PHYSICS 211B : CONDENSED MATTER PHYSICS
HW ASSIGNMENT #5

(1) Show that the Bogoliubov transformations in Eqn. (12.55) of the lecture notes preserve the fermion anticommutation relations.

(2) Show that the BCS ground state wavefunction $|\,\mathrm{G}\,\rangle$ in Eqn. (12.83) of the lecture notes is annihilated by the Bogoliubov annihilation operator $\gamma_{\boldsymbol{k}\sigma}$.

(3) Show that

$$\sum_{\boldsymbol{k},\sigma} \sigma\, c^\dagger_{\boldsymbol{k}\sigma}\, c_{\boldsymbol{k}\sigma} = \sum_{\boldsymbol{k},\sigma} \sigma\, \gamma^\dagger_{\boldsymbol{k}\sigma}\, \gamma_{\boldsymbol{k}\sigma} \quad .$$

(4) A *ferrimagnet* is a magnetic structure in which there are different types of spins present. Consider a sodium chloride structure in which the A sublattice spins have magnitude S_{A} and the B sublattice spins have magnitude S_{B} with $S_{\mathrm{B}} < S_{\mathrm{A}}$ (*e.g.* $S = 1$ for the A sublattice but $S = \frac{1}{2}$ for the B sublattice). The Hamiltonian is

$$\mathcal{H} = J \sum_{\langle ij \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j + g_{\mathrm{A}} \mu_0 H \sum_{i \in \mathrm{A}} S^z_i + g_{\mathrm{B}} \mu_0 H \sum_{j \in \mathrm{B}} S^z_j$$

where $J > 0$, so the interactions are antiferromagnetic.

(a) Work out the mean field theory for this model. Assume that the spins on the A and B sublattices fluctuate about the mean values

$$\langle \boldsymbol{S}_{\mathrm{A}} \rangle = m_{\mathrm{A}}\, \hat{\boldsymbol{z}} \qquad , \qquad \langle \boldsymbol{S}_{\mathrm{B}} \rangle = m_{\mathrm{B}}\, \hat{\boldsymbol{z}}$$

and derive a set of coupled mean field equations of the form

$$\begin{aligned} m_{\mathrm{A}} &= F_{\mathrm{A}}(\beta g_{\mathrm{A}} \mu_0 H + \beta J z m_{\mathrm{B}}) \\ m_{\mathrm{B}} &= F_{\mathrm{B}}(\beta g_{\mathrm{B}} \mu_0 H + \beta J z m_{\mathrm{A}}) \end{aligned}$$

where z is the lattice coordination number ($z = 6$ for NaCl) and $F_{\mathrm{A}}(x)$ and $F_{\mathrm{B}}(x)$ are related to Brillouin functions. Show graphically that a solution exists, and fund the criterion for broken symmetry solutions to exist when $H = 0$, *i.e.* find T_{c}. Then linearize, expanding for small m_{A}, m_{B}, and H, and solve for $m_{\mathrm{A}}(T)$ and $m_{\mathrm{B}}(T)$ and the susceptibility

$$\chi(T) = -\frac{1}{2} \frac{\partial}{\partial H} (g_{\mathrm{A}} \mu_0 m_{\mathrm{A}} + g_{\mathrm{B}} \mu_0 m_{\mathrm{B}})$$

in the region $T > T_{\mathrm{c}}$. Does your T_{c} depend on the sign of J? Why or why not?

(b) Work out the spin wave theory and compute the spin wave dispersion. (You should treat the NaCl structure as an FCC lattice with a two element basis.) Assume a classical ground state $|\,\mathrm{N}\,\rangle$ in which the spins are up on the A sublattice and down on the B sublattice, and choose

$$
\begin{array}{ll}
\underline{\text{A Sublattice}} & \underline{\text{B Sublattice}} \\
S^+ = a^\dagger\,(2S_{\mathrm{A}} - a^\dagger a)^{1/2} & S^+ = -(2S_{\mathrm{B}} - b^\dagger b)^{1/2}\,b \\
S^- = (2S_{\mathrm{A}} - a^\dagger a)^{1/2}\,a & S^+ = -b^\dagger\,(2S_{\mathrm{B}} - b^\dagger b)^{1/2} \\
S^z = a^\dagger a - S_{\mathrm{A}} & S^z = S_{\mathrm{B}} - b^\dagger b
\end{array}
$$

How does the spin wave dispersion behave near $\boldsymbol{k} = 0$? Show that the spectrum crosses over from quadratic to linear when $|\boldsymbol{k}a| \approx |S_{\mathrm{A}} - S_{\mathrm{B}}|/\sqrt{S_{\mathrm{A}} S_{\mathrm{B}}}$.

(5) In real solids crystal field effects often lead to anisotropic spin-spin interactions. Consider the anisotropic Heisenberg antiferromagnet in a uniform magnetic field,

$$
\mathcal{H} = J \sum_{\langle ij \rangle} (S_i^x\,S_j^x + S_i^y\,S_j^y + \Delta\,S_i^z\,S_j^z) + h \sum_i S_i^z
$$

where the field is parallel to the direction of anisotropy. Assume $\Delta \geq 0$ and a bipartite lattice.

(a) Think first about classical spins. In a small external field, show that if the anisotropy Δ is not too large that the lowest energy configuration has the spins on the two sublattices lying predominantly in the (x, y) plane and antiparallel, with a small parallel component along the direction of the field. This is called a canted, or 'spin-flop' structure. What is the angle θ_{c} by which the spins cant out of the (x, y) plane? What do I mean by not too large? (You may assume that the lowest energy configuration is a two sublattice structure, rather than something nasty like a four sublattice structure or an incommensurate one.)

(b) Now work out the quantum spin wave theory. To do this, you'll have to rotate the quantization axes of the spins to their classical directions. This means taking

$$
\begin{aligned}
S^x &\to \cos\theta\,S^x + \sin\theta\,S^z \\
S^y &\to S^y \\
S^z &\to -\sin\theta\,S^x + \cos\theta\,S^z
\end{aligned}
$$

with $\theta = \pm\theta_0$, depending on the sublattice in question. How is θ_0 related to θ_{c} above? This may seem like a pain in the neck, but really it isn't so bad. Besides, you shouldn't complain so much. And stand up straight – you're slouching. And brush your teeth.

(c) Compute the spin wave dispersion and find under what conditions the theory is unstable.