Physics 218C

Lecture 5C → Saturation Mechanisms

→ Zonal Modes

→ Why Zonal Modes – Saturation?

→ Calculation: { Eikonal → Phase
Wave Kinetics

N.B. Recall shear ↔ eikonal

Continues

→ System { Wave Population
Zonal Modes

→ Predator–Prey

→ FAQ

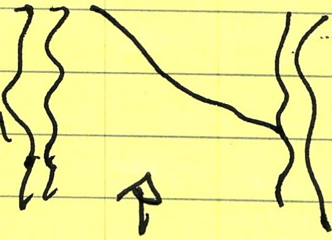
i) Saturation?

What determines $\langle \tilde{v}_r \tilde{n} \rangle$, $\langle \tilde{v}_r^2 \rangle$, etc.?

→ Mechanism – Scaling connection!

Now: — Driven system — source + sink

Heating Deposition Region



→ Boundary

↓

sink

Confinement Region

Source

→ profile

⇒ akin to fixed flux convection

Q

— MLT is not energy balance at stationarity

i.e. $$\frac{\partial n}{\partial t} + \underbrace{\tilde{V}\cdot\nabla\tilde{n} = -V_r}_{NL \sim Lin} \frac{\partial\langle n\rangle}{\partial r}$$

⇒ MLT is entry to nonlinear regime.

⇒ not a "saturation".

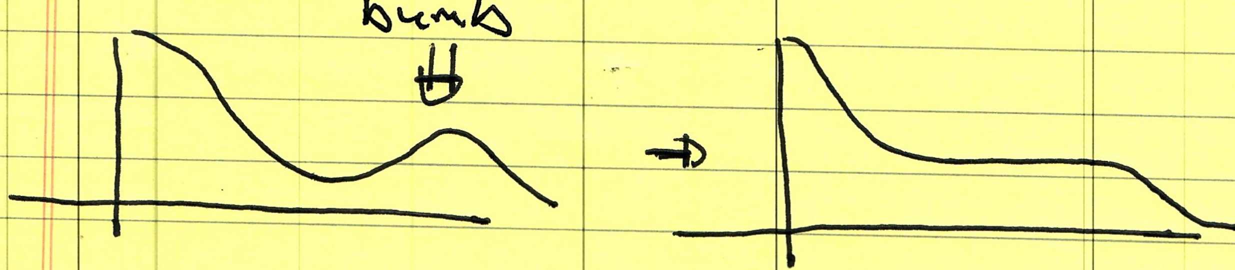
⇒ Saturation Mechanisms — A Coarse-Grained Zoology

— "where does the energy go"

a.) turn off coupling to free energy, remove free energy

⇒ "quasilinear +"

i.e. B-O-T



plateau formation - 1D QL
(218a)

also: 2D plateau $\rightarrow$ drift waves.
$\leftrightarrow$ heating !

Relevance to driven system ?

but also:

- Shearing: $\dfrac{\tilde{V}_r}{\omega}\dfrac{\partial\langle n\rangle}{\partial r} \rightarrow \dfrac{\tilde{V}_r}{\omega - k_\theta V_E' x}\dfrac{\partial\langle n\rangle}{\partial r}$

shear self-generated

shearing drive mechanism

reduced efficiency of extracting free energy

i.e. $\Rightarrow$ differential rotation of response.

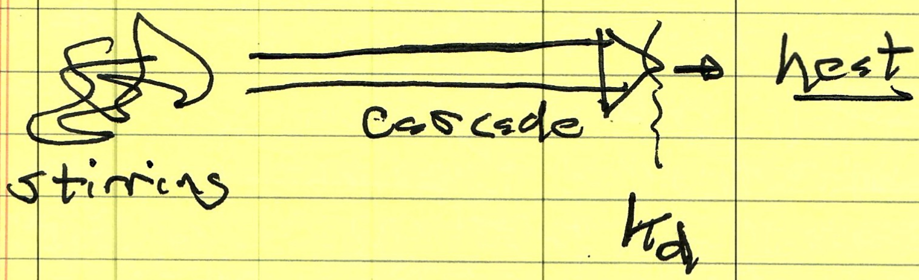
- NL frequency shifts

etc.

$$\boxed{\text{Energy removed } (\langle f\rangle \text{ re-configured}) \text{ or decoupled.}}$$

b.) Couple to Dissipation

⇒ Classic : $\underline{K41}$ — 3D



Where does the energy go → heat?

$$\epsilon = \frac{v(\ell)^3}{\ell} = \frac{\nu}{\ell_d^2} v(\ell_d)^2 \qquad \begin{cases} v(\ell) \sim \epsilon^{1/3} \ell^{1/3} \\ \ell_d \sim \dfrac{\nu^{3/4}}{\epsilon^{1/4}} \end{cases}$$

Heating: $\underbrace{\nu \left| \frac{\partial \tilde{v}}{\partial x} \right|^2 \sim \frac{\nu}{\ell_d^2} \tilde{v}(\ell_d)^2}$

viscous heating rate

$$\sim \frac{\nu}{\left(\frac{\nu^{3/4}}{\epsilon^{1/4}}\right)^2} \; \frac{\nu^2 \cancel{\ell_d^2}}{\ell_d^4} \sim \epsilon$$

$\uparrow$ energy input rate

— physics is coupling to damping by nonlinear transfer

— prototype of saturation by :

- mode-mode coupling
- NL wave particle interaction, Compton scattering

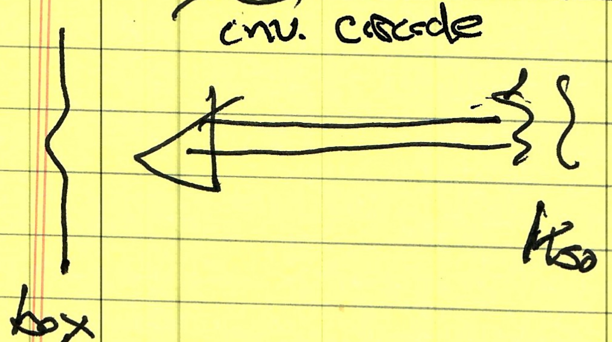
⋮ etc.

⇒ in all case, should identify the damping.

⇒ weakly damped d-o-f most effective at absorbing energy. Heavily damped modes difficult to excite and couple to.

In K41, $k \sim k_d$ are sinks, not $k \gg k_d$.

⇒ For inverse cascade (2D fluid):



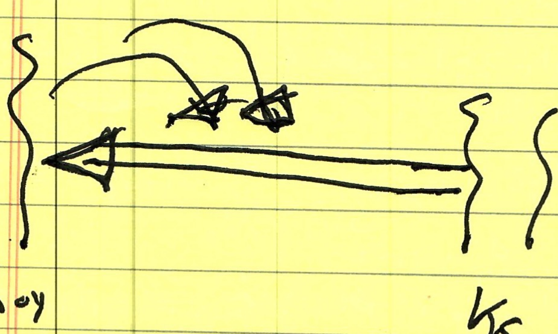
→ local process

→ need some scale independent damping ⇒ critical to process drag

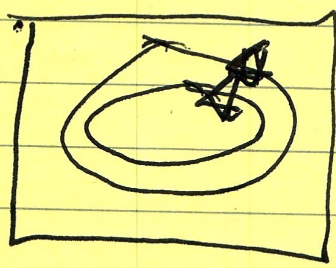
i.e. $\partial_t \nabla_\perp^2 \phi + \mu \nabla_\perp^2 \phi$

contrast $-\nu \nabla_\perp^2 \nabla_\perp^2 \phi$

If μ weak:



i.e.



large shear flows in box develop; can back-react on smaller scales.

$\Rightarrow$ Coupling to "dissipation" includes losses thru / at boundary. $\rightsquigarrow$ $\boxed{\text{spatial coupling}}$

$\Leftrightarrow$ saturation by
- turbulence spreading,
- avalanching
- $\vdots$

c.) Saturation by Coupling to "Harmless" D-O-F.

→ Distinction: Damped vs "Harmless" (not mutually exclusive)

Damped ≡ perturbation decays

"Harmless" ≡ perturbation converted to scales which don't degrade confinement

In MFE: "Harmless" ≡ $\left.\begin{matrix} n \\ m \end{matrix}\right\} = 0$

⇒ zonal mode.

V_r, $B_r \to 0$ ⇒ no transport.

Of course dissipation of Harmless DOF's ultimately disposes of energy.

$\boxed{\text{Harmless D-O-F's} \Leftrightarrow \text{symmetry.}}$

Which brings us to:

c(a)

→ Zonal Modes

- $\phi(r)$ → flow
  $n(r)$
  $T(r)$
  ⋮

  Thermodynamic variables → characterize $\langle f \rangle$ with poloidal, toroidal symmetry

i.e. $\phi_{Z(r)}$, n_{Zr}, T_{Zr} etc.

- Flow + ⇒ e.s. potential, $V_{\underline{E}\times\underline{B}}$

  Zonal density ↔ CTEM, H-W

  Electron Temperature ↔ CTEM

  Ion Temperature ↔ ITG

  etc.

  "corrugations"

→ Flow, Flow shear is $\underline{E}\times\underline{B}$!

- Zonal flow coupling exploits all three saturation channels !

i.e. – $k_\theta = 0$ for zonal modes ⇔ transport → 0

⇒ coupling to harmless D-O-F

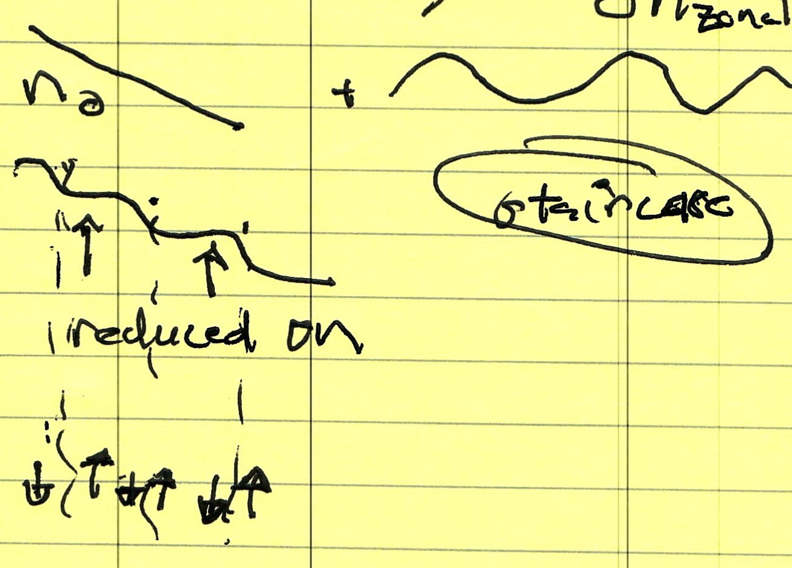
⇒ zonal flows

– dissipation coupling

$k_\theta = 0 \rightarrow \omega \rightarrow 0$ so can couple to modest dissipative drag.

– remove coupling to free energy:

i.e. corrugation



and shear:

holds steep gradient.

So, why zonal modes?

→ exploit several channels to saturation and Zonal modes:

→ modes of minimal inertia

i.e. $\partial_t (1 + k_\perp^2 \rho_s^2)\, \hat{\phi}_k$ ⟷ drift wave

vs $\partial_t\, k_\perp^2 \rho_s^2\, \hat{\phi}_k$ ⟷ zonal mode

→ modes of minimal (i.e. zero) transport

$k_\theta = 0$ ⇒ symmetry

→ modes of minimal dissipation – easily excited

i.e. $\omega = 0 - i\mu$

⇒ $\boxed{\text{Zonal modes are repository, and a natural one, for free energy released by micro-instabilities.}}$

N.B.:

- ZF generation: $\left\{\begin{array}{l}\text{inhomogeneous} \\ \text{PV mixing} \\ + \\ \text{1 direction symmetry}\end{array}\right.$ [McIntyre + Wood]

i.e.

$$\partial_t \langle \nabla_\perp^2 \phi \rangle = -\frac{\partial}{\partial r} \langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle + \cdots$$

$$\downarrow$$

$\partial_r \left(\underline{\text{PV Flux}} \right) \rightarrow$ inhomogeneous mixing

1 direction symmetry $\Leftrightarrow$ Taylor identity

i.e. $\langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle = \partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle$

$\Rightarrow$ ZF

- Open questions: low k_θ shears?

i.e. $k_\theta \neq 0 \rightarrow$ transport

but

$\partial^2 \phi / \partial x \partial y \neq 0$ & significant $\Rightarrow$ shearing

How low k_θ shears from convective cells can contribute to turbulence regulation!

- Energy Flow:

i.e. CTEM, EDW

$\langle f_e \rangle$ — $1/L_n$, $1/L_p$ etc.

↓ instability

$\boxed{\text{Waves}}$ — $\underline{k}$

↓ interaction

$\boxed{\text{Stresses, Fluxes}}$ — "envelope" → scale $|\hat{\phi}|^2 \to \underline{q}$

↓ transport, modulation $|\underline{q}| < |\underline{k}|$

$\boxed{\text{Zonal modes}}$ → $\langle f_e \rangle$

↓

$\boxed{\text{heat, dissip.}}$

$\boxed{\text{Zonal modes return energy to } \langle f_e \rangle \text{ via (radial) modulations of thermodynamic quantities.}}$

- modulation of thermodynamic parameters
- coupling to dissipation

→ (iii) The Calculation

Recall for EDW/CTEM:

$$\mathcal{E} = \int d^3x \, \rho\, c_s^2 \left[\left(\frac{\tilde{n}}{n} \right)^2 + \rho_s^2 \left(\nabla_\perp \frac{|e|\tilde{\phi}}{T} \right)^2 \right]$$

⇒

weak, for $\left\{ \begin{array}{l} \alpha > 1 \quad \text{CDW} \\ \text{EDW} \end{array} \right.$

$$\partial_t \mathcal{E} = - \int d^3x \left[\langle \tilde{v}_r \tilde{n} \rangle \, \partial_r \langle n \rangle \right]$$

$$- \int d^3x \, \langle V_E \rangle' \, \langle \tilde{V}_{r_E} \tilde{V}_{y_E} \rangle \qquad + \text{Dissip}$$

focus.

$$- \int d^3x \, \langle \tilde{\phi} \, \partial_t \tilde{n} \rangle$$

↳ drive

How calculate Reynolds power?

$$\langle \tilde{V}_{r_E} \tilde{V}_{y} \rangle = \left\langle + \frac{c^2}{B_0^2} \tilde{E}_\theta \tilde{E}_r \right\rangle$$

$$= - \frac{c^3}{B_0^2} \sum_k \underbrace{k_r k_\theta \, |\tilde{\phi}_k|^2}_{\langle k_r k_\theta \rangle \text{ correlation}}$$

⇒ Reynolds stress set by $\langle k_r k_\theta \rangle$

→ requires radially propagating wave.

Issue → $\begin{cases} \text{Symmetry !} \\ \text{alignment } E_r\, E_\theta \end{cases}$

The Point: shear tends to align k_r, k_θ !

Recall shearing coordinates:

$$\frac{dk}{dt} = -\frac{\partial}{\partial \underline{x}} \left(\cancel{\omega} + k_\theta \langle V_E \rangle \right)$$

$$\frac{dk_r}{dt} = -k_\theta \langle V_E \rangle'$$

so

$$k_r = k_r^{(0)} - k_\theta \langle V_E \rangle' t$$ shearing coordinates

Eddy tilting:

then with $t \lesssim \tilde{\tau}_c$

$$\langle k_r k_\theta \rangle = \langle k_r^{(0)} k_\theta \rangle - k_\theta^2 \langle V_E \rangle' \tilde{\tau}_c$$

tilting induced alignment.

How describe systematically?

⇒ $\boxed{\text{Wave kinetics}}$

⇒ idea is to calculate the response of the wave population to shear

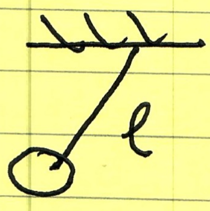
→ will capture tilting, and other physics

⇒ exploit adiabatic invariance

ie adiabatic invariant ≡ an approx conserved quantity, due time scale separation i.e. $\omega_{ZF} \simeq \Omega \simeq 0$ $\Rightarrow$ shear develops slowly

$\omega_{DW} \gg \Omega_{ZF}$

akin



$\downarrow g$ $\qquad \omega = \sqrt{g/\ell}$

$$\frac{1}{\ell}\frac{d\ell}{dt} \ll \omega$$

then $E/\omega \equiv$ adiabatic invariant

$\rightarrow$ action

$\Rightarrow$ For waves likewise motivate eq:

$N = \mathcal{E}/\omega$

↳ wave frequency

↳ energy density

N ↓ Action Density

Note QM analogy: $\mathcal{E} = N\omega$

and continuing in that vein:

$$N = N(\underline{k}, \underline{x}, t)$$

$\left.\begin{array}{l}\underline{x}\\ \underline{k}\end{array}\right\}$ Hamiltonian variables

Density / Distribution function of waves

$\underline{k} \to$ direction, momentum

$\underline{x} \to$ packet position

t

so

$$\frac{dN}{dt} = \frac{\partial}{\partial t} N + \underline{v}_{gr} \cdot \underline{\nabla} N + \frac{d\underline{k}}{dt} \cdot \nabla_{\underline{k}} N = 0$$

aka Vlasov Eqn.

and for flow,
$$\frac{d\underline{x}}{dt} = \underline{v}_{gr} + \underline{V}$$

$$\frac{d\underline{k}}{dt} = -\frac{\partial}{\partial \underline{x}} (\omega + \underline{k} \cdot \underline{V})$$

cf Landau and Lifshitz "Fluids" $\Rightarrow$ acoustic waves

so finally:

$$\boxed{\frac{\partial N}{\partial t} + (\underline{v}_{gr} + \underline{V}) \cdot \nabla N - \frac{\partial}{\partial \underline{x}} (\omega + \underline{k} \cdot \underline{V}) \cdot \frac{\partial N}{\partial \underline{k}} = 0}$$

drop brackets.

⇒ Wave Kinetic Eqn.

(recall from Langmuir Turbulence 218a)

- Can calculate response N to shear:

i.e.

$$N = \langle N(\underline{k}) \rangle + \tilde{N}$$

and treat $\hat{V}_E$ as seed / test.

- approximate linear growth + non-action-conserving interactions by $C(N)$ on RHS

Point: Modulation by shear perturbs the base state!

$$\frac{dN}{dt} = C(N)$$

$$\cong \gamma N - \underbrace{\Delta\omega}_{\text{characteristic decay rate}} N^2$$

⇐ condensed form of complex collision integral for 3 wave interactions

- obviously, can also approach via

Envelope Theory

i.e. Langmuir Turbulence

(add diffraction)

Wave Kinetics $\Leftrightarrow$ Zakharov Equations (Envelope Eqn)

$\downarrow$ $\downarrow$

BGK Mode/Trapping (calc Vlasov)   Soliton (1D) Collapse (3D) (finite time singularity)

Drift - Zonal Flow

Wave Kinetics

$\downarrow$

Modulational Inst.

$\downarrow$

Trapping

(see P.D., et. al 2005 review)

Envelope Equation

$\downarrow$

Modulational Instability

$\downarrow$

$\Rightarrow$ Soliton Formation

(see Guercan, P.D. 2014 review)

· So – how does spectrum evolve?

⇒ Quasi-linear Theory for $\langle N \rangle$!

$$\frac{\partial \langle N \rangle}{\partial t} = + \frac{\partial}{\partial k_r} \left\langle k_\theta \tilde{V}_E' \tilde{N} \right\rangle + \langle C(N) \rangle$$

⇑

flux in k induced by shearing

then,

$$\frac{\partial \tilde{N}}{\partial t} + v_{gr} \cdot \nabla \tilde{N} + |\delta| \tilde{N} = k_\theta \tilde{V}_E' \frac{\partial \langle N \rangle}{\partial k_r}$$

(a la' Plasma Wave)

so

$$\tilde{N}_{q_r, \Omega} = \frac{i q_r \tilde{V}_{Eq} \, \partial \langle N \rangle / \partial k_r}{-i \left(\Omega_q - q_r v_{gr} + i |\delta| \right)}$$

so

$$\frac{\partial \langle N \rangle}{\partial t} = \underbrace{\frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r} \langle N \rangle}_{\text{diffusion in } k} + \langle C(N) \rangle$$

where:

$$D_k = \sum_{\underline{k}_n} \frac{k_r^2 |\tilde{V}_{E,k}|^2 |\gamma| k_\theta^2}{(\Omega - \underline{k} \cdot v_g)^2 + |\gamma|^2}$$

diffusion in k_r:

→ "random shearing"

→ $\frac{dk_r}{dt} = -\frac{\partial}{\partial x}(k_\theta \tilde{V}_E')$ as Langevin

$$\left(\frac{dv}{dt} = \frac{q}{m} \tilde{E} \right)$$

$\langle k_r^2 \rangle \uparrow$ via random walk.

→ origin of irreversibility ?

$\Rightarrow$ ray chaos!

(akin wave-particle resonance)

$$\Omega/q \sim v_{gr}$$

$\left\{\begin{array}{l}\text{resonance}\\ \text{overlap.}\end{array}\right.$

$(\Omega \to 0)$

easy

(clearer for GAM)

For energy,

$$\frac{\partial \langle N \rangle}{\partial t} \Rightarrow \frac{\partial \langle \mathcal{E} \rangle}{\partial t} = \omega \frac{\partial \langle N \rangle}{\partial t}$$

$$\frac{\partial}{\partial t}\langle \mathcal{E} \rangle = \int d^3k\, \omega_k \frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r} \langle N \rangle + \int \langle C(N) \rangle \omega$$

$$= -\int d^3k \underbrace{\left(\frac{\partial \omega_k}{\partial k_r}\right)}_{\text{radial group velocity}} D_k \underbrace{\frac{\partial}{\partial k_r} \langle N \rangle}_{\text{spectral slope}}$$

+ S.T., etc.

N.B.: S.T.? un-resolved $\Rightarrow$ resolved

For sign $\partial\langle\varepsilon\rangle/\partial t$?

$$\rightarrow \frac{\partial\omega}{\partial k_r} = \frac{-2k_r k_\theta V_*}{(1+k_\perp^2\rho_s^2)^2}$$

$$\rightarrow N = \frac{\varepsilon}{\omega_k} = \frac{(1+k_\perp^2\rho_s^2)^3 |\phi_k|^2}{k_\theta V_*}$$

and k_θ const $\Rightarrow$ zonal symmetry

so

$$\frac{\partial\langle\varepsilon\rangle}{\partial t} = +\int d^3k \, \frac{(+2k_r k_\theta V_*)}{(1+k_\perp^2\rho_s^2)^2} \frac{\partial}{\partial k_r}\left(\frac{\left[(1+k_\perp^2\rho_s^2)^2 |\hat{\phi}_k|^2\right]}{\cancel{k_\theta V_*}/\cancel{(1+k_\perp^2\rho_s^2)}} \right)$$

$$[1+k_\perp^2\rho_s^2]^2 |\phi_k|^2 = \text{Potential enstrophy} = |q_k|^2$$

$$\boxed{\frac{d\langle\mathcal{E}\rangle}{dt} = \int d^3k \, \frac{2k_r}{(1+k_\perp^2\rho_s^2)^2} \frac{\partial}{\partial k_r}\left[|\tilde{\phi}_k|^2\right]}$$

- $\frac{d\langle\mathcal{E}\rangle}{dt} < 0$ for $\frac{\partial}{\partial k_r}\left[|\tilde{E}_k|^2\right] < 0$

- universally: { spectra have robust negative power law slope

so

$\frac{\partial\langle\mathcal{E}_w\rangle}{\partial t} < 0$ $\boxed{\text{Modulational Instability}}$

where does the energy go?

⇒ Flow! → show this!
(c.f. Energy conservation in QLT)

For flow:

$$\frac{\partial}{\partial t}\langle V_E\rangle = -\frac{\partial}{\partial r}\langle \tilde{V}_r \tilde{V}_\theta\rangle - \mu\langle V_E\rangle$$

$\Omega = |\tilde{E}|^2$

then, for stress

$$\delta\langle\tilde{v}_r\tilde{v}_\theta\rangle \cong \frac{k_r k_\theta}{(1+k_\perp^2\rho_s^2)}\,\delta\Omega$$

$\mapsto \langle\tilde{v}_r\tilde{v}_\theta\rangle$ induced by modulating

and, to emphasize modulation:

$$\partial_t\,\delta\langle V_\theta\rangle_E = -\frac{\partial}{\partial r}\sum \frac{c^2}{4 B_0^2}\frac{k_r k_\theta\,\delta\Omega}{(1+k_\perp^2\rho_s^2)} - \mu\,\langle\delta V_E\rangle$$

$\Rightarrow$ $\delta\Omega \sim \delta\langle V_E\rangle'$

$\therefore$ $\rightarrow$ Z.F. growth due to shearing of waves

$\rightarrow$ recovers $\partial_t\langle\mathcal{E}_F\rangle = -\partial_t\langle\mathcal{E}_W\rangle$ ✓

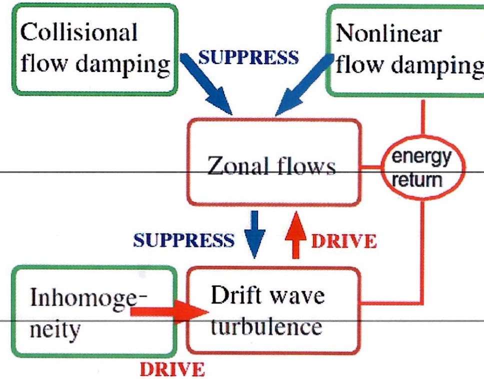
$\rightarrow$ Reynolds work/power and flow shearing by self-generated flow are simply relabeling $\rightarrow$ books balance.

This brings us to the

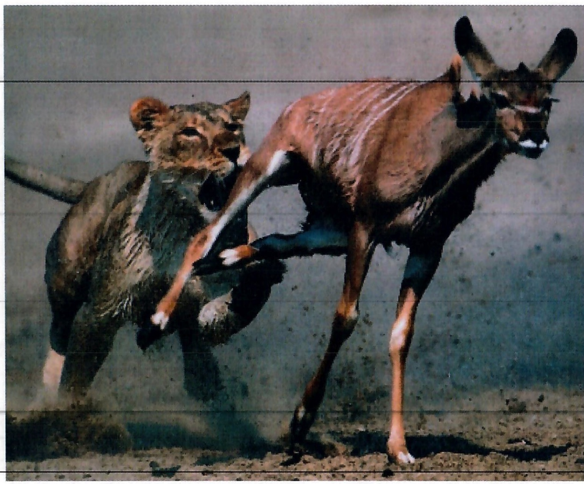
Predator - Prey ..... TRC

Feedback Loops I

- Closing the loop of shearing and Reynolds work



- Spectral 'Predator-Prey' equations



Prey → Drift waves, <*N*>

$$\frac{\partial}{\partial t}\langle N\rangle - \frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r}\langle N\rangle = \gamma_k \langle N\rangle - \frac{\Delta\omega_k}{N_0}\langle N\rangle^2$$

Predator → Zonal flow, $|\phi_q|^2$

$$\frac{\partial}{\partial t}|\phi_q|^2 = \Gamma_q\left[\frac{\partial\langle N\rangle}{\partial k_r}\right]|\phi_q|^2 - \gamma_d|\phi_q|^2 - \gamma_{NL}[|\phi_q|^2]|\phi_q|^2$$