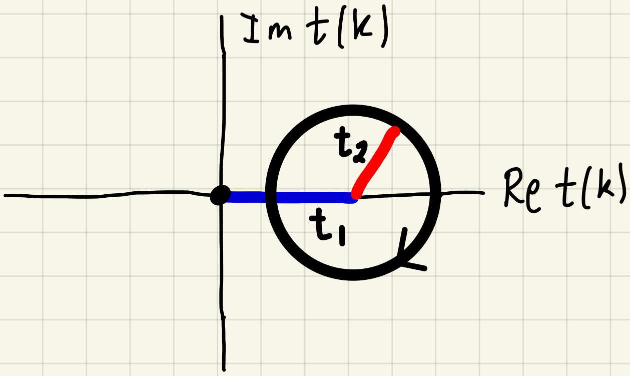
Solving for the cell functions, we find

$$u_{A\pm}(k) = \frac{1}{\sqrt{2}} \quad , \quad u_{B\pm}(k) = \mp \frac{1}{\sqrt{2}} \frac{t^*(k)}{|t(k)|} = \mp \frac{1}{\sqrt{2}} e^{-i\Theta(k)}$$

where $\Theta(k) = \arg t(k)$ with $t(k) = t_1 + e^{-ik\tilde{a}} t_2$. Thus

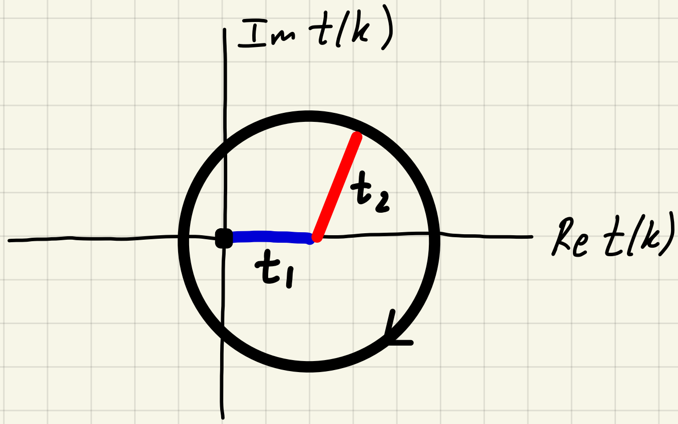
$$P_\pm = \frac{1}{4\pi} \oint d\Theta = \frac{1}{2} W$$

where W is the winding number of $\Theta(k)$ around the Brillouin zone.



$|r| = |t_1/t_2| > 1$

$W = 0$, $\exp(2\pi i P_\pm) = +1$



$|r| = |t_1/t_2| < 1$

$W = -1$, $\exp(2\pi i P_\pm) = -1$

Thus, the topologically trivial phase with no winding and $\exp(2\pi i P_\pm) = +1$ has no $E=0$ edge states, while the topologically nontrivial phase with $\exp(2\pi i P_\pm) = -1$ has an exponentially localized $E=0$ edge state (in the TL) at each of the edges.

- **Dirac equation**

Dirac Hamiltonian (1928): $H = c\vec{\alpha}\cdot\vec{p} + \beta mc^2$

Bohr: "What are you working on, Mr. Dirac?"

Dirac: "I am trying to take the square root of something."

$$\vec{\alpha} = \{\Gamma^1, \Gamma^2, \Gamma^3\} \quad , \quad \beta = \Gamma^4 \qquad \text{(all Hermitian)}$$

all anticommuting, with $(\Gamma^\mu)^2 = 1$. I.e.

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu} \qquad \text{(Clifford algebra)}$$

Possible sol$^{\underline{n}}$: rank-4 matrices

$$\Gamma^1 = X \otimes \mathbb{1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad , \quad \Gamma^2 = Y \otimes \mathbb{1} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\Gamma^3 = Z \otimes X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad , \quad \Gamma^4 = Z \otimes Y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

to which we can add

$$\Gamma^5 \equiv -\Gamma^1\Gamma^2\Gamma^3\Gamma^4 = Z \otimes Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Here $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices. Excercise : show $\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}$.

Now suppose $H = \vec{d} \cdot \vec{\Gamma}$ where $\vec{d} = (d_1, d_2, d_3, d_4, d_5) \in \mathbb{R}^5$. Then

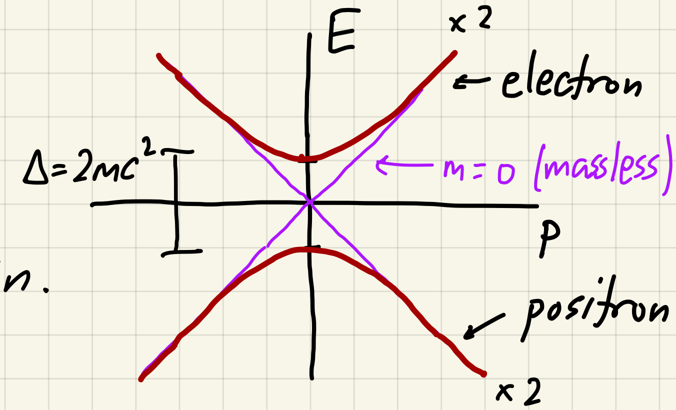
$$H^2 = d_\mu d_\nu \Gamma^\mu \Gamma^\nu = \frac{1}{2} d_\mu d_\nu \{\Gamma^\mu, \Gamma^\nu\}$$
$$= \frac{1}{2} d_\mu d_\nu \times 2\delta^{\mu\nu} = \vec{d}^{\,2} \cdot \mathbb{1}$$

Thus we must have four eigenvalues arranged in two doublets:

$$\lambda_{1,2} = +|\vec{d}| \quad , \quad \lambda_{3,4} = -|\vec{d}|$$

With $\vec{d} = (cp_x, cp_y, cp_z, mc^2, 0)$, we have

$$|\vec{d}| = \sqrt{c^2\vec{p}^{\,2} + m^2c^4}$$



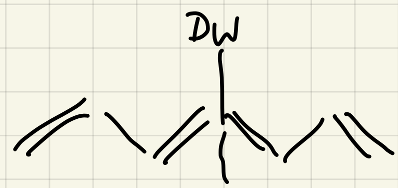
The double degeneracy is due to spin.

Massless $(m=0)$ case : **Dirac cone**

In $d=1$ space dimension, there are two Dirac matrices, X and Z :

$$H = cpX + mc^2 Z = \begin{pmatrix} mc^2 & cp \\ cp & -mc^2 \end{pmatrix}$$

$$\lambda_{1,2} = \pm\sqrt{c^2p^2 + m^2c^4} \equiv E_\pm(p)$$



- Bound states at a domain wall

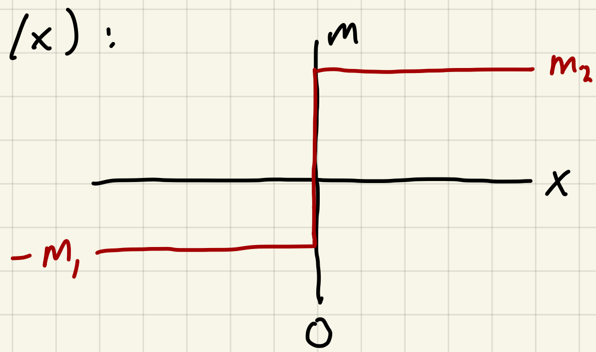
Write

$$H = -i\hbar c\,\partial_x X + m(x)c^2 Z = \begin{pmatrix} m(x)c^2 & -i\hbar c\frac{\partial}{\partial x} \\ -i\hbar c\frac{\partial}{\partial x} & -m(x)c^2 \end{pmatrix}$$

where $[c] = L/T$ but c may not be the speed of light.

Consider a **domain wall** in $m(x)$:

$$m(x) = \begin{cases} -m_1 & \text{if } x<0 \\ +m_2 & \text{if } x>0 \end{cases}$$



Let's solve the Schrödinger equation separately for $x \gtrless 0$, and impose continuity at $x=0$. We assume a bound state for which $\psi(x \to \pm\infty) = 0$.

- $x > 0$: $\vec{\psi}(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = \begin{pmatrix} \psi_1^{(>)} \\ \psi_2^{(>)} \end{pmatrix} e^{-\gamma_> x}$

$$H\vec{\psi}(x) = \left(i\hbar c\gamma_> X + m_2 c^2 Z\right)\vec{\psi}(x) = E\vec{\psi}(x)$$

A nontrivial sol$^{\underline{n}}$ requires

$$\det\left(i\hbar c\gamma_> X + m_2 c^2 Z - E\mathbb{1}\right) = 0$$

$$\Rightarrow \det\begin{pmatrix} m_2 c^2 - E & i\hbar c\gamma_> \\ i\hbar c\gamma_> & -m_2 c^2 - E \end{pmatrix} = E^2 - m_2^2 c^4 + \hbar^2 c^2 \gamma_>^2 = 0$$

Thus, $\gamma_> = \pm\sqrt{m_2^2 c^4 - E^2}/\hbar c$

When $m_2 c^2 < |E|$, we have $\gamma_> \in i\mathbb{R}$, corresponding to a plane wave solution. When $m_2 c^2 > |E|$, we have a normalizable real sol$^{\underline{n}}$, choosing the positive root for $\gamma_>$. The corresponding eigenvector satisfies

$$\frac{\psi_1^{(>)}}{\psi_2^{(>)}} = -\frac{i\hbar c\gamma_>}{m_2 c^2 - E}$$

For $x < 0$, write $\vec{\psi}(x) = \begin{pmatrix} \psi_1^{(<)} \\ \psi_2^{(<)} \end{pmatrix} e^{+\gamma_< x}$

Substituting into the Schrödinger eqn yields

$$\det\begin{pmatrix} -m_1c^2-E & -i\hbar c\gamma_< \\ -i\hbar c\gamma_< & +m_1c^2-E \end{pmatrix} = E^2 - m_1^2c^4 + \hbar^2c^2\gamma_<^2 = 0$$

hence $\gamma_< = \pm\sqrt{m_1^2c^4 - E^2}/\hbar c$. With $m_1c^2 > |E|$, choosing the positive root for $\gamma_<$, we obtain a real, normalizable solution, with

$$\frac{\psi_1^{(<)}}{\psi_2^{(<)}} = -\frac{i\hbar c\gamma_<}{m_1c^2+E}$$

Continuity at $x=0$ then requires

$$\frac{\gamma_<}{m_1c^2+E} = \frac{\gamma_>}{m_2c^2-E} \Rightarrow \sqrt{\frac{m_1c^2+E}{m_1c^2-E}} = \sqrt{\frac{m_2c^2-E}{m_2c^2+E}}$$

When $E=0$ we have a solution! The solution is

$$\vec{\psi}(x) = \sqrt{\frac{c}{\hbar}\,\frac{m_1m_2}{m_1+m_2}}\begin{pmatrix}1\\ i\end{pmatrix} e^{-|m(x)c\,x|/\hbar}$$

Note this is a bound state, exponentially localized about the domain wall at $x=0$.

General $m(x)$: For a general function $m(x)$, we have a zero energy sol$^{\underline{n}}$ $\vec{\psi}(x)$ provided

$$\left(-i\hbar c\, X\,\partial_x + m(x)c^2 Z\right)\psi(x) = 0$$

Thus $\partial_x \vec{\psi}(x) = -\frac{c}{\hbar} m(x) Y \vec{\psi}(x)$. Now in order to have a sol$^{\underline{n}}$, we must have that $\vec{\psi}(x)$ is an eigenstate of Y. We write

$$\vec{\psi}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i\eta \end{pmatrix} f(x)$$

where $\eta = \pm 1$. Then

$$\partial_x f(x) = -\eta \frac{c}{\hbar} m(x) f(x)$$

$$f(x) = \underbrace{A}_{\text{normalization constant}} \exp\left\{ -\eta \frac{c}{\hbar} \int_0^x dx' \, m(x') \right\}$$

In order to have a normalizable sol$^{\underline{n}}$, we must choose

$$\eta = \operatorname{sgn}\left[m(\infty) - m(-\infty) \right]$$

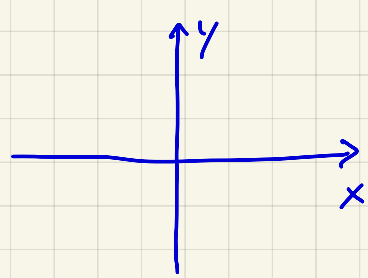
A bound state solution then exists whenever $m(\infty) m(-\infty) < 0$.

- **Helical edge states in $d=2$**

Schrödinger eqn: $\left(-i\hbar c \, \Gamma^1 \partial_x - i\hbar c \, \Gamma^2 \partial_y + m(x) c^2 \Gamma^4 \right) \vec{\psi} = E \vec{\psi}$

(i) $\vec{\psi}(x,y) = A f(x) e^{i k_y y} (\alpha \vec{\xi}_1 + \beta \vec{\xi}_2)$ with $\Gamma^2 \vec{\xi}_{1,2} = \vec{\xi}_{1,2}$ and $E = \hbar c k_y$. Note $\Gamma^2 = Y \otimes 1$ so

$$\vec{\xi}_1 = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \vec{\xi}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ i \end{pmatrix}$$



$m = m(x)$

(ii) $\vec{\Psi}(x,y) = B\,g(x)e^{ik_y y}(\gamma\vec{\xi}_3 + \delta\vec{\Psi}_4)$ with $\Gamma^2\vec{\xi}_{3,4} = -\vec{\xi}_{3,4}$

and $E = -\hbar c k_y$. Note

$$\vec{\xi}_3 = \begin{pmatrix} 1 \\ 0 \\ -i \\ 0 \end{pmatrix} \quad , \quad \vec{\xi}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -i \end{pmatrix}$$

Thus for sol^n (i) with $E = +\hbar c k_y$, we have

$$-i\hbar c\,\partial_x f(x)\,\Gamma^1(\alpha\vec{\xi}_1 + \beta\vec{\xi}_2) + m(x)c^2 f(x)\,\Gamma^4(\alpha\vec{\xi}_1 + \beta\vec{\xi}_2) = 0$$

But

$$\Gamma^1\vec{\xi}_1 = i\vec{\xi}_3 \quad , \quad \Gamma^4\vec{\xi}_1 = i\vec{\xi}_4$$
$$\Gamma^1\vec{\xi}_2 = i\vec{\xi}_4 \quad , \quad \Gamma^4\vec{\xi}_2 = -i\vec{\xi}_3$$
$$\Gamma^1\vec{\xi}_3 = -i\vec{\xi}_1 \quad , \quad \Gamma^4\vec{\xi}_3 = i\vec{\xi}_2$$
$$\Gamma^1\vec{\xi}_4 = -i\vec{\xi}_2 \quad , \quad \Gamma^4\vec{\xi}_4 = -i\vec{\xi}_1$$

and so the equation for sol^n (i) becomes

$$\hbar c\,\partial_x f\,(\alpha\vec{\xi}_3 + \beta\vec{\xi}_4) + m(x)c^2 f\,(-i\beta\vec{\xi}_3 + i\alpha\vec{\xi}_4) = 0$$

which requires

$$\frac{\beta}{\alpha} = -\frac{i\alpha}{i\beta} = -\frac{\alpha}{\beta} \Rightarrow \beta^2 = -\alpha^2$$

So we may take

$$\alpha = 1,\ \beta = i: \qquad \partial_x \ln f = -\frac{c}{\hbar}m(x)$$
$$\alpha = 1,\ \beta = -i: \qquad \partial_x \ln f = +\frac{c}{\hbar}m(x)$$

For sol$^{\underline{n}}$ (ii) with $E = -\hbar c k_y$, we have

$$-i\hbar c\, \partial_x g\, \Gamma^1 (\gamma \vec{\xi}_3 + \delta \vec{\xi}_4) + m(x) c^2 g\, \Gamma^4 (\gamma \vec{\xi}_3 + \delta \vec{\psi}_4) = 0$$

$$\Rightarrow \quad -\hbar c\, \partial_x g (\gamma \vec{\xi}_1 + \delta \vec{\xi}_2) + m(x) c^2 g \,(-i\delta \vec{\xi}_1 + i\gamma \vec{\xi}_2) = 0$$

and we conclude

$$\frac{\delta}{\gamma} = \frac{i\gamma}{-i\delta} = -\frac{\gamma}{\delta} \Rightarrow \delta^2 = -\gamma^2$$

So we may take

$$\gamma = 1, \ \delta = i \ : \quad \partial_x \ln g = +\frac{c}{\hbar} m(x)$$

$$\gamma = 1, \ \delta = -i \ : \quad \partial_x \ln g = -\frac{c}{\hbar} m(x)$$

Thus, the solutions are:

(i) $E = \hbar c k_y$, $\Upsilon = +1$

$$\vec{\psi}_1(x,y) = A e^{ik_y y} e^{-\frac{c}{\hbar}\int^x dx' m(x')} (\vec{\xi}_1 + i\vec{\xi}_2) \leftarrow \begin{pmatrix} 1 \\ i \\ i \\ -1 \end{pmatrix}$$

$$\vec{\psi}_2(x,y) = A e^{ik_y y} e^{+\frac{c}{\hbar}\int^x dx' m(x')} (\vec{\xi}_1 - i\vec{\xi}_2) \leftarrow \begin{pmatrix} 1 \\ -i \\ i \\ 1 \end{pmatrix}$$

(ii) $E = -\hbar c k_y$, $\Upsilon = -1$

$$\vec{\psi}_3(x,y) = B e^{ik_y y} e^{+\frac{c}{\hbar}\int^x dx' m(x')} (\vec{\xi}_3 + i\vec{\xi}_4) \leftarrow \begin{pmatrix} 1 \\ i \\ -i \\ 1 \end{pmatrix}$$

$$\vec{\psi}_4(x,y) = B e^{ik_y y} e^{-\frac{c}{\hbar}\int^x dx' m(x')} (\vec{\xi}_3 - i\vec{\xi}_4) \leftarrow \begin{pmatrix} 1 \\ -i \\ -i \\ -1 \end{pmatrix}$$

Thus, if $m(\infty) > 0 > m(-\infty)$, we have normalizable sol$^{\underline{ns}}$ $\vec{\psi}_1$ and $\vec{\psi}_4$, while if $m(\infty) < 0 < m(-\infty)$, we have normalizable sol$^{\underline{ns}}$ $\vec{\psi}_2$ and $\vec{\psi}_3$. The time-dependence is

(i) $e^{ik_y y} e^{-iEt/\hbar} = e^{ik_y(y-ct)}$: up-mover, $\Upsilon = +1$

(ii) $e^{ik_y y} e^{-iEt/\hbar} = e^{ik_y(y+ct)}$: down-mover, $\Upsilon = -1$

- **Lecture 4 (Jan 14)** : **Adiabatic theorem and Berry's phase**

Consider a Hamiltonian $H(\vec{\lambda})$ dependent on a set of parameters $\vec{\lambda} = \{\lambda_1, \dots, \lambda_K\}$, with eigenfunctions $\{\varphi_n(\vec{\lambda})\}$:

$$H(\vec{\lambda})\,|\varphi_n(\vec{\lambda})\rangle = E_n(\vec{\lambda})\,|\varphi_n(\vec{\lambda})\rangle$$

Now let $\vec{\lambda} = \vec{\lambda}(t)$ be time-dependent. The adiabatic theorem says that if $\vec{\lambda}(t)$ evolves very slowly, such that $\Delta E_n \cdot \tau \gg \hbar$, where τ is the time scale of the variation, i.e. $\tau = |\vec{\lambda}|/|\dot{\vec{\lambda}}|$, and $\Delta E_n = E_{n+1} - E_n$ is the gap between consecutive levels, then the solutions to the time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = H(\vec{\lambda}(t))\,|\Psi(t)\rangle$$

are proportional to the instantaneous adiabatic WFs, with

$$|\Psi_n(t)\rangle = e^{i\gamma_n(t)}\, e^{-i\hbar^{-1}\int^t dt'\, E_n(\vec{\lambda}(t'))}\, |\varphi_n(\vec{\lambda}(t))\rangle$$