Physics 218c

Lecture 3c – PV and Drift Waves – Part Ic

Recall: Derived Hasegawa-Wakatani Model

$$\rho_s^2 \frac{d}{dt} \nabla_\perp^2 \phi = D_\parallel \nabla_\parallel^2 \left(\tilde{\phi} - \frac{T}{|e|} \frac{\tilde{n}}{n} \right)$$

$$\frac{d}{dt} \delta n + \frac{\tilde{v}_r}{n_0} \partial_r \langle n \rangle = D_\parallel \nabla_\parallel^2 \left(\tilde{\phi} - \frac{T}{|e|} \delta n \right)$$

→ drift instability, due $\begin{cases} \text{dissipation} \\ \omega \leftrightarrow \phi \text{ phase} \end{cases}$

$k_\parallel^2 v_{th}^2 / \omega \nu \neq 0$

⇒ $\langle \tilde{v}_r \tilde{n} \rangle \neq 0$.

$$\frac{\tilde{n}}{n} = \frac{|e|\hat{\phi}}{T} + \tilde{h}$$

$\tilde{h}$ non-adiabatic electrons.

→ $\alpha \lessgtr 1$ regimes

Usual DW regime is $\alpha > 1$

→ Ohm's Law balance is fundamental.

→ $\alpha \to \infty$ recovers Hasegawa - Mima

→ often written with viscosity, particle diffusion

<u>Now</u>

- important class of modes →
<u>Zonal Modes</u>

- $k_{\parallel} = 0$, $k_{\theta} = 0$ distinguished by symmetry

$n_Z = \delta n(r)$ → dynamic density profile

i.e. $\langle n \rangle = n_0(r) + \delta n(r)$

$n_0(r)$ → fixed

$\delta n(r)$ → zonal density perturbation

$\langle n \rangle$ → what is measured or seen.

$\delta n(r)$ ⇒ feedback of fluctuations on profile i.e. <u>transport</u>

cont'd

$\nabla_\perp^2 \phi_Z = \partial_r^2 \phi(r)$ → Zonal vorticity
(polarization)

⇒ $V_{E\times B}$ → "Zonal flow"

~ particle flow at $E \times B$ velocity ($\neq$ Mass flow $\int d^3v \, \underline{v} f$).

n.b. arises naturally via vorticity!

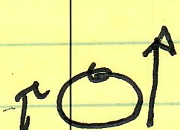
Zonal Flow ⇒ $E\times B$ shearing

$$\frac{d}{dt} = \partial_t + \underline{V}_E \cdot \underline{\nabla} + \underline{\tilde{V}} \cdot \underline{\nabla}$$

$$= \partial_t + \langle V_E(r) \rangle \partial_y + \underline{\tilde{V}} \cdot \nabla$$

↓

sheared $\underline{E} \times \underline{B}$ flow



$\langle V_E(r) \rangle = \langle \partial_r \phi \rangle$

⇒ limits response.

$\langle \nabla_\perp^2 \phi \rangle =$ [redacted] $\bar{V}_E' + \langle \nabla_r^2 \tilde{\phi}(r) \rangle$

fixed,
large scale

For zonal density, flow evolution, zonally average H-W Eqns.

$\Rightarrow$

$$\partial_t \langle n \rangle + \partial_r \langle \tilde{v}_r \tilde{n} \rangle = S_n + D \nabla_r^2 \langle n \rangle$$

$$\partial_r \langle \tilde{v}_r \tilde{h} \rangle$$

→ particle flux evolves ~~zonal density~~ density profile

→ $\langle \tilde{v}_r \tilde{n} \rangle$ calculated by quasi-linear theory, previously

→ QL good, as dynamics dissipative

→ Concern: NL frequency shift, due zonal perturbations, shear

i.e. fate $\omega - \omega_* \rightarrow \omega - \omega_* [1 + \delta n']$

→ zonal density (corrugation) feeds back on density profile, which evolves mode. (i.e. QL).

in QL

$$\partial_t \delta n = \partial_r D_n \partial_r \delta n$$

$$\frac{k_\parallel^2 v_{th}^2}{\nu_{ei}} > \omega \ldots$$

$$\omega \simeq \omega_{\underline{k}}.$$

$$D_n \simeq \sum_{\underline{k}} |\tilde{v}_{r\underline{k}}|^2 \left[\frac{k_\perp^2 \rho_s^2}{1+k_\perp^2\rho_s^2}\right] \frac{\omega_{\underline{k}} \nu_{ei}}{k_\parallel^2 v_{th}^2}$$

$\cancel{\alpha}$ → dissipative phase shift

→ collisions

Compare:

$$\partial_t \langle f\rangle = \partial_v D_v \partial_v \langle f\rangle$$

$$D_v = \sum_{\underline{k}} \frac{q^2}{m^2} |E_{\underline{k}}|^2 \frac{|\gamma_{\underline{k}}|}{(\omega-kv)^2+|\gamma_{\underline{k}}|^2}$$

~ $\pi\,\delta(\omega-kv)$ at st. state

~ resonant phase shift

→ irreversibility via chaos

(zonal avg. vorticity)

Also have:

$$\partial_t \langle \nabla_r^2 \phi \rangle + \partial_r \langle \tilde{V}_r \nabla_\perp^2 \tilde{\phi} \rangle = \mu \nabla_r^2 \langle \nabla_r^2 \phi \rangle$$

polarization
charge

- zonal flow evolution clear.

- key → vorticity flux

  Must treat on equal footing with

  mean field density evolution.

- but, what is the physics of the vorticity flux?

$$\langle \tilde{V}_r \nabla_\perp^2 \tilde{\phi} \rangle = \langle \partial_y \tilde{\phi} \, (\partial_x^2 \tilde{\phi} + \partial_y^2 \tilde{\phi}) \rangle$$

$$= \langle \partial_y \tilde{\phi} \, \partial_x^2 \tilde{\phi} \rangle$$

i.e.
$i k_y k_y^2 \rightarrow$
odd in k_y

$$\langle \ \rangle = \sum_k$$

then

$$\langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle = \langle \partial_x (\partial_y \tilde{\phi}\, \partial_x \tilde{\phi}) \rangle$$
$$- \langle \partial_{y,x} \tilde{\phi}\, \partial_x \tilde{\phi} \rangle$$

$\Downarrow$
0
odd, k_y

$$\doteq \langle \partial_x (\partial_y \tilde{\phi}\, \partial_x \tilde{\phi}) \rangle$$
$$= \partial_x \left(\langle \partial_y \tilde{\phi}\, \partial_x \tilde{\phi} \rangle \right)$$

$\Downarrow$
$E \times B$ Reynold's stress
i.e. $\langle \tilde{v}_r \tilde{v}_\theta \rangle$

then,

$$\langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle \equiv \text{Reynolds } \underline{\text{Force}} \ (E \times B)$$

i.e. $\boxed{\text{Vorticity flux drives } \underline{E \times B} \text{ flow}}$.

— the Physics!

N.B.

→ $\underline{1}$ direction of symmetry utilized.

→ McIntyre and R. Wood - Theorem:
↳ $\langle v_r \nabla^2 \phi \rangle \neq 0$
⇒ PV mixing and 1 direction of symmetry
⇒ zonal flow formation.

⇝ [Welcome to the Taylor Identity
(G. I. Taylor, 1915)

links vorticity flux ⇔ Reynolds stress

- important

- generalizes to Eliassen - Palm relations in GFD.

⇝ Extensions - left as HW.

a) $\langle \tilde{B}_r \tilde{J}_{\parallel} \rangle = ?$

- Magnetic Taylor Identity

b) Relate a) to charge balance

i.e.

$$\partial_t \langle \nabla_\perp^2 \phi \rangle = -\partial_r \left[\langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle - \langle \tilde{B}_r \tilde{J}_{\parallel} \rangle \right]$$

meaning?

→ Relate zonal modes $\left\{ \begin{array}{c} \delta n \\ \text{Vort.} \end{array} \right\}$ and relaxation?

This brings us back to PV;

- work in limit of $\nu = 0$
- add H-W eqns

$\Rightarrow \quad \frac{d}{dt} \rightarrow \tilde{v}$ only

$$\frac{d}{dt}\left(\delta n - \rho_s^2 \nabla_\perp^2 \tilde{\phi}\right) + \tilde{v}_r \partial_r \left(\frac{\langle n \rangle}{n_0} - \frac{\rho_s^2}{n_0} \langle \nabla_\perp^2 \phi \rangle\right)$$

$$- \nu \nabla_\perp^2 \left(\delta n - \rho_s^2 \nabla_\perp^2 \tilde{\phi}\right) = 0$$

$$PV \Big|_{H-W} = \underset{\text{const}}{\delta n} - \rho_s^2 \nabla_\perp^2 \phi \equiv q$$

$$\equiv n_0 + \delta n - \rho_s^2 \nabla_\perp^2 \phi$$

$$= \underbrace{n_0 + \phi}_{} + \underset{\text{non-Boltzmann}}{h} - \rho_s^2 \nabla_\perp^2 \phi$$

NB: change $n \rightarrow$ change flow shear.

$q \equiv$ total charge, GC + polarization

$\downarrow$ PV. $\quad$ GC $\downarrow n$ $\quad$ polarization $\downarrow -\nabla_\perp^2\phi$

$$\frac{d}{dt}\tilde{q} + \tilde{v}_r \partial_r \langle q \rangle - \nu \nabla_\perp^2 \tilde{q} = 0$$

→ charge conservation.

Now, $\boxed{\langle \tilde{q}^2/2 \rangle \equiv \text{Potential Enstrophy}}$

so

spreading → turbulent transport potential enstrophy.

$$\left[\partial_t \frac{\langle \tilde{q}^2 \rangle}{2} + \partial_r \frac{\langle \tilde{v}_r \tilde{q}^2 \rangle}{2} + \langle \tilde{v}_r \tilde{q} \rangle \partial_r \langle q \rangle + \nu \frac{\langle (\nabla_\perp \tilde{q})^2 \rangle}{2} = 0\right.$$

$\langle \tilde{v}_r \tilde{q} \rangle$ → PV flux

production.

Charge balance relation.

N.B. – $\langle \tilde{q}^2 \rangle \leftrightarrow \langle \delta f^2 \rangle$

– 2D: $\langle \tilde{q}^2 \rangle$ and $\mathcal{E} = \langle \tilde{n}^2 \rangle + \langle (\nabla \tilde{\phi})^2 \rangle$ conserved → selective decay for minimum enstrophy, dual cascade and

$$- \langle \tilde{v}_r \tilde{q} \rangle \, \partial_r \langle q \rangle \rightarrow \text{production}$$

$$= \Big[\langle \tilde{v}_r \underset{n}{\tilde{n}} \rangle - \rho_s^2 \langle \tilde{v}_r \nabla_\perp^2 \tilde{\phi} \rangle \Big] \partial_r \langle q \rangle$$

$$= \Big[\langle \tilde{v}_r \tilde{n} \rangle - \rho_s^2 \partial_x \langle \tilde{v}_r \tilde{v}_\theta \rangle \Big] \partial_r \langle q \rangle$$

using Taylor Identity

so

$$\langle \tilde{v}_r \tilde{n} \rangle - \rho_s^2 \partial_x \langle \tilde{v}_r \tilde{v}_\theta \rangle \qquad \text{transport potential enstrophy}$$

$$= \frac{-1}{\partial_r \langle q \rangle} \left[\partial_t \frac{\langle \tilde{q}^2 \rangle}{2} + \partial_r \frac{\langle \tilde{v}_r \tilde{q}^2 \rangle}{2} + \nu \frac{\langle (\nabla \tilde{q})^2 \rangle}{2} \right]$$

For stationary state, $\partial_t \to 0$ $\quad \partial_r \langle q \rangle \neq 0$ (shear flow instability)

$\Rightarrow$

$$\underbrace{\langle \tilde{v}_r \tilde{n} \rangle}_{\text{particle flux}} - \rho_s^2 \partial_x \underbrace{\langle \tilde{v}_r \tilde{v}_\theta \rangle}_{\text{Reynolds force,}} = \frac{-1}{\partial_r \langle q \rangle} \left[\partial_r \frac{\langle \tilde{v}_r \tilde{q}^2 \rangle}{2} + \nu \frac{\langle (\nabla \tilde{q})^2 \rangle}{2} \right]$$

so

- at steady state,

$$\langle \tilde{v}_r \tilde{n} \rangle \cong \rho_s^2 \partial_x \langle \tilde{v}_r \tilde{v}_\theta \rangle$$

$$+ \, O\left(\text{spreading} + \text{dissipation} \right)$$

⇒ relates particle flux and ZF drive (vorticity flux).

⇒ indicates importance of zonal flows, due PV conservation.

One can go further:

$$\partial_t \langle V_E \rangle + \mu \langle V_E \rangle = -\partial_r \langle \tilde{v}_r \tilde{v}_\theta \rangle$$

$$= \langle \tilde{v}_r \rho_s^2 \nabla_\perp^2 \tilde{\phi} \rangle$$

so

$$\partial_t \langle \frac{\tilde{q}^2}{2} \rangle + \partial_r \langle \tilde{v}_r \frac{\tilde{q}^2}{2} \rangle + \left[\langle \tilde{v}_r \tilde{n} \rangle - \partial_t \langle V_E \rangle - \mu \langle V_E \rangle \right]$$

$$* \, \frac{\partial \langle q \rangle}{\partial r} + \nu \langle (\nabla \tilde{q})^2 \rangle = 0$$

then for static $\partial\langle q\rangle/\partial r$, and
$q = \nabla^2\phi - n$:

$$\boxed{\partial_t\left\{\frac{\langle\tilde{q}^2\rangle}{\partial\langle q\rangle/\partial r} + \langle V_E\rangle\right\} = \langle\tilde{V}_r\tilde{\underline{n}}\rangle - \frac{1}{\frac{\partial\langle q\rangle}{\partial r}}\left[\langle(\nabla\tilde{q})^2\rangle + \partial_r\langle\tilde{V}_r\tilde{q}^2\rangle\right] - \mu\langle V_E\rangle}$$

Charney - Drazin Thm ! (variant)

- ① + ②
  → Zonal flow driven by time evolution of WMD (Wave Mom. Dens.)

  i.e. $k\theta\langle\tilde{q}^2\rangle / k\theta\,\partial\langle q\rangle/\partial r$ or; wave activity density

  Pseudomomentum.

→ Absent ③ - ⑤, accelerate flow only if change WMD → fluctuation intensity

– ③ particle flux (turbulent) drive
\+
⑤ driven flow at st. state.

– $\partial\langle\tilde{v}\rangle/\partial r \neq 0$ ⇒ Rayleigh-Kuo (shear flow inst.)

④
– spreading enters balance.

C-D theorem illustrates constraint of PV conservation on zonal flow production and relation to transport.

HW: Derive C-D theorem for forced Charney equation. Compare to H-W case.

Lecture 3d – Electromagnetics and Reduced MHD.

Now, electromagnetics ....

What is the simplest model ?

⇒ Reduced MHD

↳ Drift Alfven, 4 field (3 versions: Hasegawa, Drake, Hazeltine), 6 field (Xu) .....

N.B. Above list : Reduced MHD + H-W

⇒ everything else.

⇒ Key to Reduced MHD : time scales

cf: [2/8 b notes Kulsrud]

3 Modes MHD :

→ Fast → Magnetosonic : $\omega^2 = k_\perp^2 (V_A^2 + C_s^2)$

→ Intermediate → Shear Alfven : $\omega^2 = k_\parallel^2 V_A^2$

→ Slow → acoustic (parallel) : $\omega^2 = k_\parallel^2 C_s^2$

+ Entropy → $\omega = 0$.

Point: Eliminate Magnetosonic time scale!

$$\omega \ll \omega_{M.S.}$$

How?

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v} = -\frac{\nabla p}{\rho} + \frac{\underline{J}\times\underline{B}}{c}$$

$$\frac{d\underline{v}}{dt} = -\underline{\nabla} w - \frac{1}{\rho}\nabla\frac{B^2}{8\pi} + \frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi\rho}$$

$$= -\underline{\nabla}\left(\frac{p}{\rho} + \frac{B^2}{8\pi\rho}\right) + \frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi\rho}$$

so

$$\cancel{\frac{d}{dt}\underline{\nabla}\cdot\underline{v}}_{0} = -\nabla^2\left(\frac{p}{\rho} + \frac{B^2}{8\pi\rho}\right)$$

$$\Rightarrow \quad \delta p \sim -\frac{\underline{B}\cdot\delta B}{4\pi} \quad \Rightarrow \text{perturbed pressure balance.}$$

i.e. $\left\{\begin{array}{l} \tau \gg \tau_{MS} \Rightarrow \text{pressure balance.} \\ \\ \perp \text{ incompressibility} \end{array}\right.$

Reduced MHD → Simplifying the representation:

→ strong magnetization — anisotropy

$\delta B_{\parallel} \to 0$; $\delta B_{\perp} = \nabla A_{\parallel} \times \hat{z}$

$\delta V_{\perp} = \nabla \phi \times \hat{z}$.

→ $\nu > \nu_{MS}$

Aside

→ $\boxed{\text{Reduced MHD}}$ → Reduced Representation for strong straight B_0
→ eliminates fast mode

Note: ① full MHD : 3 $\underline{V}$ components
2 $\underline{B}$ " " ($\underline{\nabla} \cdot \underline{B} = 0$)
ρ, p

⇒ 7 components

② if $\underline{\nabla} \cdot \underline{V} = 0$ ⇒ 4 components
($\rho = \text{const}$, p from $\underline{\nabla} \cdot \underline{V} = 0$)

③ strongly magnetized system ⇒ Reduced MHD
⇒ scalar equations for ϕ, ψ (2 scalar fields)

Now:
- assume strong B_z (strong magnetization → gyrokinetics) → later

"strong" ⟺ $\rho v^2 \sim p \ll B_z^2/8\pi$

so motion strongly anisotropic, and small scales generated in ⊥ direction only, as strong B_z inhibits line bending. (energy to perturb strong, high energy density field).

⇒ order : $B_z \sim \nabla_\perp \sim 1$

$B_\perp \sim \partial_z \sim O(\epsilon)$

Take $\rho \sim 1$, as $\underline{\nabla} \cdot \underline{V} = 0$ enforced by strong B_z:

$V_\perp^2 \sim p \sim B_\perp^2$ (i.e. equipartition of energy)
(springiness)

$\Rightarrow V_\perp \sim \epsilon$, $p \sim \epsilon^2$, $\partial_t \sim \underline{V}_\perp \cdot \underline{\nabla}_\perp \sim \epsilon$

and pressure balance ($\underline{\nabla} \cdot \underline{V} = 0$ / incompressibility)

$\delta(B_z^2) \sim 2B_z \delta(B_z) \sim p$

i.e. $\boxed{\omega \ll k(c_s^2 + V_A^2)^{1/2}}$ (ezborn)

idea is to order out the fast mode

$\Rightarrow \delta B_z \sim \epsilon^2$.

$\therefore$ to lowest order $\Rightarrow$ $B_z = \text{const}$.

Now then: ($\underline{\nabla} \cdot \underline{B} = 0$)

$$\boxed{\begin{aligned} \underline{B} &= \hat{z} \times \underline{\nabla}\psi + B_z \hat{z} \\ &= \nabla A_\parallel \times \hat{z} + B_z \hat{z} \qquad \psi = -A_\parallel \end{aligned}}$$

B rep. by single scalar potential

parallel comp. of vector pot.

$\underline{\nabla} \cdot \underline{B} = \partial_z \tilde{B}_z = \epsilon^3 \to 0$.

Similarly,

$\partial_z p \sim O(\epsilon^3)$, $\Rightarrow$ $\cancel{V_z}^{\,0} \ll V_\perp$
$J_\perp B_\perp \sim \epsilon^3$ neglect V_z.

Now, $\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi = -\frac{\underline{V}\times\underline{B}}{c}$

$$\Rightarrow \quad +\frac{1}{c}\frac{\partial \underline{A}}{\partial t} = \frac{\underline{V}\times\underline{B}}{c} - \underline{\nabla}\phi \qquad (*)$$

$$B_z = (\underline{\nabla}_\perp \times \underline{A}_\perp)\cdot\hat{z}$$

so $\partial_t A_\perp \sim \epsilon^3$ (a la $\partial_z \rho_z$)

$\therefore \quad \nabla_\perp\phi \cong \left(\frac{\underline{V}\times\underline{B}}{c}\right)_\perp$, in $(*)$ $\quad V_\perp \leftrightarrow \nabla\phi$.

{Inductive piece negligible.

$$\Rightarrow \quad \boxed{\underline{V}_\perp = c\,\frac{\hat{z}\times\underline{\nabla}\phi}{B_z}}$$

$\perp$ velocity

$\Rightarrow$ {motion $\perp$ is $\underline{E}\times\underline{B}$

Now, taking parallel component of $(*)$ $\qquad \psi \leftrightarrow A_\parallel$

(units!)

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial t}\psi + \underline{V}\cdot\underline{\nabla}\psi = B_z\partial_z\phi}$$

so have (vector potential / flux) equation:

$$\boxed{\frac{\partial \psi}{\partial t} + \underline{V}\cdot\underline{\nabla}\psi = B_z\,\partial_z\phi}$$

$\underline{V}\cdot\nabla\psi$ from

$\underline{B}\cdot\nabla\phi \rightarrow$

$B_0\,\partial_z\phi + \nabla\psi\times\hat{z}\cdot\nabla\phi$

equation of evolution of magnetic flux.

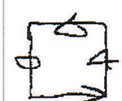
$= B_z \hat{z} + \hat{z}\times\nabla\psi$

94.

or, alternatively, $\boxed{\frac{\partial\psi}{\partial t} - \underline{B}\cdot\nabla\phi = 0}$.

Finally, for ϕ, write:

$\perp$ motion



cells of $E\times B$ drift, ('spin up' rate?)

$$\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V} = -\frac{\nabla p}{\rho_0} + \frac{\underline{J}\times\underline{B}}{c}$$

$(\nabla\times)\cdot\hat{z} \Rightarrow$ vorticity component ($\parallel \hat{z}$) evolution.

Dynamics in $\perp$ plane, $\nabla\phi\times\hat{z} = \underline{V}_\perp$, $\underline{\omega} \to \hat{z}\cdot\underline{\omega}$

$$\frac{\partial}{\partial t}\omega_z + \underline{V}_\perp\cdot\nabla\omega_z = \underbrace{-\cancel{\nabla\times\frac{\nabla p}{\rho_0}}}_{0} + \hat{z}\cdot\nabla\times\left(\frac{\underline{J}\times\underline{B}}{c}\right)$$

$$= \underline{B}\cdot\nabla J_z - \cancel{\underline{J}\cdot\nabla B_z} \qquad \delta B_z \sim \epsilon^3$$

$$\cong \underline{B}\cdot\nabla J_z$$

$\therefore$

$$\boxed{\frac{\partial}{\partial t}\omega_z + \underline{V}\cdot\nabla\omega_z = \underline{B}\cdot\nabla J}$$

but:

$$\omega_z = \hat{z}\cdot\nabla\times\underline{V} = \nabla^2\phi$$

$$J_z = \hat{z}\cdot(\nabla\times\underline{B})\frac{c}{\cancel{4\pi}} = \nabla^2\psi$$

So → Waves → time scales → Reduced MHD

so finally have:

$$\boxed{\frac{\partial}{\partial t}\nabla^2\phi + \underline{V}\cdot\underline{\nabla}\,\nabla^2\phi = B_z\frac{\partial}{\partial z}\nabla^2\psi + \underline{\tilde{B}}\cdot\underline{\nabla}\,\nabla^2\psi}$$

$B = B_0\hat{c}_{\parallel}$

Finally, have reduced MHD equation:

$E_{\parallel} = \eta J_{\parallel}$

$$\frac{\partial\psi}{\partial t} + \underline{V}\cdot\underline{\nabla}\psi = B_z\,\partial_z\phi + \eta\nabla^2\psi$$

$$\frac{\partial}{\partial t}\nabla^2\phi + \underline{V}\cdot\underline{\nabla}\,\nabla^2\phi - \nu\nabla^2\nabla^2\phi = \underline{\tilde{B}}\cdot\underline{\nabla}\,\nabla^2\psi + B_z\frac{\partial}{\partial z}\nabla^2\psi$$

vorticity in plane $\perp \hat{c}_{\parallel}$.

- note have reduced MHD to 2 scalar evolution equations

- does this look familiar?

- 2D dynamics + shear Alfven wave.

- nonlinearity → 2D dynamics.

even stronger!

$B_0 \partial_z \to 0$
$\partial_z \psi = 0$
$\partial_z \phi = 0$ l.O. + ~~avg z~~.

– for $\boxed{2D \quad MHD}$:

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t} \nabla^2\phi + \underline{V}\cdot\underline{\nabla}\, \nabla^2\phi = \underline{B}\cdot\underline{\nabla}\, \nabla^2\psi + \nu \nabla^2 \nabla^2\phi \\ &\frac{\partial \psi}{\partial t} + \underline{V}\cdot\underline{\nabla}\psi = \eta \nabla^2\psi \end{aligned}}$$

– ① Conservation Laws, etc. (HW)
0D

$$\frac{d}{dt} E = 0_{(\text{to } \eta, \nu)}, \quad E = \int d^3x \left[\frac{(\nabla\phi)^2}{2} + \frac{(\nabla\psi)^2}{2} \right]$$

② $H = A \cdot B \cong \underset{\text{const.}}{B_z}\psi$ $\qquad \int d^2x\, A^2 = \text{MSMP}$ (2D)
0D

$$\Rightarrow H = \int d^3x\, B_z \psi, \quad \frac{dH}{dt} = 0, \text{ to } O(\eta)$$

Ohm's Law (flux advection) is simple statement of helicity conservation.

? $\frac{\partial}{\partial t}\psi + \nabla\cdot\Gamma_\psi = \eta\nabla^2\psi$ form Γ_ψ s/t $\left\{ \begin{array}{l} H \text{ conserved} \\ E_M \text{ dissipated} \end{array} \right.$

③ $$\underline{K = \int d^3x\, \underline{V}\cdot\underline{B} = \int d^3x\, (\nabla\phi \cdot \nabla\psi)}$$

also conserved, to dissipation.

Alfven wave propagation cnbalance.

Alternative Approach:

① $\hat{n}\cdot(\underline{E} = \eta \underline{J})$

→ as before!

② $\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot\underline{J} = 0$, continuity!

$\rho = (n_i - n_e)\, Z$

and QN ⇒

$$\boxed{\underline{\nabla}\cdot\underline{J} = 0}$$ → generic!

⇒

$$\boxed{\underline{\nabla}_\perp\cdot\underline{J}_\perp = -\nabla_\parallel J_\parallel}$$

[inductive and electrostatic]

$$E_\parallel = -\frac{1}{c}\frac{\partial A_\parallel}{\partial t} - \nabla_\parallel \phi$$

$$+ \cancel{\nabla_\parallel P}$$

advective

$$\left[n_0 m_i \frac{d \underline{V}_0}{dt} = n_0 \underline{E} - \underline{\nabla} P + n_0 q \frac{\underline{V}\times\underline{B}_0}{c} \right.$$

→ $O(\omega/\Omega)$ expansion, Ω low $P(T_i)$.

$\underline{J}_\perp \simeq (n_i \underline{V}_E - n_0 \underline{V}_E)\, Z + n_i Z\, \underline{V}_{pol}$

$E\times B$ current, cancels. polarization current → ions ($m_i \gg m_e$)

$$\boxed{\underline{\nabla}_\perp\cdot(n_0 \underline{V}_{pol}) = -\frac{1}{Z}\nabla_\parallel J_\parallel}$$

→ vort eqn

$= {}^{-1}/Z \left(\partial_z \tilde{J}_\parallel + \tilde{B}_\perp \cdot \nabla_\perp \tilde{J}_\parallel \right)$

and back to vorticity eqn!

⇒ can extend to H-W, H-M, 3 field, ITG...

→ Now, can relate routes to RMHD:

So can come to RMHD by different orders of strong field and fluid approx.

Now, extensions...