Physics 216/116
Note 5a
[redacted] — Instabilities 1a (Life after spheres)

I.) → Convection / Rayleigh–Benard

[redacted] – convection: heat → ∇T → motion
broadly relevant
– ideal physics – Schwarzschild criterion
– dissipation and Rayleigh, Prandtl number
– Rayleigh-Benard Equations.
– Boundary conditions and $Ra_{crit.}$

II.) → Rotating Convection

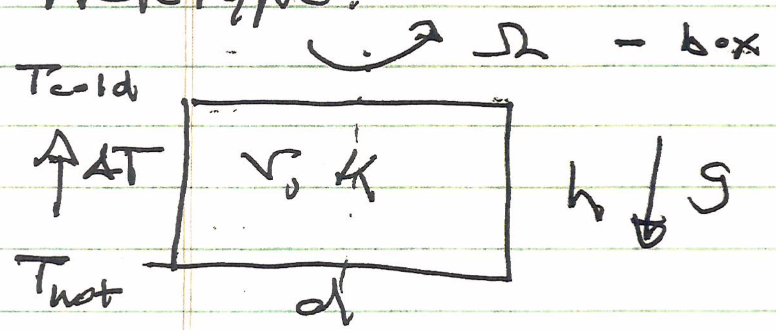
– freezing-in law with rotation
– Taylor-Proudman Theorem, implications for cells
– rotating convection
– physics of inertial waves
– relation magneto-convection, etc.

Convection (Rayleigh–Benard)

~ thousands papers
~ central to key problems of
  heat transport, general circulation

Prototype:

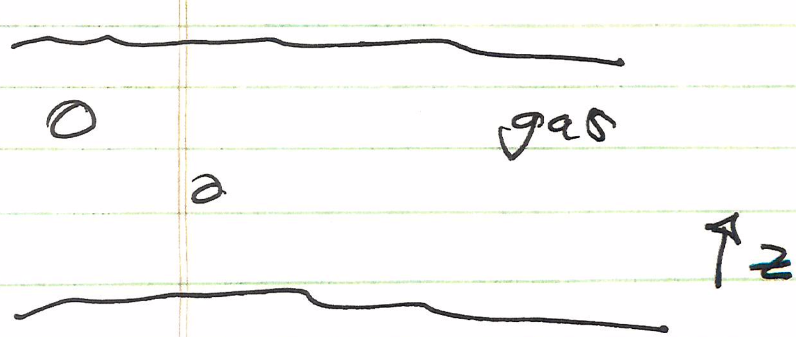
Ω – box can rotate



~ no slip B.C.

- critical ΔT or Ra for instability
- pattern structure
- effect rotation

i.)
Heuristics – Ideal Fluid / Gas
i.e. stellar atmosphere
→ Schwarzschild Criterion (ideal)



$$\frac{dp}{dz} = -\rho g$$

(eqbrm → hydrostatic)

$(g>0)$

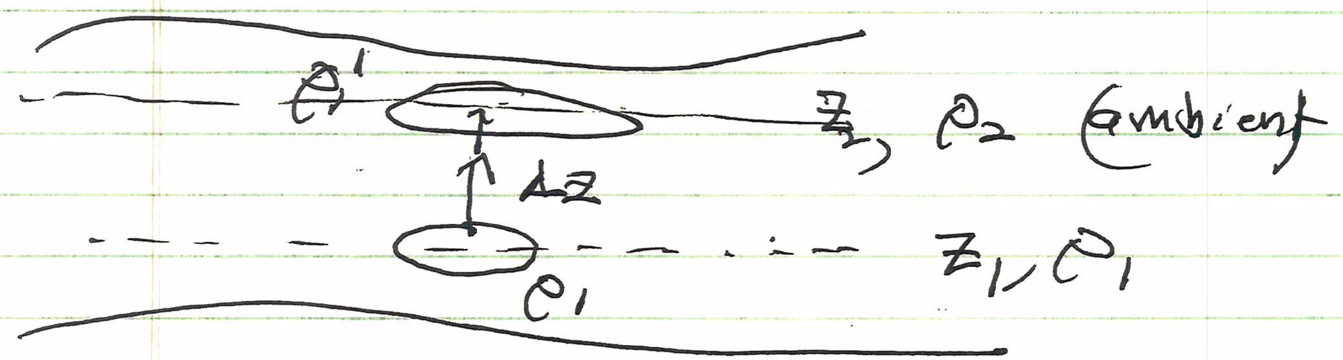
$d\theta/\partial z < 0$, $\partial P/\partial z < 0$.

$P\rho^{-\gamma} \simeq \text{const}$ → eqn. state (ideal gas)

Basic idea:

- virtual displacement of slug/blob ρ_1 upward to ρ_1' (after thermodynamic equilibration)
- $\rho_1' < \rho_2$ → blob buoyant, rises, unstable

  $\rho_1' > \rho_2$ → blob sinks, stable



For infinitesimal displacement:

Now, $\rho_2 = \rho_1 + \frac{\partial\rho}{\partial z}\Delta z$

What is ρ_1' → density of $\left\{\begin{matrix}\text{perturbed}\\ \text{displaced}\end{matrix}\right\}$ blob ρ_1?

Point:

- blob ρ_1 equilibrates pressure with surroundings → ρ_1'
- why? $\frac{\Delta z}{c_s} \ll \tau_{rise}$

i.e. rise time is long, slow
→ "incompressible"

then
$$P_1\rho_1^{-\gamma} = \text{const} = P_1'\rho_1'^{-\gamma}$$

but $P_1' = P_2$ → i.e. surroundings of test blob
($\delta p = 0$)
↳ incompressible

$$P_2 = \left(P_1 + \frac{dP}{dz}\Delta z\right)$$

so

$$P_1 \rho_1^{-\gamma} = \left(P_1 + \frac{dP_1}{dz}\Delta z\right)\rho_1'^{\,-\gamma}$$

$$(\rho_1'/\rho_1)^{\gamma} = \left(1 + \frac{\Delta z}{P}\frac{dP_1}{dz}\right)$$

$$\rho_1' = \rho_1\left(1 + \frac{\Delta z}{P_1}\frac{dP_1}{dz}\right)^{1/\gamma}$$

$$= \left(1 + \frac{1}{\gamma}\frac{\Delta z}{P_1}\frac{dP_1}{dz}\right)\rho_1$$

$$\rho_2 = \left(1 + \frac{1}{\rho_1}\Delta z\frac{d\rho_1}{dz}\right)\rho_1$$

$$\rho_1' < \rho_2 \iff \frac{1}{\gamma}\frac{\Delta z}{P_1}\frac{dP_1}{dz} < \frac{\Delta z}{\rho_1}\frac{d\rho}{dz}$$

buoyand

$$\boxed{\frac{1}{\gamma}\frac{1}{P_1}\frac{dP_1}{dz} < \frac{1}{\rho_1}\frac{d\rho}{dz}}$$

or, as both gradients negative

$$\left|\frac{1}{\gamma}\frac{1}{P_1}\frac{dP_1}{dz}\right| > \left|\frac{1}{\rho_1}\frac{d\rho}{dz}\right|$$

Now, $\sigma = ( ) \ln P\rho^{-\gamma}$

$$\frac{dS}{dz} = ( )\left[\frac{1}{P}\frac{dP}{dz} - \frac{\gamma}{\rho}\frac{d\rho}{dz}\right]$$

buoyancy $\rightarrow$ $\frac{dS}{dz} < 0$. $\rightarrow$ free energy available!

$$\frac{dS}{dz} \begin{matrix} < \\ = \\ > \end{matrix} 0 \quad \begin{matrix} \text{superadiabatically} \\ \text{adiabatically} \\ \text{subadiabatically} \end{matrix} \Big/ \text{ stratified}$$

$$\frac{dS}{dz} < 0 \rightarrow \frac{1}{P}\frac{dP}{dz} < \frac{\gamma}{\rho}\frac{d\rho}{dz}$$

$P = k_B \rho T$

$$\boxed{\frac{1}{T}\frac{dT}{dz} < \frac{(\gamma - 1)}{\rho}\frac{d\rho}{dz}}$$

– γ captures essential thermal properties

– $\gamma - 1$ specifies how steep ∇T must be relative to density.

(iv) Scales

— "Incompressibility"

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\underline{V}) = 0$$

$$\underset{①}{\partial_t\tilde{\rho}} + \underset{②}{V_z\, d\rho/dz} + \underset{③}{\rho_0\,\underline{\nabla}\cdot\underline{\tilde{V}}} = 0$$

$$\frac{1}{\rho_0}\tilde{\rho}/T \quad , \quad \frac{\tilde{V}_z}{L_\rho} \quad , \quad k_z\tilde{V}_z$$

$T \gg (k_\perp c_s)^{-1}$ → time long relative to sound transit time

$\underbrace{}_{=\text{, sound}}$

(i.e. $1/T \ll k c_s$) $\frac{\tilde{\rho}}{\rho_0} \sim \frac{\tilde{V}}{c_s}$

[incompressible — no density perturbation]

drop ①

$L_\rho \sim L_s > k_z^{-1}$ → drop ②

$\underline{\nabla}\cdot\underline{V} \cong 0$ for

- wavelengths < scale height
- $|\tilde{V}| \ll c_s$
- $\tilde{T} \gg \lambda/c_s$

→ simplest subsonic extension,

$\nabla \cdot (\rho \underline{V}) = 0$ → incompressible mass flow. (anelastic).

$\underline{\nabla} \cdot \underline{V} + \frac{\tilde{V}_z}{\bar{\rho}} \frac{d\rho}{dz} = 0$ → decouples sound wave

→ retains finite scale height

→ modifies freezing-in law.

To identify scales, convenient to work with s, in full analysis (as in ideal fluid):

$$\partial_t S + \underline{V} \cdot \underline{\nabla} S = 0$$

$$s = \underset{\text{mean}}{\langle s \rangle} + \tilde{s}$$

↳ fluctuation

$$\frac{\partial \tilde{s}}{\partial t} + \tilde{v} \frac{\partial \langle s \rangle}{\partial z} = 0$$

$$s \sim \ln (P \rho^{-\gamma}) \sim \ln (T \rho^{-(\gamma-1)})$$

so

$$\tilde{s} \text{ [redacted]} = \left[\frac{\tilde{T}}{T_0} - (\gamma - 1)\frac{\tilde{\rho}}{\rho_0} \right]$$

Now, $\underline{\nabla} \cdot \underline{v} \simeq 0$, so $\delta p = 0$.

$$\left[\frac{\partial \underline{v}}{\partial t} = -\frac{\underline{\nabla} p}{\rho} + \underline{g} \right.$$

$$\partial_t \underline{\nabla} \cdot \underline{v} = -\frac{\nabla^2 \tilde{p}}{\rho_0} + \cancel{\underline{\nabla} \cdot \underline{g}}$$

(scales)

$$\left. \begin{aligned} \underline{\nabla} \cdot \underline{v} \simeq 0 \Rightarrow \quad & \nabla^2 \tilde{p} = 0 \\ & k^2 \tilde{p}_k = 0 \\ & \frac{\tilde{\rho}}{\rho} = -\frac{\tilde{T}_k}{T_0} \end{aligned} \right]$$

$$\frac{\tilde{\rho}}{\rho_0} = -\frac{\tilde{T}}{T_0}$$

$$\boxed{\tilde{s} \text{ [redacted]} = \gamma \frac{\tilde{T}}{T_0}}$$

$\Rightarrow$ entropy perturbation tied to temperature perturbation, alone.

For estimation:

$$\frac{\partial \tilde{V}_z}{\partial t} = -\frac{\partial_z \tilde{p}}{\rho_0} - g\frac{\tilde{\rho}}{\rho_0}\hat{z}$$

$\delta p \to 0$

$$\cong g\frac{\tilde{T}}{T_0}\hat{z}$$

$$\gamma \frac{\partial}{\partial t}\tilde{T}/T_0 = -\tilde{V}_z \frac{dS_0}{dz}$$

so $$\frac{\tilde{T}}{T_0} \sim \frac{1}{\gamma}\tau_b \tilde{V}_z \frac{dS_0}{dz}$$

so $$\boxed{\frac{1}{\tau_b^2} \sim \frac{g}{\gamma}\frac{dS_0}{dz}}$$ → buoyancy time scale

Now, consider dissipation:

viscosity:

$$\partial_t \to \partial_t - \nu\nabla^2, \quad 1/\tau_\nu \sim \frac{\nu}{\ell^2}$$

thermal:

$$\partial_t \to \partial_t - \kappa\nabla^2, \quad 1/\tau_H \sim \kappa/\ell^2$$

→ Diffusion effects will smear out heat parcel if:

$$1/\tau_r\tau_\kappa \gtrsim 1/\tau_b^2$$

i.e. parcel needs free energy sufficient to overcome dissipation

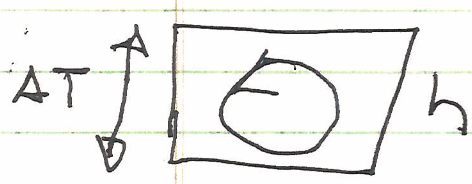
$$\frac{\tau_r\tau_\kappa}{\tau_b^2} \sim \frac{g}{\gamma}\frac{\partial\langle s\rangle}{\partial z}\,\ell^4/\nu\kappa$$

$$\boxed{Ra = \frac{g}{\gamma}\frac{\partial\langle s\rangle}{\partial z}\,\ell^4/\nu\kappa}$$

Rayleigh #

key dimensionless number in fluid mechanics.

For fluid in box:



$\rho_0 \equiv$ const

$$\delta\rho = -\alpha\,\delta T$$

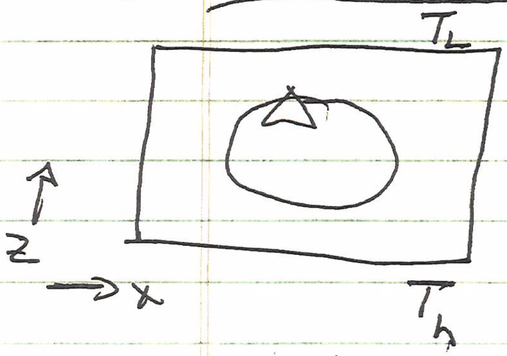
α: coeff of thermal expansion

$$\boxed{Ra = g\,\Delta T\,\alpha\,\frac{h^3}{\nu\kappa}}$$

clearly need $Ra > Ra_{crit}$ for convective instability.

- free energy
- sufficient to overcome damping

iii) Calculation



- consider vorticity $\perp$ to x, z
$\Rightarrow \omega_y$

- $\underline{V} = \nabla\phi \times \hat{y}$

$$\omega_y = -\partial_x^2\phi - \partial_y^2\phi$$

eqbrm: hydro
ρ_0 const
$\nabla T = 0$, + b.c.

Now, $\rho = \rho_0 + \tilde{\rho}$

where $\tilde{\rho} = -\rho_0 \alpha \tilde{T}$

$$\alpha = \frac{-1}{\rho}\frac{\partial \rho}{\partial T}\bigg)_p \equiv \text{coeff thermal expansion}$$

$$p_0 = -\rho_0 g z \quad \text{(hydrostatics)}$$
$$= -\rho_0 g z$$

N.B. incompressibility:

$$\Delta P \sim \rho g \underset{\text{height}}{h}$$

$$P = \rho c_s^2$$

$$\frac{\Delta P}{P} \sim \frac{\Delta \rho}{\rho_0} \sim \frac{gh}{c_s^2}$$

need $\frac{\Delta \rho}{\rho_0} \ll 1$ for validity of $\left\{ \begin{array}{l} \rho_0 = \text{const} \\ \text{hydrostatics} \end{array} \right.$

so $\frac{\Delta \rho}{\rho_0} \ll 1 \rightarrow \frac{gh}{c_s^2} \ll 1$ and $\frac{gh}{c_s^2} \ll \alpha \Delta T$

(thermally induced strat.)

Now,

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v} = -\frac{1}{\rho} \underline{\nabla} P + \nu \nabla^2 \underline{v} - g \hat{z}$$

$$P = P_0 + \tilde{P}$$
$$\rho = \rho_0 + \tilde{\rho}$$

$$\frac{\underline{\nabla} P}{\rho} = \frac{\underline{\nabla} P_0}{\rho_0} + \frac{\underline{\nabla} \tilde{P}}{\rho_0} - \frac{\underline{\nabla} P_0}{\rho_0^2} \tilde{\rho}$$

$$= -g \hat{z} + \frac{\underline{\nabla} \tilde{P}}{\rho_0} - g \hat{z} \alpha \tilde{T}$$

$$\frac{\nabla P}{\rho} = \underbrace{-g\vec{z} + \nabla\left(\frac{\tilde{P}}{\rho_0}\right)}_{} - g\alpha\tilde{T}\vec{z}$$

equilibrium $\quad \nabla\cdot\underline{U}=0 \quad$ buoyancy

$$\frac{\partial \underline{U}}{\partial t} + \underline{U}\cdot\nabla\underline{U} = -\frac{\nabla P}{\rho} - g\vec{z}$$

equilibrium

$$0 = -\frac{\nabla P_0}{\rho_0} - g\vec{z} = g\vec{z} - g\vec{z} = 0$$

pert

$$\boxed{\frac{\partial \tilde{U}}{\partial t} = -\nabla\left(\frac{\tilde{P}}{\rho_0}\right) + g\alpha\tilde{T}\vec{z}}$$

$\hat{y}\cdot\nabla\times \quad$ and $\quad \underline{U} = \nabla\phi\times\vec{y}$

$$-\frac{\partial}{\partial t}\nabla^2\phi = g\alpha\frac{\partial}{\partial x}\left(\frac{\tilde{T}}{T_0}\right) - \nu\nabla^2(-\nabla^2\phi)$$

$$\partial_t\frac{\tilde{T}}{T_0} = -\tilde{U}_z\frac{d\bar{T}_0}{dz} + \kappa\nabla^2 T$$

$\tilde{U}_z = \partial_x\phi$

Now, universal notation:

$\hat{U}_z \to w$ $\qquad \beta = -\frac{dT}{dz} = \frac{\Delta T}{h}$

$\frac{\tilde{T}}{T_0} \to \Theta$

$\delta\rho \equiv -\alpha\delta T$

$\omega_z \to \omega$

see Chandrasekhar

and $\hat{z} \cdot (\nabla \times \nabla \times \text{NSE}) \Rightarrow$

$$\boxed{\begin{aligned} \frac{\partial}{\partial t} \nabla^2 W &= g\alpha \left(\partial_x^2 \Theta \right) + \nu \nabla^2 \nabla^2 W \\ \frac{\partial \Theta}{\partial t} &= \beta W + \kappa \nabla^2 \Theta \end{aligned}}$$

standard form.

For local theory: dispersion relation.

$$(-i\omega + \nu k^2)(-i\omega + \kappa k^2)$$

$$= g\alpha\beta \frac{k_x^2}{k^2} \qquad k^2 = k_x^2 + k_z^2$$

N.B. : $\begin{cases} \nabla_h^2 \to \partial_x^2 + \cancel{\partial_y^2} \\ \nabla^2 = \partial_x^2 + \partial_z^2 \end{cases}$

and : $\begin{cases} T_0 = T_b - \beta z \\ \beta = \Delta T / h \end{cases}$

can de-dimensionalize :

length $\to h$
time $\to h^2/\kappa$

$\underline{v} \to h/(h^2/\kappa) = \kappa/h$

$\kappa^2/h^2 \leftarrow P/\rho_0$

$T \to \kappa\nu/\alpha g h^3$

$\Rightarrow$ de-dim

$$\boxed{\begin{aligned} \partial_t \nabla^2 W &= P \nabla^4 W + \partial_x^2 \Theta \\ \partial_t \Theta &= \nabla^2 \Theta + Ra\, W \end{aligned}}$$

2 param : $Ra = g\alpha\beta h^3/\nu\kappa \to$ Rayleigh #

$P = \nu/\kappa \to$ Prandtl #
relative strength dissip'n.

Game now becomes:

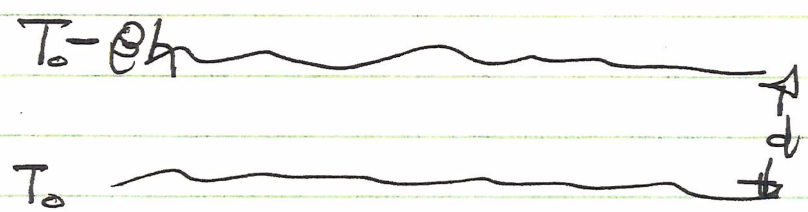
- take $Pr \sim 1$.
- scan k_x s/t
- Ra_{crit} for onset instability.

$\Rightarrow$ <u>Laborious;</u>

What are boundary conditions?

$\Rightarrow$ B.C. set effect of dissp'n and thus Ra crit

Assume wide tank (don't concern lateral)



T fixed: $\tilde{\Theta} = 0$ at $z = 0, h$

walls: $\tilde{w} = \tilde{U}_z = 0$ at $z = 0, h$

but 6^{th} order system (4 for w, 2 for Θ)

$\rightarrow$ need 2 more.

Can envision two scenarios for two more b.c.'s:

1. no-slip
2. stress free (Rayleigh 1916, 2 free boundaries)

1. No-Slip (rigid)

$$\Rightarrow \tilde{V}_h \Big|_{0,h} = 0$$

{ horizontal/tangential velocity vanishes at 0, h

but, work with W ?!

$\underline{\nabla} \cdot \underline{V} = 0$

$$\nabla_h \tilde{V}_h + \partial_z \tilde{V}_z = 0$$

as all ∇_h of $\tilde{V}_h$ vanish as $\tilde{V}_h$ vanishes

$$\Leftrightarrow \quad k \times \tilde{V}_h(\underline{z}) \Big|_{0,h} = 0 \quad \text{as} \quad V_{h}(z) \Big|_{0,h} = 0$$

$$\Rightarrow \nabla_h \tilde{V}_h = 0$$

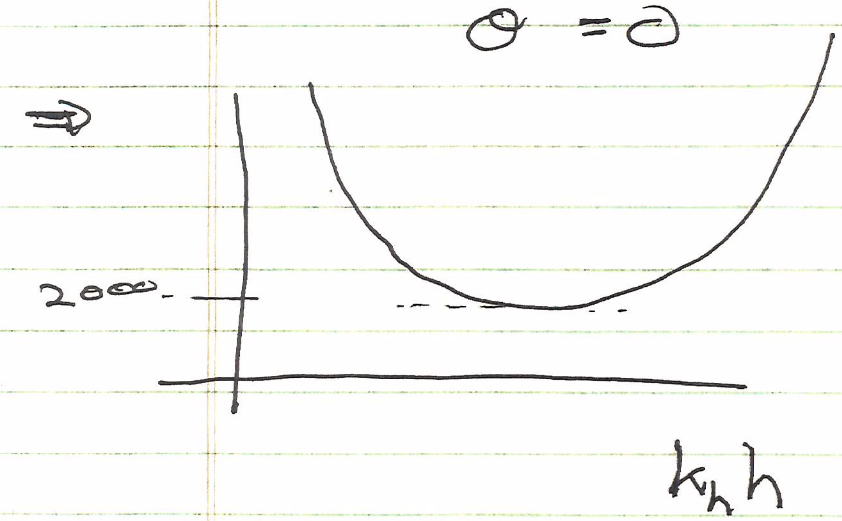
so $\boxed{\partial_z W = 0}$.

B.C.'s : $\partial_z W \Big|_{0,h} = 0$

$W = 0 \Big|_{0,h}$

$\Theta = 0$

$\Rightarrow$



Chandra. Fig. 39

$Ra_{crit} \sim 2000$

- Where does the shape come from?

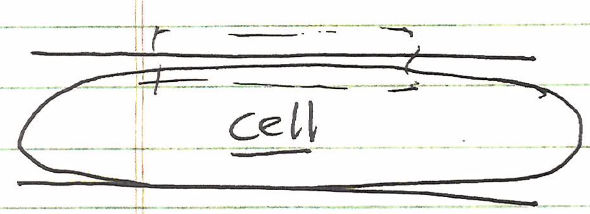
- high k? - $\nu k^2 = \nu(k_h^2 + k_v^2)$

$\kappa k^2 = \kappa(k_h^2 + k_v^2)$

rising k_h → rising dissipation due diffusion

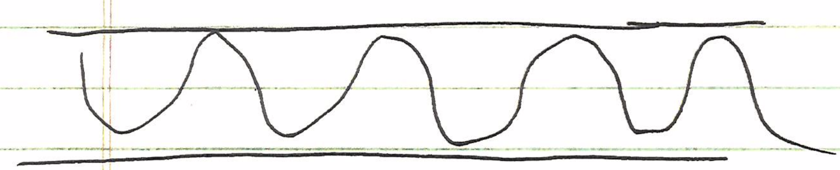
→ rising Ra_{crit}.

– low k ? → top, bottom boundary layer due no slip.



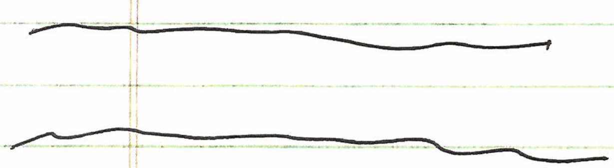
→ dissipation due viscous effects + $\tilde{V}_h = 0$ condition at 0, h.

i.e compare:



→ greater curvature but less effect $\tilde{V}_h = 0$.

② stress free:



Free surface at top, bottom → no stress.

$$\underline{\underline{\tau}} = -\eta \, \partial_z V_h$$

↓

shear stress delivered to surface.

so need, $\partial_z \tilde{V}_h \big|_{0,h} = 0$.

Have: $\partial_h V_h = -\partial_z V_z$

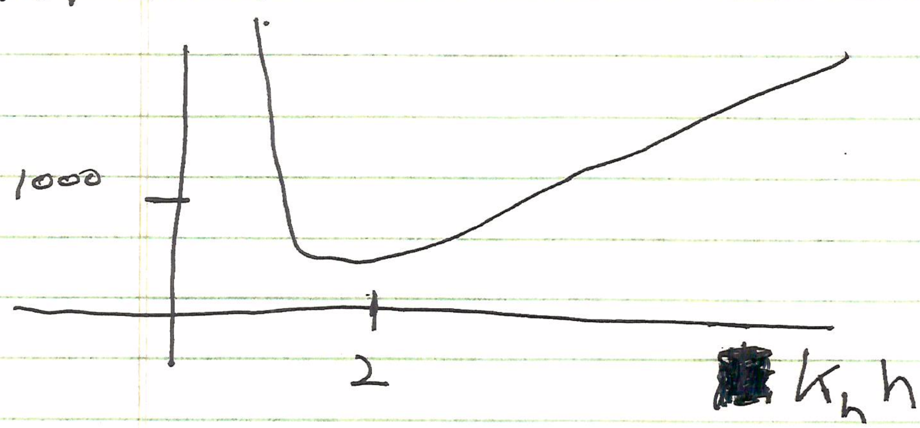
$$\partial_z \partial_h V_h = -\partial_z^2 V_z$$

$$\partial_h \partial_z V_h = 0 = -\partial_z^2 V_z$$

so

$$\boxed{\left.\partial_z^2 W\right|_{0,\,h} = 0}$$

and have:



$$Ra_{crit} \sim \frac{27\pi^4}{4}$$

for $k_h h_{crit} \sim \pi/\sqrt{2}$

→ substantially lower Ra_{crit} due stress free b.c. → no longer fighting no-slip condition.

- high k → increased dissipation

  low k → effects layers at boundary.