Physics 218 C

Lecture 3 a - Part of

→ Left - Overs

(vs TTF Dinner)

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stew
⋮

a.) Reduced MHD

b.) Dynamics of Zonal Flows in DWT

→ An Introduction

On Reduced MHD

Reduced MHD → Simplifying the representation:

→ strong magnetization — anisotropy.

$\delta B_{\parallel} \to 0$; $\delta B_{\perp} = \nabla A_{\parallel} \times \hat{z}$

$\delta V_{\perp} = \nabla \phi \times \hat{z}$. 9b

→ $\tau > \tau_{MS}$

Aside

→ $\boxed{\text{Reduced MHD}}$ → Reduced Representation for strong ⊙ straight B_0

→ eliminates fast mode

Note: ① full MHD : 3 $\underline{V}$ components

2 $\underline{B}$ " " ($\underline{\nabla} \cdot \underline{B} = 0$)

ρ, p

⇒ 7 components

② if $\underline{\nabla} \cdot \underline{V} = 0$ ⇒ 4 components

(ρ = const, p from $\underline{\nabla} \cdot \underline{V} = 0$)

③ strongly magnetized system ⇒ Reduced MHD

⇒ scalar equations for ϕ, ψ (2 scalar fields)

Now:

- assume strong B_z (strong magnetization → gyrokinetics) ↳ later

"strong" ⟷ $\rho v^2 \sim p \ll B_z^2/8\pi$

so motion strongly anisotropic, and small scales generated in ⊥ direction only, as strong B_z inhibits line bending. (energy to perturb strong, high energy density field).

⇒ Order : $B_z \sim \nabla_{\perp} \sim 1$

$B_{\perp} \sim \partial_z \sim O(\epsilon)$

Take $\rho \sim 1$, as $\underline{\nabla} \cdot \underline{V} = 0$ enforced by strong B_z:

$V_\perp^2 \sim p \sim B_\perp^2$ (i.e. equipartition of energy)
(springiness)

$\Rightarrow \quad V_\perp \sim \epsilon, \quad p \sim \epsilon^2, \quad \partial_t \sim \underline{V}_\perp \cdot \underline{\nabla}_\perp \sim \epsilon$

and pressure balance ($\underline{\nabla} \cdot \underline{V} = 0$ / [illegible] incompressibility)

$\delta(B_z^2) \sim 2 B_z \delta(B_z) \sim p$

$\Rightarrow \delta B_z \sim \epsilon^2$.

i.e. $\omega \ll k(c_s^2 + V_A^2)^{1/2}$ (eqbrm)

[idea is to order out the **first mode**]

∴ to lowest order $\Rightarrow$ $B_z = \text{const}$.

Now then: ($\underline{\nabla} \cdot \underline{B} = 0$)

$$\boxed{\begin{aligned} \underline{B} &= \hat{z} \times \underline{\nabla} \psi + B_z \hat{z} \\ &= \nabla A_\parallel \times \hat{z} + B_z \hat{z} \qquad \psi = -A_\parallel \end{aligned}}$$

B rep. by single scalar potential

parallel comp. of vector pot.

$\underline{\nabla} \cdot \underline{B} = \partial_z \tilde{B}_z = \epsilon^3 \rightarrow 0$.

Similarly;

$\partial_z p \sim O(\epsilon^3)$, $\Rightarrow$ $\cancelto{0}{V_z} \ll V_\perp$
$J_\perp B_\perp \sim \epsilon^3$

neglect V_z.

Now, $\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi = -\frac{\underline{V}\times\underline{B}}{c}$

$\Rightarrow \quad +\frac{1}{c}\frac{\partial A}{\partial t} = \frac{\underline{V}\times\underline{B}}{c} - \underline{\nabla}\phi \qquad \circledast$

$B_z = (\underline{\nabla}_\perp \times \underline{A}_\perp)\cdot\hat{\underline{z}}$

so $\partial_t A_\perp \sim \epsilon^3$ (ala $\underline{\partial_z p_z}$)

∴ $\nabla_\perp\phi \cong \left(\frac{\underline{V}\times\underline{B}}{c}\right)_\perp$, in $\circledast$ $\underline{V}_\perp \Leftarrow \nabla\phi$.

{Inductive piece negligible.

$\Rightarrow \boxed{\underline{V}_\perp = c\frac{\hat{\underline{z}}\times\underline{\nabla}\phi}{B_z}}$

⊥ velocity

→ {motion ⊥ is $\underline{E}\times\underline{B}$.

Now, taking parallel component of $\circledast$.

(units!)

$\Rightarrow \boxed{\frac{\partial}{\partial t}\psi + \underline{V}\cdot\underline{\nabla}\psi = B_z\partial_z\phi}$

so have (vector potential / flux) equation:

$$\boxed{\frac{\partial\psi}{\partial t} + \underline{V}\cdot\underline{\nabla}\psi = B_z\partial_z\phi}$$

$\underline{V}\cdot\nabla\psi$ from

$B\cdot\nabla\phi \rightarrow$

$B_0\partial_z\phi + \nabla\psi\times\hat{z}\cdot\nabla\phi$

Equation of evolution of magnetic flux.

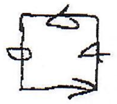
$$= B_z \hat{z} + \hat{z} \times \underline{\nabla}\psi$$

or, alternatively, $$\boxed{\frac{\partial \psi}{\partial t} - \underline{B}\cdot\underline{\nabla}\phi = 0} \qquad 94.$$

Finally, for ϕ, write:

$$\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V} = -\frac{\underline{\nabla}p}{\rho_0} + \frac{\underline{J}\times\underline{B}}{c}$$

⊥ motion



cells of E×B drift, ('spin up' rate?)

$(\underline{\nabla}\times\ )\cdot\hat{z} \Rightarrow$ vorticity component $(\parallel \hat{z})$ evolution.

Dynamics on ⊥ plane, $\nabla\phi\times\hat{z} = \underline{V}_\perp$. $\underline{\omega} \to \hat{z}\cdot\underline{\omega}$

$$\frac{\partial}{\partial t}\omega_z + \underline{V}_\perp\cdot\underline{\nabla}\,\omega_z = \overset{0}{\cancel{-\underline{\nabla}\times\frac{\underline{\nabla}p}{\rho_0}}} + \hat{z}\cdot\underline{\nabla}\times\left(\frac{\underline{J}\times\underline{B}}{c}\right)$$

$$= \underline{B}\cdot\underline{\nabla}J_z - \cancel{\underline{J}\cdot\underline{\nabla}B_z} \qquad \delta B_z \sim \epsilon^3$$

$$\cong \underline{B}\cdot\underline{\nabla}J_z$$

$$\therefore \quad \boxed{\frac{\partial}{\partial t}\omega_z + \underline{V}\cdot\nabla\,\omega_z = \underline{B}\cdot\underline{\nabla}J}$$

but:

$$\omega_z = \hat{z}\cdot\underline{\nabla}\times\underline{V} = \nabla^2\phi$$

$$J_z = \hat{z}\cdot(\underline{\nabla}\times\underline{B})\frac{c}{4\pi} = \nabla^2\psi$$

So → Waves → time scales → Reduced MHD

so finally have:

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}\nabla^2\phi + \underline{v}\cdot\underline{\nabla}\,\nabla^2\phi &= B_z\frac{\partial}{\partial z}\nabla^2\psi\\ &\quad + \underline{\tilde{B}}\cdot\underline{\nabla}\,\nabla^2\psi\end{aligned}}$$

$B = B_0\hat{e}_{\parallel}$

Finally, have reduced MHD equation:

$$\boxed{\begin{aligned}&\frac{\partial\psi}{\partial t} + \underline{v}\cdot\underline{\nabla}\psi = B_z\,\partial_z\phi + \eta\nabla^2\psi\\[1em] &\frac{\partial}{\partial t}\nabla^2\phi + \underline{v}\cdot\underline{\nabla}\,\nabla^2\phi - \nu\nabla^2\nabla^2\phi\\ &\qquad = \underline{\tilde{B}}\cdot\underline{\nabla}\,\nabla^2\psi + B_z\frac{\partial}{\partial z}\nabla^2\psi\end{aligned}}$$

$E_{\parallel} = \eta J_{\parallel}$

vorticity in plane ⊥ $\hat{e}_{\parallel}$.

- note have reduced MHD to 2 scalar evolution equations
- does this look familiar?
- 2D dynamics + shear Alfven wave.
- nonlinearity → 2D dynamics.

even stronger!

$B_0 \partial_z \to 0$

$\partial_z \psi = 0$

$\partial_z \phi = 0$

l.o. + [illegible].

15.

- for $\underline{\underline{2D}}$ MHD:

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t}\nabla^2\phi + \underline{V}\cdot\underline{\nabla}\,\nabla^2\phi = \underline{B}\cdot\underline{\nabla}\,\nabla^2\psi + \nu\nabla^2\nabla^2\phi \\ &\frac{\partial\psi}{\partial t} + \underline{V}\cdot\underline{\nabla}\psi = \eta\nabla^2\psi \end{aligned}}$$

0D - ① Conservation Laws, etc. (HW)

$$\frac{d}{dt}E = \underset{(\text{to } \eta, \nu)}{0}, \quad E = \int d^3x\left[\frac{(\nabla\phi)^2}{2} + \frac{(\underline{\nabla}\psi)^2}{2}\right]$$

② $H = A\cdot B \cong \underset{\substack{\downarrow \\ \text{const.}}}{B_z}\psi$ $\quad \int d^2x\, A^2 = \text{MSMP}$ (2D)

0D

$$\Rightarrow \quad H = \int d^3x\, B_z\psi, \quad \frac{dH}{dt} = 0, \text{ to } O(\eta)$$

Ohm's Law (flux advection) is simple statement of helicity conservation.

? $\frac{\partial}{\partial t}\psi + \nabla\cdot\Gamma_\psi = \eta\nabla^2\psi$ form Γ_ψ s/t $\left\{\begin{array}{l}H \text{ conserved} \\ E_M \text{ dissipated}\end{array}\right.$

③ $$\underline{K = \int d^3x\, \underline{V}\cdot\underline{B} = \int d^3x\,(\nabla\phi\cdot\underline{\nabla}\psi)}$$

also conserved, to dissipation.

Alfven wave [illegible] turbulence.

Reduced MHD — Brief

See Strauss, '76 (PoF) for full details

→ the point: − strong $\langle \underline{B} \rangle$, ~ straight

− low frequency $(\omega < \omega_{MS})$

− $\langle B \rangle$ ~ unperturbed

− $\nabla \cdot \underline{V} = 0$

$\underline{\nabla} \cdot \underline{V} = 0 \Rightarrow$ 2 component $\underline{V}$

$\underline{\nabla} \cdot \underline{B} = 0 \Rightarrow$ 2 components $\underline{B}$

$\underline{E} + \frac{\underline{V} \times \underline{B}}{c} \simeq 0 \quad \Rightarrow \quad \underline{V}_{\perp} = +\frac{c}{B} \underline{E}_{\perp} \times \vec{z}$

$\delta B_{\parallel} \to 0$

$$\underline{E}_{\perp} = -\underline{\nabla}_{\perp} \phi - \frac{1}{c} \cancel{\frac{\partial A_{\perp}}{\partial t}}$$

$$\underline{V}_{\perp} = -\frac{c}{B} \underline{\nabla}_{\perp} \phi \times \vec{z}$$

$\underline{\nabla} \times \underline{A} = 0$

$\underline{B}_{\perp} = \nabla A_{\parallel} \times \vec{z}$

$\underline{B}_{\perp} = \nabla A_{\parallel} \times \vec{z}$

Then,

$$\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi = \eta \underline{J}$$

$$E_{\parallel} = -\frac{1}{c}\frac{\partial A_{\parallel}}{\partial t} - \nabla_{\parallel}\phi = \eta J_{\parallel}$$

$$\underline{B} = B_0\hat{z} + \underline{B}_{\perp}$$

$\Rightarrow$

$$-\frac{1}{c}\frac{\partial A_{\parallel}}{\partial t} - \frac{(B_0\hat{z} + \tilde{B}_{\perp})\cdot\nabla\phi}{|\underline{B}_0\hat{z} + \tilde{\underline{B}}_{\perp}|} = \eta J_{\parallel}$$

$\therefore$

$$-\frac{1}{c}\frac{\partial A_{\parallel}}{\partial t} - \partial_z\phi - \tilde{B}_{\perp}\nabla\phi = \eta J_{\parallel}$$

and

$$\boxed{-\frac{1}{c}\frac{\partial A_{\parallel}}{\partial t} - \partial_z\phi - \underline{\nabla}A_{\parallel}\times\hat{z}\cdot\nabla\phi = \eta J_{\parallel}}$$

$\Rightarrow$

$$\boxed{\frac{\partial\psi}{\partial t} + \underline{v}\cdot\nabla\psi = \partial_z\phi + \eta\nabla^2\psi}$$

Reduced Ohm's Law

Now,

strong field

+

$v_\perp$ only → set by ϕ

so, $\frac{\underline{B}_0}{|B_0|} \cdot \underline{\nabla} \times \underline{v} = \hat{z} \cdot \underline{\nabla} \times \underline{v}$ → is the key to dynamics

Now, $$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v}\right) = -\underline{\nabla} P + \frac{\underline{J} \times \underline{B}}{c}$$

$$\underline{v} \cdot \underline{\nabla}\, \underline{v} = \underline{\nabla} \frac{v^2}{2} - \underline{v} \times \underline{\omega}$$

$$\rho\left(\frac{\partial v}{\partial t}\right) = -\underline{\nabla}\left(P + \rho \frac{v^2}{2}\right) + \rho\, \underline{v} \times \underline{\omega} + \frac{\underline{J} \times \underline{B}}{c}$$

$\rho \equiv \rho_0$ (incompr/Boussinesq)

$$\frac{\partial \underline{v}}{\partial t} = -\underline{\nabla}\left(\frac{P}{\rho} + \frac{v^2}{2}\right) + \underline{v} \times \underline{\omega} + \frac{\underline{J} \times \underline{B}}{c\rho}$$

$$= -\underline{\nabla}\left(\frac{P}{\rho} + \frac{B^2}{8\pi\rho} + \frac{v^2}{2}\right) + \underline{v} \times \underline{\omega} + \frac{\underline{B} \cdot \underline{\nabla}\, \underline{B}}{4\pi\rho}$$

$$= -\underline{\nabla}\left(P/\rho + \frac{B^2}{8\pi} + \frac{v^2}{2}\right) + \underline{v} \times \underline{\omega} + \frac{\underline{B}_0 \partial_z \tilde{B}_\perp}{4\pi\rho} + \frac{\underline{\tilde{B}}_\perp \cdot \underline{\nabla}_\perp \underline{\tilde{B}}_\perp}{4\pi\rho}$$

(no B_0 curv)

so $\underline{\nabla} \times$

$$\frac{\partial \underline{\omega}}{\partial t} = \underline{\nabla} \times \underline{v} \times \underline{\omega} + \frac{B_0}{4\pi\rho_0} \partial_z \underline{\nabla} \times \underline{\tilde{B}_\perp}$$

$$+ \frac{\underline{\tilde{B}_\perp}}{4\pi\rho_0} \cdot \nabla_\perp \left(\underline{\nabla} \times \underline{\tilde{B}_\perp} \right)$$

$$= -\underline{v} \cdot \underline{\nabla} \omega + \underline{\omega} \cdot \underline{\nabla} \underline{v} + \frac{B_0}{4\pi\rho} \partial_z \underline{\nabla} \times \underline{\tilde{B}_\perp}$$

$$+ \frac{\underline{\tilde{B}_\perp}}{4\pi\rho} \cdot \nabla_\perp \left(\underline{\nabla} \times \underline{\tilde{B}_\perp} \right)$$

$\hat{z} \cdot (\ ) \Rightarrow$

$$\Rightarrow \frac{d}{dt} \omega_z = \cancel{\underline{\omega} \cdot \underline{\nabla} v_z} + \frac{B_0}{4\pi\rho_0} \partial_z \tilde{J}_{\parallel} + \frac{\nabla A_\parallel \times \hat{z}}{4\pi\rho_0} \cdot \nabla \tilde{J}_{\parallel}$$

$$\omega_z \rightarrow \nabla^2 \phi$$

so
$$\boxed{\frac{d}{dt} \nabla^2 \phi = \partial_t \nabla^2 \phi + \underline{v} \cdot \underline{\nabla} \nabla^2 \phi = \frac{B_0}{4\pi\rho_0} \partial_z \tilde{J}_z + \frac{\tilde{B}}{4\pi\rho_0} \cdot \nabla \tilde{J}_z}$$

→ Vorticity eqn!

Alternative Approach:

①

$\hat{n} \cdot \left(\underline{E} = \eta \underline{J} \right)$

→ as before!

②

$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0$, continuity!

$\rho = (n_i - n_e) e$

and QN ⇒

$$\boxed{\underline{\nabla} \cdot \underline{J} = 0}$$ → generic!

⇒

$\underline{\nabla}_\perp \cdot \underline{J}_\perp = -\nabla_\parallel J_\parallel$.

advective

$$\left[\; n_0 m_i \frac{d\underline{V}_0}{dt} = n_0 \underline{E} - \underline{\nabla} P + n_0 e \frac{\underline{V} \times \underline{B}_0}{c} \right.$$

→ $O(\omega/\Omega)$ expansion, Ω
low P (T_i) .

$\underline{J}_\perp \simeq (n_i \underline{V}_E - n_e \underline{V}_E) e + n_i e \underline{V}_{pol}$

$E \times B$ current, cancels. polarization current → ions
$(m_i \gg m_e)$

$$\boxed{\underline{\nabla}_\perp \cdot \left(n_0 \underline{V}_{pol} \right) = -\frac{1}{e} \nabla_\parallel J_\parallel}$$ → vort eqn

$$= -1/e \left(\partial_z \tilde{J}_\parallel + \tilde{\underline{B}}_\perp \cdot \underline{\nabla}_\perp J_\parallel \right)$$

and back to vorticity eqn.!

⇒ can extend to H-W, H-M, 3 field, ITG....

→ Now, can relate routes to RMHD:

[So can come to RMHD by different orders of strong field and fluid approx.

Now, extensions:

Dynamics of Zonal Flows in DWT

→ An Introduction

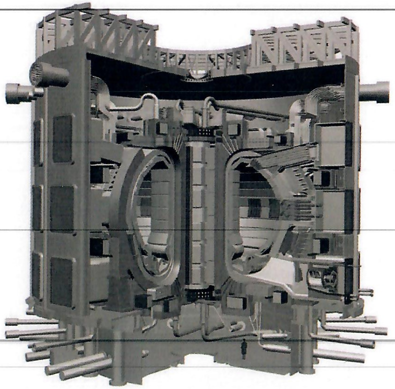
What regulates radial extent? ➔ Shear Flows ‘Natural’ to Tokamaks

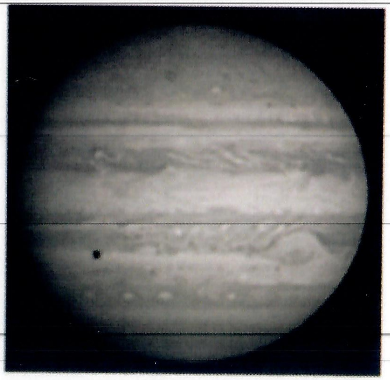
- Zonal Flows Ubiquitous for:

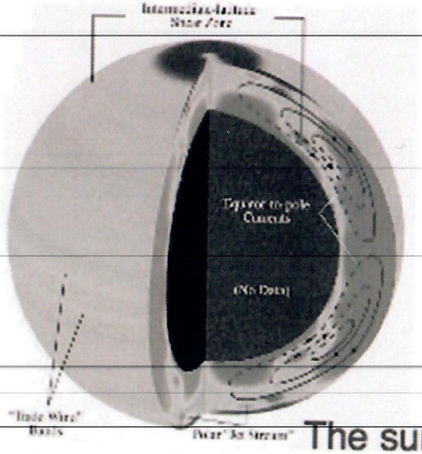
  ~ 2D fluids / plasmas $R_0 < 1$

  Rotation $\vec{\Omega}$, Magnetization $\vec{B}_0$, Stratification

  Ex: MFE devices, giant planets, stars…





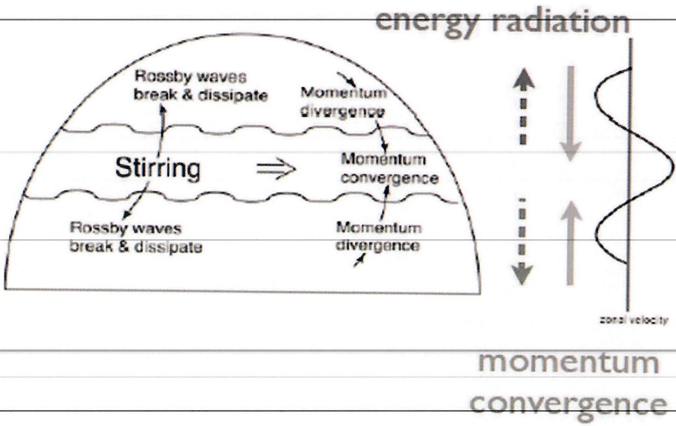


The sun

Heuristics of Zonal Flows a): How Form?

Simple Example: Zonally Averaged Mid-Latitude Circulation

- classic GFD example: Rossby waves + Zonal flow (c.f. Vallis '07, Held '01)
- Key Physics:



Rossby Wave:

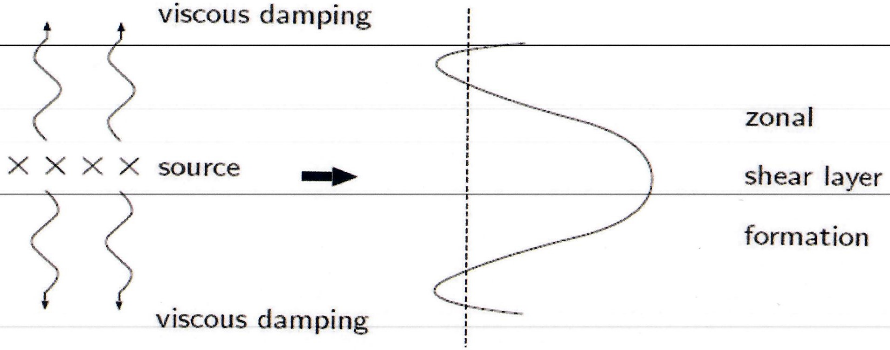
$$\omega_k = -\frac{\beta k_x}{k_\perp^2}$$

$$v_{gy} = 2\beta \frac{k_x k_y}{(k_\perp^2)^2}, \quad \langle \tilde{v}_y \tilde{v}_x \rangle = \sum_{\vec{k}} -k_x k_y \left| \hat{\phi}_{\vec{k}} \right|^2$$

$\therefore v_{gy} v_{phy} < 0$ → Backward wave!

➔ Momentum convergence at stirring location

- …“the central result that a rapidly rotating flow, when stirred in a localized region, will converge angular momentum into this region.” (I. Held, ’01)
- Outgoing waves $\Rightarrow$ incoming wave momentum flux

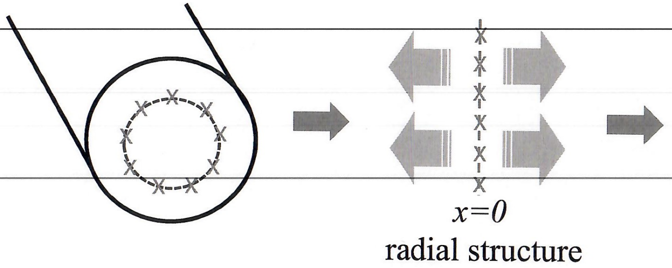


- Local Flow Direction (northern hemisphere):
  - eastward in source region
  - westward in sink region
  - set by $\beta > 0$
  - Some similarity to spinodal decomposition phenomena
    → Both ‘negative diffusion’ phenomena

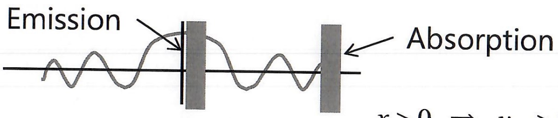
Wave-Flows in Plasmas

MFE perspective on Wave Transport in DW Turbulence

- localized source/instability drive intrinsic to drift wave structure



  - couple to damping ↔ outgoing wave



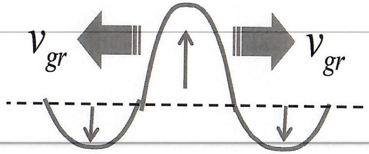
  - $v_{gr} = -2\rho_s^2 \frac{k_\theta k_r v_*}{(1+k_\perp^2 \rho_s^2)^2}$

$x > 0 \Rightarrow v_{gr} > 0$

$v_* < 0 ,\ k_r k_\theta > 0$

$$\langle v_{rE} v_{\theta E} \rangle = -\frac{c^2}{B^2} |\phi_{\bar{k}}|^2 k_r k_\theta < 0$$

- outgoing wave energy flux → incoming wave momentum flux → counter flow spin-up!



- zonal flow layers form at excitation regions

Zonal Flows I

- What is a Zonal Flow?
  - $n = 0$ potential mode; $m = 0$ (ZFZF), with possible sideband (GAM)
  - toroidally, poloidally symmetric *ExB* shear flow
- Why are Z.F.'s important?
  - Zonal flows are secondary (nonlinearly driven):
    - modes of minimal inertia (Hasegawa et. al.; Sagdeev, et. al. '78)
    - modes of minimal damping (Rosenbluth, Hinton '98)
    - drive zero transport ($n = 0$)
  - natural predators to feed off and retain energy released by gradient-driven microturbulence

Zonal Flows II

- Fundamental Idea:
  - Potential vorticity transport + 1 direction of translation symmetry
    → Zonal flow in magnetized plasma / QG fluid
  - Kelvin's theorem is ultimate foundation

- G.C. ambipolarity breaking → polarization charge flux → Reynolds force
  - Polarization charge $\Rightarrow$ $-\boxed{\rho^2}\nabla^2\phi = \boxed{n_{i,GC}(\phi) - n_e(\phi)}$

    polarization length scale ↲ ↳ ion GC ↳ electron density

  - so $\Gamma_{i,GC} \neq \Gamma_e$ $\Rightarrow$ $\rho^2\left\langle \tilde{v}_{rE}\nabla_\perp^2\tilde{\phi}\right\rangle \neq 0$ $\Leftrightarrow$ 'PV transport'

    ↳ polarization flux → What sets cross-phase?

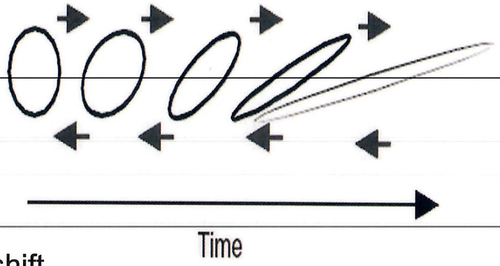
  - If 1 direction of symmetry (or near symmetry):

    $-\rho^2\left\langle \tilde{v}_{rE}\nabla_\perp^2\tilde{\phi}\right\rangle = -\partial_r\left\langle \tilde{v}_{rE}\tilde{v}_{\perp E}\right\rangle$ (Taylor, 1915)

    $-\partial_r\left\langle \tilde{v}_{rE}\tilde{v}_{\perp E}\right\rangle$ $\Rightarrow$ Reynolds force $\Rightarrow$ Flow

Zonal Flows Shear Eddys I

- Coherent shearing: (Kelvin, G.I. Taylor, Dupree’66, BDT‘90)
  - radial scattering + $\langle V_E \rangle'$ → hybrid decorrelation
  - $k_r^2 D_\perp \to \quad (k_\theta^2 \langle V_E \rangle'^2 D_\perp / 3)^{1/3} = 1/\tau_c$
  - Akin shear dispersion
  - shaping, flux compression: Hahm, Burrell ’94



- Other shearing effects (linear):
  - spatial resonance dispersion:

Response shift and dispersion

$$\omega - k_\parallel v_\parallel \Rightarrow \omega - k_\parallel v_\parallel - k_\theta \langle V_E \rangle'(r - r_0)$$

  - differential response rotation → especially for kinetic curvature effects

→ N.B. Caveat: Modes can adjust to weaken effect of external shear

(Carreras, et. al. ‘92; Scott ‘92)

Shearing II

- Zonal Shears: Wave kinetics (Zakharov et. al.; P.D. et. al. '98, et. seq.)

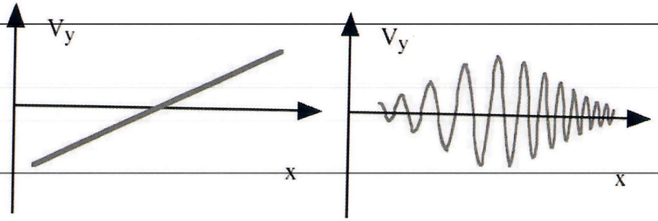
  Coherent interaction approach (L. Chen et. al.)

- $dk_r / dt = -\partial(\omega + k_\theta V_E)/\partial r$; $V_E = \langle V_E \rangle + \widetilde{V}_E$

  Mean shearing $: k_r = k_r^{(0)} - k_\theta V_E' \tau$

  Zonal Random shearing $: \langle \delta k_r^2 \rangle = D_k \tau$

  $$D_k = \sum_q k_\theta^2 \left| \widetilde{V}'_{E,q} \right|^2 \tau_{k,q}$$



- Wave ray chaos (not shear RPA) underlies D_k → induced diffusion
- Induces wave packet dispersion
- Applicable to ZFs and GAMs

- Mean Field Wave Kinetics

$$\frac{\partial N}{\partial t} + (\vec{V}_{gr} + \vec{V}) \cdot \nabla N - \frac{\partial}{\partial r}(\omega + k_\theta V_E) \cdot \frac{\partial N}{\partial \vec{k}} = \gamma_{\vec{k}} N - C\{N\}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\langle N \rangle - \frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r}\langle N \rangle = \gamma_{\vec{k}} \langle N \rangle - \langle C\{N\} \rangle}$$

Zonal shearing → computed using modulational response

Shearing III

- Energetics: Books Balance for Reynolds Stress-Driven Flows!
- Fluctuation Energy Evolution – Z.F. shearing

$$\int d\vec{k}\,\omega\left(\frac{\partial}{\partial t}\langle N\rangle-\frac{\partial}{\partial k_r}D_k\frac{\partial}{\partial k_r}\langle N\rangle\right)\Rightarrow\frac{\partial}{\partial t}\langle\varepsilon\rangle=-\int d\vec{k}V_{gr}(\vec{k})D_{\vec{k}}\frac{\partial}{\partial k_r}\langle N\rangle \qquad V_{gr}=\frac{-2k_rk_\theta V_*\rho_s^2}{\left(1+k_\perp^2\rho_s^2\right)^2}$$

Point: For $d\langle\Omega\rangle/dk_r<0$, Z.F. shearing damps wave energy

- Fate of the Energy: Reynolds work on Zonal Flow

Modulational Instability

$$\partial_t\delta V_\theta+\partial\left(\delta\left\langle\tilde{V}_r\tilde{V}_\theta\right\rangle\right)/\partial r=-\gamma\delta V_\theta$$

$$\delta\left\langle\tilde{V}_r\tilde{V}_\theta\right\rangle\sim\frac{k_rk_\theta\delta\Omega}{(1+k_\perp^2\rho_s^2)^2}$$

N.B.: Wave decorrelation essential:
Equivalent to PV transport
(c.f. Gurcan et. al. 2010)

Modulation → inhomogeneity
in PV mixing

- Bottom Line:
  - Z.F. growth due to shearing of waves
  - “Reynolds work” and “flow shearing” as relabeling → books balance
  - Z.F. damping emerges as critical; MNR ‘97

Approaches to Modulation

~ Weak, Wave Turbulence Problems

→ Quasi-particle, Wave Kinetics ➔ δN

See: P.D. Itoh, Itoh, Hahm '05 PPCF

→ Envelope Theory, Generalized NLS ➔ ψ

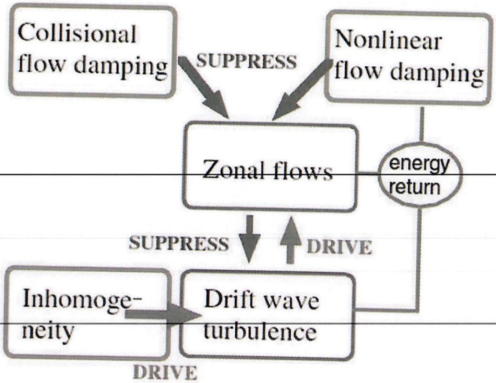
See: O.D. Gurcan, P.D. '2014 J. Phys. A.

N.B.: Representation of PV mixing and its inhomogeneity is crucial

Feedback Loops I

- Closing the loop of shearing and Reynolds work

- Spectral 'Predator-Prey' equations



Collisional flow damping

SUPPRESS

Nonlinear flow damping

Zonal flows

energy return

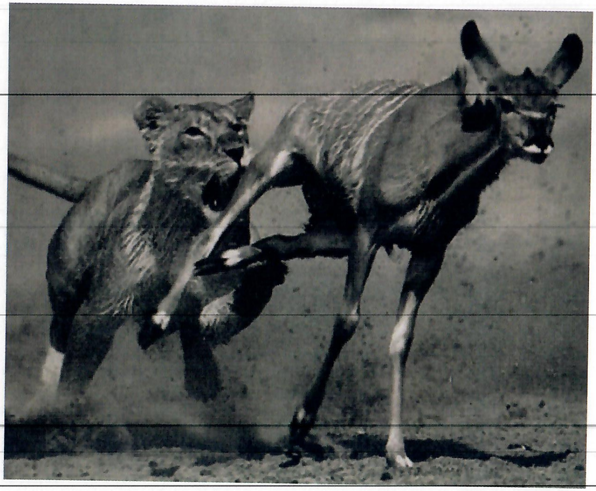
SUPPRESS

DRIVE

Inhomogeneity

Drift wave turbulence

DRIVE



Prey → Drift waves, <*N*>

$$\frac{\partial}{\partial t}\langle N\rangle - \frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r}\langle N\rangle = \gamma_k \langle N\rangle - \frac{\Delta\omega_k}{N_0}\langle N\rangle^2$$

Predator → Zonal flow, $|\phi_q|^2$

$$\frac{\partial}{\partial t}|\phi_q|^2 = \Gamma_q\left[\frac{\partial\langle N\rangle}{\partial k_r}\right]|\phi_q|^2 - \gamma_d|\phi_q|^2 - \gamma_{NL}[|\phi_q|^2]|\phi_q|^2$$

Feedback Loops II

- Recovering the 'dual cascade':
  - Prey → <N> ~ <Ω> ⇒ induced diffusion to high k_r { ⇒ Analogous → forward potential enstrophy cascade; PV transport
  - Predator → $|\phi_q|^2 \sim \left\langle V_{E,\theta}^2 \right\rangle$ { ⇒ growth of *n=0, m=0* Z.F. by turbulent Reynolds work; ⇒ Analogous → inverse energy cascade

- Mean Field Predator-Prey Model

  (P.D. et. al. '94, DI²H '05)

$$\frac{\partial}{\partial t} N = \gamma N - \alpha V^2 N - \Delta\omega N^2$$

$$\frac{\partial}{\partial t} V^2 = \alpha N V^2 - \gamma_d V^2 - \gamma_{NL}(V^2) V^2$$

System Status

State	No flow	Flow ($\alpha_2 = 0$)	Flow ($\alpha_2 \neq 0$)
N (drift wave turbulence level)	$\frac{\gamma}{\Delta\omega}$	$\frac{\gamma_d}{\alpha}$	$\frac{\gamma_d + \alpha_2 \gamma \alpha^{-1}}{\alpha + \Delta\omega \alpha_2 \alpha^{-1}}$
V^2 (mean square flow)	0	$\frac{\gamma}{\alpha} - \frac{\Delta\omega \gamma_d}{\alpha^2}$	$\frac{\gamma - \Delta\omega \gamma_d \alpha^{-1}}{\alpha + \Delta\omega \alpha_2 \alpha^{-1}}$
Drive/excitation mechanism	Linear growth	Linear growth	Linear growth Nonlinear damping of flow
Regulation/inhibition mechanism	Self-interaction of turbulence	Random shearing, self-interaction	Random shearing, self-interaction
Branching ratio $\frac{V^2}{N}$	0	$\frac{\gamma - \Delta\omega \gamma_d \alpha^{-1}}{\gamma_d}$	$\frac{\gamma - \Delta\omega \gamma_d \alpha^{-1}}{\gamma_d + \alpha_2 \gamma \alpha^{-1}}$
Threshold (without noise)	$\gamma > 0$	$\gamma > \Delta\omega \gamma_d \alpha^{-1}$	$\gamma > \Delta\omega \gamma_d \alpha^{-1}$