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# THE THEORY OF THE STABILITY OF NON-UNIFORM PLASMA AND ANOMALOUS DIFFUSION†

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**Abstract**—A review of the theory of the stability of a low-pressure ($p \ll H^2/8\pi$) plasma contained in a magnetic field is given, with regard to the excitation of drift waves due to non-uniformity of the plasma. In the first part are considered drift waves in a low-temperature plasma, when a hydrodynamic description is justified. It is shown that dissipative effects (in particular, the effect of finite electrical conductivity) play an important part in the stability of the plasma. On the basis of a rigorous linear theory of stability, a semi-quantitative discussion of non-linear effects in an unstable plasma and of anomalous diffusion is presented. It has been possible along these lines to obtain a natural explanation of the phenomenon of Bohm diffusion and of the 'critical' value of magnetic field. The results are compared with the available experimental data.

In the second part of the paper, drift waves in a high-temperature plasma are considered. An analysis of linear stability theory is presented, using the collisionless Boltzmann equation. A quasi-linear method has been adopted to calculate the influence of drift waves on particle diffusion in an unstable plasma.

## 1. INTRODUCTION

VARIOUS aspects of the theory of magnetic containment of a plasma, including the well-studied problem of plasma stability on an ideal magnetohydrodynamic medium model, have been discussed in books[1] and review articles[2]. The basic cause of the instability of the magnetic equilibrium of a plasma may be thought of as the 'effective' acceleration due to a gravitational force, arising in the absence of a real gravitational force from the curvature of the magnetic field lines.

The results of an experimental study of plasma stability cannot, however, always be explained in terms of the ideal theoretical models. There has therefore been, in recent years, an intense study of the influence of dissipative effects on plasma stability. For example, a theory of this sort has been applied to the question of the stability of the so-called positive column of a gas discharge and has given an explanation of several anomalous phenomena occurring in low-temperature weakly ionized plasmas[3], when the cause of instability was an electric current, flowing along the lines of force of the superimposed magnetic field.

Until recently, however, there remained unexplained the nature of the *experimentum crucis* of magnetic containment, the phenomenon of so-called anomalous Bohm diffusion of a plasma across a magnetic field. In 1949 it was discovered experimentally[4] that the diffusion coefficient of a plasma, across a magnetic field, substantially exceeds the value obtained from classical kinetic theory. Bohm suggested that the reason for this anomaly was instability of an unknown kind which was causing the plasma to become turbulent, and he postulated a coefficient of anomalous diffusion

$$D_{\perp} = \frac{cT}{16eH} \tag{1}$$

where $H$ is the magnetic field strength; $T$ is the temperature of the plasma; $c$ is the

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velocity of light *in vacuo*; $e$ is the electronic charge. Since that time there have been many experiments to establish the nature of this instability and the character of the turbulence caused by it, but none has succeeded in elucidating the anomaly. (The best that has been done has been to obtain numerically (!) approximate values of the diffusion coefficient for special cases[5] by making additional assumptions.) On the other hand, there has been a succession of experiments to study plasma diffusion[6] and they have frequently led to contrary results, sometimes obtaining satisfactory agreement with the classical theory. We shall, in particular, show that in a fully ionized low-pressure plasma, placed in a strong magnetic field ($H^2 \gg 8\pi p$ where $p$ is the plasma pressure), instability does indeed exist when the magnetic field lines are straight and uniform along their length. This instability is due only to the presence of a density gradient in the plasma. Examination of the turbulence arising from this instability leads to a diffusion coefficient close to (1). It will also be shown that condition (1) is not universal and that régimes do exist in which the diffusion coefficient may vary, for instance, as $1/H^2$.

It is well known that a plasma which is not in thermodynamic equilibrium is unstable as regards the excitation of oscillations of a different sort. Such oscillations often exert a large influence on the particle distribution in both velocity and ordinary space. In a uniform plasma, placed in a magnetic field, there are seven classes of oscillation (Alfvén, Langmuir etc.). In a non-uniform plasma, the dielectric properties may be altered substantially even for only small spatial gradients. This mainly applies to the low-frequency oscillations (having frequencies small compared with the ion cyclotron frequency). Here there occur classes of oscillation the phase velocity of which coincides with the velocity of the plasma drift due to the magnetic field and the pressure gradient; these are 'drift waves' [see, for example, GALEEV *et al.*[7]]. The 'slowness' of drift waves leads one to expect that they will exert a considerable influence on plasma stability. For example, the passage of a relatively low-velocity particle beam through the plasma may be quite sufficient to excite them. In an introduction it is natural to precede the systematic exposition of the theory of drift wave excitation and instability by a non-rigorous qualitative argument. Let us choose a one-dimensional geometry for the initial equilibrium configuration of the plasma in the magnetic field: $\mathbf{H}_0 = \{0, 0, H_0\}$ is the magnetic field, which is taken along the $z$ axis. Let the gradients of the unperturbed quantities be directed along the $x$ axis; $n_0 = n_0(x)$ is the density; $T_0 = T_0(x)$ is the temperature; $\mathbf{E} = \{E_{0x}\ 0, 0\}$ is the electric field strength, determined from the condition

$$-\frac{\mathrm{d}}{\mathrm{d}x}\, n_0 T_0 = -e n_0 E_{0x}$$

which expresses the absence of a macroscopic ion velocity, in equilibrium.

The properties of the drift waves may be deduced qualitatively in the following way. The dispersion equations for the frequencies of the drift waves are most simply derived in the co-ordinate system for which the unperturbed electric field equals zero ($E_{0x} = 0$). Let the phase velocity of the waves be greater than the ion thermal velocity $V_{Ti}$ and less than the electron thermal velocity $V_{Te}$. Then provided that the perturbation electric field can be derived from a scalar potential (curl $\mathbf{E} = 0$) the electrons, moving along the lines of force, are able to redistribute themselves according to the Boltzmann law: $n = n_0(x)e\varphi/T$ ($n$ is the perturbation of plasma density, $\varphi$ is the electrical potential of the perturbations). Choosing perturbations of the form $\exp(i\omega t + ik_y y + ik_z z)$ and making use of the conditions of irrotational field and quasi-neutrality, and of the equation of continuity for the ions

$$i\omega n + c\frac{E_y}{H_0}n_0'(x) = 0 \tag{2}$$

we obtain

$$\omega = \omega_e \equiv k_y \frac{cTn_0'}{eH_0 n_0}. \tag{3}$$

On increasing $k_z$ these waves go over into the so-called ion sound waves.

From equation (3) it follows that the frequency of the oscillations increases as the wavelength decreases. When, however, the wavelength attains the order of the ion Larmor radius, this increase ceases. This is because the effective electric field, averaged over the particle orbit, decreases; therefore the mean drift velocity of the particles, $cE_y/H$ in equation (2), falls. An exact analysis shows that this decrease of field just compensates for the increase of frequency due to the increase in $k_y$, so that $\omega \approx V_{Ti}n_0'/n$ for $k_y r_i \gg 1$. We shall show later that these drift waves are excited in a non-uniform low-density plasma even without a temperature gradient, there being only a density gradient. In the case of a high-temperature plasma, where one may neglect particle collisions, the excitation mechanism for drift waves consists of a resonant interaction between the wave and electrons moving along the magnetic lines of force with a velocity equal to the phase velocity of the wave $\omega/k_{\|}$.

In a low-temperature plasma for which hydrodynamics may be used, an imaginary part of the frequency will occur due to the various dissipative effects: viscosity, thermal conductivity and so on. In a uniform plasma these factors lead to a damping of waves. In a non-uniform plasma the situation may be otherwise. Let us, for example, consider the influence of finite electrical resistance, arising from a frictional force between the electrons and the ions. It has been found that this may cause the excitation of drift waves (3). In fact, using the assumptions on which the existence of the oscillations (3) is based, considering for simplicity that the ions are cold and taking into consideration the frictional force of the electrons on the ions, we obtain, for the motion of electrons along and of ions perpendicular to the magnetic field, the equations

$$-ik_z nT_0 - en_0E_z - m_e n_0 V_{ez}\nu_e = 0 \tag{4}$$

$$m_i n_0 \frac{d\mathbf{V}_i}{dt} = en_0\mathbf{E}_\perp + \frac{e}{c}n_0(\mathbf{V}_i \times \mathbf{H}_0). \tag{5}$$

Here $\nu_e$ is the effective electron–ion collision frequency.

Adding to these the condition curl $\mathbf{E} = 0$ and the equations of continuity, (2), and of quasi-neutrality

$$ik_y\left(i\frac{\omega}{\omega_H}\cdot\frac{cE_y en_0}{H_0} + \frac{\omega}{\omega_H}\cdot\frac{k_y cTn}{eH_0}\right) + ik_z j_z = 0 \tag{6}$$

we obtain the following dispersion equation:

$$\omega^2 = i\omega_s\omega - i\omega_s\omega_e \tag{7}$$

where

$$\omega_s = \left(\frac{k_z}{k_y}\right)^2 \omega_{He}\omega_{Hi}/\nu_e.$$

Thus it is evident that there is instability in a plasma of variable density having finite conductivity. If $\omega_s \gg \omega_e$ (the friction is small) then $\mathrm{Re}\,\omega \approx \omega_e$; $\mathrm{Im}\,\omega \approx \omega_e^2/\omega_s$. In the opposite case ($\omega_e \gg \omega_s$) $\mathrm{Re}\,\omega \approx \mathrm{Im}\,\omega \approx \sqrt{\omega_e\omega_s}$.

It is not difficult to understand physically how the frictional force acts in this example. In the absence of friction, and for intermediate frequencies ($V_{Ti} \ll \omega/k_z \ll V_{Te}$), the electrons, moving along the lines of force, are able to become redistributed according to the Boltzmann law and the pressure forces and electric field do no work. In the presence of friction the electron mobility falls. Because of this, the original density fluctuation continues to grow, leading to the development of instability. The net work done by the pressure forces of the expanding plasma is now positive over a period of the oscillations, and it is easily verified that it exceeds the frictional loss.

From the discussion given here it follows that in both limiting cases, a high temperature and a low-temperature plasma, there are physical mechanism which produce a growth of drift oscillations with time (the growth rates are the frequencies of the drift waves).

A detailed investigation of the anomalies in transport phenomena, occurring because of drift instabilities, meets with great difficulties; it is, nevertheless, precisely the results of such an investigation which must be our aim. An attempt has been made to derive an anomalous diffusion coefficient by combining the results of linear stability-theory with some, in our view, reasonable estimates of the effects of non-linearity. As has already been remarked, this will lead to a natural explanation of the Bohm diffusion coefficient (in a low-temperature plasma). For a high-temperature plasma, where collisions are infrequent, the so-called quasi-linear method[8,9] has been used to calculate the diffusion coefficient. This method takes into account the effect on the particle distribution function of the waves which arise from the instability.

Waves arising from an instability tend to alter the electron distribution function in such a way as to suppress the instability. If however the electron–electron collisions are sufficiently frequent to allow a Maxwellian velocity distribution to be re-established, the instability will not be suppressed although the anomalous diffusion caused by it will be substantially reduced.

## 2. THE HYDRODYNAMIC THEORY OF STABILITY OF A NON-UNIFORM LOW-TEMPERATURE PLASMA

### (i) *Linearized equations. Investigation of the problem in terms of eigenvalues*

To investigate the stability of a non-uniform plasma having finite conductivity we shall use the two-fluid hydrodynamic equations for electrons and ions in a strong magnetic field ($p \ll H^2/8\pi$). We shall assume, as in the example above, that the perturbed motion does not disturb the quasi-neutrality of the plasma ($n_{0i} = n_{0e}$, $n_i = n_e$). In addition we shall neglect the perturbation $H_z$ in Maxwell's equations for the electric and magnetic fields of the perturbations; this is justified if $8\pi n_0 T_0 \ll H_0^2$.

For the frequencies appropriate to the problem ($V_{Te} \gg \omega/k_z \gg V_{Ti}$) we shall neglect both the motion of the ions along $H_0$ and the inertia of the electrons; this latter means that we limit ourselves to those oscillations having frequencies much less than the electron collision frequency with neutral particles and ions. Then the linearized set of two-fluid hydrodynamic equations[10] in the frame of reference for which the unperturbed electric field $E_0 = 0$, has the form

$$ik_yE_z - ik_zE_y = -\frac{i\omega}{c}H_x, \qquad ik_yE_z - \frac{\mathrm{d}E_y}{\mathrm{d}x} = 0; \tag{8}$$

$$\frac{\mathrm{d}H_y}{\mathrm{d}x} - ik_yH_x = \frac{4\pi_{jz}}{c}, \qquad \frac{\mathrm{d}H_x}{\mathrm{d}x} + ik_yH_y = 0, \qquad (\mathrm{div}\,\mathbf{j} = 0); \tag{9}$$

$$-ik_z(n_0T_e + nT_0) - en_0E_z - 0{\cdot}71\,ik_zn_0T_e + R_{\|} + \frac{eV_{0y}H_x}{c} = 0; \tag{10}$$

$$-\boldsymbol{\nabla}_{\perp}p_e - e\mathbf{E}_{\perp}n_0 - \frac{en_0}{c}(\mathbf{V}_e \times \mathbf{H}) - \frac{en}{c}(\mathbf{V}_0 \times \mathbf{H}_0) = 0; \tag{11}$$

$$\tfrac{3}{2}n_0(i\omega T_e + V_{0e}T_eik_y + V_{xe}T_{0e}') - T_{0e}(i\omega n + ik_yV_{0e}n + V_xn_0')$$
$$= -k_z^2\kappa T_e + \frac{5}{2}\cdot\frac{cT_0}{eH_0}ik_y(nT_{0e}' - T_en_0'); \tag{12}$$

$$i\omega n + \mathrm{div}\,(n\mathbf{V}_0) + \mathrm{div}\,(n_0\mathbf{V}) = 0; \tag{13}$$

$$m_in_0[i\omega\mathbf{V}_i + (\mathbf{V}_{0i}\,.\,\boldsymbol{\nabla})\mathbf{V}_i] = en_0\mathbf{E} + \frac{en}{c}(\mathbf{V}_{0i} \times \mathbf{H}_0)$$
$$+ \frac{en_0}{c}(\mathbf{V}_i \times \mathbf{H}_0) - \boldsymbol{\nabla}p_i - \mathrm{div}\,\boldsymbol{\pi}_{\perp} + R_i; \tag{14}$$

$$\tfrac{3}{2}n_0(i\omega T_i + V_{0i}T_iik_y + V_{xi}T_{0i}') - T_{0i}(i\omega n + ik_yV_{0i}n + V_{xi}n_0')$$
$$= \frac{5}{2}\cdot\frac{cT_{0i}}{eH_0}(T_in_0' - nT_{0i}'); \tag{15}$$

$$\pi_{yy} = -\pi_{xx} = \frac{nT_{0i}}{2\omega_{Hi}}\left(\frac{\partial V_{xi}}{\partial y} + \frac{\partial V_{yi}}{\partial x}\right);$$
$$\pi_{xy} = \pi_{yx} = \frac{nT_i}{2\omega_{Hi}}\left(\frac{\partial V_{xi}}{\partial x} - \frac{\partial V_{yi}}{\partial y}\right). \tag{16}$$

Here $j$ is the perturbation of the current density. In the equation of motion for the electrons (10) along the magnetic field lines, both the thermal force $0{\cdot}71\ ik_zn_0$ arising from the presence of a temperature gradient and the frictional force $R_{\|}$ of the electrons with the ions (or with the neutral particles in the case of a weakly ionized plasma) have been taken into account. The frictional force has not been included in the equation of motion of the electrons across the magnetic field; this implies that $\omega_{He}\nu_e^{-1} \gg 1$. The equation of heat balance (12) for the electrons has been included since there arises a perturbation in the temperature $T_e$ in the presence of an initial temperature gradient $T_{0e}'$. In (12) $\kappa$ is the coefficient of electronic thermal conductivity. Equation (13) is the equation of continuity for the ions. The motion of the ions across the magnetic field is described by equation (14), in which the viscosity tensor, with components given in (16) has been retained. Note that the form of the viscosity tensor (16) corresponds to a hydrodynamical description of the motion across the magnetic field even when collisions between the ions have been neglected, if the

Larmor radius of the ions is small compared with the characteristic transverse dimensions [cf. for example, RUDAKOV[11]]. The quantity $R_i$ is the frictional force between the ions and the neutral particles; (the friction between the ions and the electrons scarcely affects the motion of the ions). Equation (15) is the equation of heat balance for the ions; $V_{0e}$ and $V_{0i}$ are the unperturbed electron and ion velocities.

First of all we shall consider a fully ionized plasma, where $R_i = 0$. When this is so

$$R_{\parallel} = \frac{en_0 j_z}{\sigma_{\parallel}} - \frac{1}{\sigma_{\parallel}^2} \cdot \frac{d\sigma_{\parallel}}{dT_e} T_e e n j_0 \tag{17}$$

where $j_0$ is the initial longitudinal current. The second term in (17) corresponds to an additional frictional force due to the conductivity parallel to the magnetic field changing.

Choosing perturbations of the form $\varphi(x) \exp(ik_y y + ik_z z + i\omega t)$ we obtain from (8) to (15) the following equation for long-wave perturbations of the potential ($k_\perp r_i \ll 1$) in the co-ordinate system where the ions are at rest:

$$\frac{d^2\varphi}{dx^2} - \left\{1 - i\frac{\omega_s}{\omega}\left[\left(1 - \frac{\omega_e}{\omega - \omega_i}\right) - \frac{(1+\alpha)\omega_T - i\omega_0}{\omega - \frac{2}{3}i\chi k_z^2}\right]\right\} k_y^2 \varphi = 0. \tag{18}$$

Here

$$\omega_T = k_y \frac{cT_{0e}}{eH_0} \cdot \frac{T_{0e}'}{T_{0e}}; \quad \omega_i = \frac{k_y c T_{0i}}{eH_0} \cdot \frac{n_0'}{n_0};$$

$$\chi = \frac{\kappa}{n_0}; \quad \omega_0 = \frac{k_y}{k_z} \cdot \frac{j_0 c (d\sigma_{\parallel}/dx)}{\sigma_{\parallel}^2 \cdot H_0}; \quad \alpha = 0{\cdot}71. \tag{19}$$

An analysis of (18) was carried out by MOISEEV and SAGDEEV[12] for the case involving no longitudinal current. In general, a rigorous solution of a differential equation of the type of (18) requires an accurate knowledge of the density profile. Here we shall limit ourselves to the case when $\omega_i, \omega_0, \omega_T \ll \omega_e$, and the density changes so slowly that the magnitude of $\omega_s$ may be considered constant. In the neighbourhood of points where $n_0'/n_0$ is a maximum, $\omega_e$ may be put in the form

$$\omega_e = \omega_{e0} - \beta_e x^2. \tag{20}$$

Using (20), we obtain for (18)

$$\frac{d^2\varphi}{dx^2} + (2E - Kx^2)k_y^2\varphi = 0, \tag{21}$$

where

$$E = -\frac{1}{2}\left[1 - i\frac{\omega_s}{\omega}\left(1 - \frac{\omega_e}{\omega}\right)\right]k_y^2; \tag{22}$$

$$K = -i\frac{\omega_s \beta_e}{\omega^2} k_y^2. \tag{23}$$

The solution of equation (21) is similar to that of the Schrödinger equation for the linear harmonic oscillator. We consequently have, for the eigenvalues and eigenfunctions

$$\frac{E}{\gamma^2} = \frac{n+1}{2}; \tag{24}$$

$$\varphi(x) \sim H_n(\gamma x) \exp(-\gamma^2 x^2/2). \tag{25}$$

Here $H_n(\gamma x)$ are Hermite polynomials and

$$\gamma^2 = k_y \frac{\sqrt{\omega_s \beta_e/2}}{\Omega^2 + \nu^2} [(\Omega + |\nu|) - i(\Omega - |\nu|)]. \tag{26}$$

From (25) and (26) it follows that finite solutions exist. In the quasi-classical approximation, when $\gamma$ is small, we obtain (7) from (24). The existence of localized solutions confirms that the complex potential of equation (18) [i.e. the function $\varphi$] has the character of a potential well. Although equation (21) is only correct sufficiently near to the bottom of the well, the results obtained from it remain qualitatively correct wherever localized solutions exist provided that the density only changes slowly[25]. For varying temperature it is necessary to take into account the influence of thermal conductivity on the growth rate. However, it is clear that, for $\omega \geqslant \omega_s$, if

$$\omega_s \gg k_z^2 \chi \tag{27}$$

this influence is negligible. Using the expressions for $\omega_s$, Im $\omega$ and $\chi$, we obtain from (27)

$$k_y r_i \ll 1, \tag{28}$$

i.e. the thermal conductivity changes the growth rates of the long wavelength perturbations considered hardly at all.

Let us now consider the case when there is current flowing along the magnetic lines of force[13]. If $\omega_e$, $\omega_i$, $\omega_T \ll \omega_0$, equation (18) assumes the form

$$\varphi'' - \left[1 - i\frac{\omega_s}{\omega}\left(1 + \frac{i\omega_0}{\omega - \frac{2}{3}i\chi k_z^2}\right)\right] k_z^2 \varphi = 0. \tag{29}$$

Let $\omega_0 \gg \omega_s$, $\chi k_z^2$. This is equivalent to satisfying the following inequalities:

$$V_{z0} \gg V_{Te}(\omega_{He}\tau_e)\left(\frac{k_z^2}{k_y k_R}\right)\frac{l}{\lambda_z}, \qquad (\omega_0 \gg \chi k_z^2); \tag{30}$$

$$V_{z0} \gg \frac{\omega_{Hi}}{k_R}(\omega_{He}\tau_e)^2\left(\frac{k_z}{k_y}\right)^3, \qquad (\omega_0 \gg \omega_s). \tag{31}$$

Here $V_{z0} = j_0/en_0$; $l$ is the electron mean free path; $\lambda_z = 2\pi/k_z$; $k_R \approx 2\pi/r$ ($r$ is the transverse dimension of the system). It is to be noted that, for a uniform initial magnetic field, one must take $k_z > (k_0 k_y)/k_x$, where $k_0 \approx (2\pi j_0)/cH_0$; this does not conflict with $k_z$ being small, however, since $k_0$ is small if, in accordance with the Kruskal–Shafranov criterion, we exclude the hydrodynamic instabilities of an ideal plasma from consideration.

When conditions (30) and (31) are satisfied, equation (29) gives a pure potential

$$U = 1 - \frac{\omega_s \omega_0}{\nu^2}, \qquad \omega = i\nu. \tag{32}$$

In Fig. 1 is shown a graph of $U(x)$ for an indicated variation of the temperature $T_0(x)$. Comparing the growth rates for the given case (Im $\omega \approx \sqrt{\omega_0 \omega_s}$) with those for the instability developing from drift waves (Im $\omega \approx \sqrt{\omega_e \omega_s}$), we see that for

$$V_{z0} > V_{T_e} \frac{l}{\lambda_z} \tag{33}$$

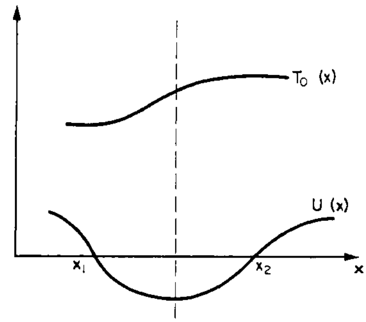


FIG. 1.—Graph of $U(x)$ when $\omega_0 \gg \omega_s, \chi k_z^2$.

current instability develops the faster. The case $\omega_e \ll \omega_0 \ll \omega_s$ corresponds to the current-convective instability considered by KADOMTSEV(14). If $\omega_s \gg \omega_e$, $\omega_0$ and the thermal conductivity are small but $\omega_0 \gtrsim \omega_e$, then $\omega = \omega_e - i\omega_0$. In this case instability also develops from the drift waves, but with a growth rate $\sim\omega_0$, i.e. the initial longitudinal current plays a fundamental part in the development of the instability.

One must take into account the effect of the neutral gas, for a weakly ionized plasma. When $\omega_0$, $\omega_T$, $\omega_i \ll \omega_e$, the equation for the perturbations of potential takes the form(13)

$$\frac{d^2\varphi}{dx^2} - \left[1 + \frac{\omega_s}{i\omega + \nu_{0i}}\left(1 - \frac{\omega_e}{\omega}\right)\right]k_y^2\varphi = 0. \tag{34}$$

Here $\nu_{0i}$ is the ion–neutral collision frequency. For the frequencies and growth rates we then have

$$\left.\begin{aligned} &\text{Re}\,\omega \equiv \Omega = \omega_e\Big/1 + \frac{\nu_{0i}}{\omega_s};\\ &\text{Im}\,\omega \equiv \nu = \omega_e^2\Big/\omega_s\left(1 + \frac{\nu_{0i}}{\omega_s}\right)^3\\ &(\omega_s + \nu_{0i} \gg \sqrt{\omega_e\omega_s}). \end{aligned}\right\} \tag{35}$$

Taking the longitudinal motion of the electrons into account in the equation of continuity, when $\omega_i$, $\omega_0$, $\omega_T \ll \omega_e$, we have for the perturbations of potential in a weakly ionized plasma(15):

$$\omega = \frac{1}{2}\left(\omega_e - i\,\frac{\omega_e^2}{4\nu_{0i}} + i\,\frac{k_z^2V_{Te}^2}{\nu_{0e}}\right),$$

$$\omega_e,\ k_zV_i \ll \nu_{0i} \ll \omega_s; \tag{36a}$$

$$\omega = \frac{\omega_s\omega_e}{\nu_{0i}} + i\left(\frac{k_z^2V_{Te}^2}{\nu_{0e}} - \frac{\omega_e^2\omega_s^2}{\nu_{0i}^3}\right). \tag{36b}$$

$$\nu_{0i} \gg \omega_s; \qquad k_\perp^2V_i^2 \ll \omega_{Hi}^2; \qquad V_i = \sqrt{\frac{T_e}{m_i}}; \qquad k_\perp^2 = k_x^2 + k_y^2;$$

$\nu_{0e}$ is the electron–neutral collision frequency.

The longitudinal motion of the ions has less stabilizing effect than does the stabilizing factor we have considered in (36).

In a strongly ionized plasma, when $\omega_s \gg \omega$, ion–acoustical oscillations ($\Omega = k_z V_i$) grow at a rate $\omega_e$.

One of the important dissipative processes, friction between different kinds of particles, has been considered above and it has been shown that on doing so a destabilizing action appears. We shall now briefly mention the peculiarities of the influence of thermal conductivity and viscosity. Thus, if one retains in the dispersion equation the terms which recognize the finite Larmor radius and the term $\chi k_z^2$, and considers the friction as being small [which is so if condition (28) is not obeyed], then a non-uniform plasma will be unstable even in the absence of a temperature gradient[7]. For long-wave perturbations, thermal conduction causes an insignificant change in the growth rates and frequencies, but viscosity, because of the transverse motion of the ions, exerts a stabilizing action[13].

Let us discuss the effect of 'shear' in the magnetic field when investigating the stability of a non-uniform plasma. We shall consider current-instability first of all. Let the magnetic lines of force lie in the $(y, z)$ plane and be straight, having an angle of inclination $\theta(x)$, relative to the $z$-axis, which is a function of $x$. This shear of the lines of force is provided by a current, flowing along them, so that the angle of inclination $\theta$ in accordance with Maxwell's equation curl $\mathbf{H} = 4\pi/c\,\mathbf{j}$ is related to $V_{e0}$ by the expression

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = -\frac{4\pi e n_0 V_{e0}}{cH_0}, \qquad V_{i0} = 0.$$

If the density and the temperature have a similar variation, and thermal conduction may be neglected, the graph for $U$ is similar to that depicted in Fig. 1. In the presence of shear,

$$k_{\parallel} = k_z + k_y \int^x \frac{\mathrm{d}\theta}{\mathrm{d}x}\,\mathrm{d}x$$

($k_{\parallel}$ is the component of the wave vector along the magnetic field). It is necessary to take thermal conduction into account when $k_{\parallel}$ increases; when this is so, $\omega_s$ also increases and $\omega_0$ decreases. The potential well becomes smaller, as may be seen from (29). When $\omega_s \gg \omega_0$, instability generally vanishes, if $\chi k_{\parallel}^2 \geqslant \omega_0$.

Let us now consider the case when $\omega_e \gg \omega_T, \omega_0, \omega_i$. With increase of $k_{\parallel}$ the situation may arise in which $\omega_s \gg \omega_e$, and this leads to a narrowing of the potential well [this follows from analysis of (18)], and so the condition $\omega_e/k_{\parallel} \gg V_i$ may also be violated. In the opposite limiting case, in a fully ionized plasma, as has already been indicated, the ion–acoustical oscillations grow at a rate $\omega_e \ll k_{\parallel} V_{Ti}$. We note that an increase in $k_x$ and a small value of the parameter $\omega_e/k_z V_i$ substantially decrease the diffusion coefficient [cf. (41)]. It follows from (36a) that in a weakly ionized plasma the effect of increasing $k_{\parallel}$ is to cause stabilization of the instability.

In conclusion, let us consider what effect the violation of the condition curl $\mathbf{E} = 0$ would have. Neglecting thermal conduction due to collisions, in the absence of an external longitudinal current and of a temperature gradient, we obtain from equations (8) to (15) the equation (in the laboratory system of co-ordinates)

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} - k_y^2\left[1 - i\omega_s \frac{\omega - \omega_i - \omega_e}{\omega(\omega - \omega_i)}\left(1 - \frac{\omega(\omega - \omega_i)}{k_z^2 V_A^2}\right)\right]\varphi = 0. \qquad (37)$$

If the perturbations are not described by a potential ($\omega \gg k_z V_A$) then we find from the qualitative criterion for stability

$$\mathrm{Im}\,\omega = \mathrm{Re}\,\frac{k_\perp^2}{k_y^2}\,.\,\frac{k_z^2 V_A^2}{\omega_s} > 0$$

i.e. the perturbations die out with time. We notice that the stabilizing effect, due to these circumstances increases with increasing temperature.

(ii) *Consideration of non-linear effects. The influence of instabilities on anomalous diffusion*

The development of instability must lead to the occurrence of a turbulent state in a plasma and to the onset of turbulent diffusion. As is usually done in turbulence theory, we shall estimate the resulting diffusion coefficient by dimensional analysis.

The diffusion coefficient may be written down in the form

$$D \approx V_\pi^2 \tau. \tag{38}$$

Here $V_\pi$ is the velocity fluctuation of the plasma; $\tau$ is the characteristic correlation time. In the case we have been considering $\tau \approx 1/\nu/$, since there is no other time scale to characterize the irreversible nature of the turbulent state. The amplitude of the fluctuations is determined from the following considerations. On the other hand, instability leads to a growth in the amplitude of the fluctuations and $(\partial V_\pi)/\partial t \sim \nu V_\pi$; on the other, non-linear terms such as div($n$**V**) cause a transfer of energy to the short wavelength part of the spectrum, where the fluctuations get damped. The balance between these two processes determines the steady-state value of the fluctuation amplitude.

$$|\nu|\,.\,n \approx \frac{V_\pi}{\lambda_\perp}\,n, \tag{39}$$

where $\lambda_\perp$ is the characteristic dimension of the turbulence elements along the $x$-axis direction.

Determining $V_\pi \approx |\nu|\lambda_\perp$ from (39), we obtain

$$D \approx |\nu|\lambda_\perp^2. \tag{40}$$

It is natural to take as $\lambda_\perp$ the wavelength of the instability $\lambda_x$. For the case ($\nu/\Omega \ll 1$) it has been possible to develop an adequately rigorous method for investigating the turbulence fluctuations and their influence on the particle distribution function.[9,16] The latter is calculated using the so-called quasi-linear method[9], the essence of which is that the distribution function can be split into slowly varying and rapidly oscillating parts and in the equation for the 'slow' part the mean square effect of the rapid oscillations is taken. The division into two frequency ranges presupposes that there operate in the plasma, from the very start, two processes having different time scales: rapid oscillations with a slowly varying amplitude. This is just the previously mentioned condition $\nu \ll \Omega$. Extending the quasi-linear method to a non-uniform plasma, in the hydrodynamic approximation, and using Boltzmann's equation for the waves[16], the following expression for the diffusion coefficient is obtained[15]:

$$D \approx \frac{\nu^2}{\omega k_\perp^2}\,. \tag{41}$$

If $\nu_k \approx \Omega_k$ then perturbations having different wavelengths will strongly interact with each other over a prolonged time interval and the method we have indicated cannot be used. One can, however, take advantage of expression (41) in order to estimate the diffusion, putting $\nu \approx \Omega$ in it. We then obtain expression (40). As an example, let us find $D$ when the ions are cold, when there is no longitudinal current and when $T_e =$ const. Then for $\omega_e \gg \omega_s$ it follows from (18) that $k_x \approx k_y$. Using the expression for Im $\omega$, we obtain

$$D \approx \frac{\sqrt{\omega_s \omega_e}}{k_y{}^2}. \tag{42}$$

We are interested in the maximum diffusion coefficient, and so we shall take the minimum allowable values of $k_y \approx (2\pi)/r$, where $r$ is the transverse dimension characteristic of the system. The ratio $k_z/k_y$ still comes into $\sqrt{\omega_s}$ and should also be taken as its maximum; from the condition $\omega_e \geqslant \omega_s$ we have the limiting acceptable value for $k_z/k_y$:

$$\left(\frac{k_z}{k_y}\right)_{\max} \lessapprox \frac{c}{V_A} \cdot \sqrt{\frac{\omega_e}{4\pi\sigma_{\parallel}}}, \tag{43}$$

where

$$V_A = H_0/\sqrt{4\pi n_0 m_i}.$$

Finally we obtain

$$D_\perp \simeq cT_{0e}/2\pi e H_0. \tag{44}$$

We notice that the diffusion coefficient given by Bohm has just this order of magnitude[4]. It is now necessary to consider the following. According to (43) the value of $(k_z/k_y)_{\max}$ will decrease with increasing magnetic field. It may turn out that, for $k_y \sim (2\pi)/r$, the magnitude of $k_z$ attains the order of $2\pi/L_{\parallel}$ ($L_{\parallel}$ is the longitudinal dimension of the system.) It is clear that one would now obtain the smallest diffusion coefficient, for the largest increase in $H_0$ for $k_y \sim 2\pi/r$. In fact, for constant $k_y$ and when $k_z$ is given a lower limit, $k_z \geqslant 2\pi/L_{\parallel}$, the value of $\omega_s$ becomes greater than $\omega_e$ with increasing $H_0$. If $\omega_s \gg \omega_e$, it is easily seen from (18) that $D$ grows with increasing $k_y$. But as has been shown, for $\omega_e \gg \omega_s$, $D$ decreases with increasing $k_y$; i.e. for $\omega_s \approx \omega_e$ the value of $D$ rises to a maximum. Thus for magnetic fields which are greater than

$$H^* \approx L_{\parallel}^{2/3}\, c(m_i m_e \nu_e T_e)^{1/3}/r^{4/3} e \tag{45}$$

(for $H \approx H^*$, $k_z \approx 2\pi/L_{\parallel}$ and $k_y \approx 2\pi/r$), the minimum value of $k_y$ is obtained from (43) as before, where now $k_z \approx 2\pi/L_{\parallel}$. The expression for the diffusion coefficient then assumes the form

$$D_\perp \approx \frac{cT}{2\pi e H_0} \cdot \frac{H^*}{H_0}. \tag{46}$$

In these discussions, the radius of the plasma column has been taken as fixed and independent of the magnetic field. We note also that drift-wave instability leads to a diffusion coefficient $\sim 1/H^2$ in particular when $\nu_{0i} \geqslant \omega_e, \omega_s$[15], and a similar situation arises for current instability when $H > H^{**}$, where

$$H^{**} \approx \frac{L_{\parallel}}{r} \cdot \frac{(k_R V_{z0})^{1/3} c m_i^{1/3} m_e^{2/3}}{e \tau_e^{2/3}}. \tag{47}$$

The very abundance of experimental data, in which the influence of greatly differing factors is often superimposed, makes it difficult to give a detailed analysis of all the experiments on plasma diffusion. This matter demands, at the very least, a separate detailed discussion. We shall confine ourselves here to the analysis of those experiments in which the influence of a longitudinal current appears to be unimportant.

In the experiments described in GUTHRIE and WAKERLING[4], ($r \approx 0{\cdot}3$ cm, $H \approx$ (3–4).$10^3$ gauss, $p \approx 5.10^{-4}$–$5.10^{-3}$ torr, $T_e \approx T_i \approx 2$ eV) $\nu_{0i} \ll \omega_s$, $\omega_e \lessdot \omega_s$. Therefore, in accordance with (35), the expression for the diffusion coefficient approximates to (46), and this is confirmed by these experiments. In the same paper[4] the presence of low-frequency oscillations was discovered, having a frequency of the order of magnitude of that of drift waves.

In several experiments, the anomalous diffusion sets in above a certain field $H_{\text{crit}}$. Thus in the experiment of ZHARINOV[17] using an argon discharge having a neutral-gas density of $7.10^{13}$ cm$^{-3}$, an electron density of $10^{11}$ cm$^{-3}$, $T_e \approx T_i \approx 1$ eV, $L_{\|} \approx$ 10 cm, anomalous phenomena set in at $H_0 \approx 2300$ oersteds. It is easy to understand the existence of a lower magnetic-field limit if it is assumed that anomalous diffusion will not be masked by classical diffusion when

$$\frac{cT}{2\pi e H_0} > D_{i\perp}. \tag{48}$$

($D_{i\perp}$ is the classical diffusion coefficient for the ions.)

From (48) we have

$$\omega_{Hi} > 2\pi \nu_{0i} \tag{49}$$

which gives an accurate qualitative and a satisfactory quantitative agreement with the experiment.

If one uses expression (36b) for the growth rate and takes the radius of the plasma column to be determined by classical diffusion (only if it has not been set by virtue of the conditions of the experiment) then we obtain the following condition for instability:

$$\omega_{Hi}^2 \cdot \frac{\sigma_{0e}}{\sigma_{0i}} \cdot \frac{l_e^2}{L_{\|}^2} \sqrt{\frac{m_i}{m_e}} > \nu_{0i}^2 \tag{50}$$

where $l_e$ is the electron mean free path; $\sigma_{0e}$, $\sigma_{0i}$ are respectively the effective collision cross section of electrons and ions with neutrals.

It is of interest to note that the measured diffusion coefficient in the Stellarator coincides with (46)[18]. The analysis of the curves of plasma containment time given by STODIEK *et al.*[18] shows that Bohm diffusion evidently changes to $1/H^2$ diffusion at $H_0 \gtrsim 2.10^4$ gauss. From (45) we find that, for the parameters of the Stellarator ($n \approx 10^{14}$ cm$^{-3}$, $L_{\|} \approx 1$ m, $r \approx 1$ cm, $T_e \geqslant 5$ eV), the magnetic fields at which the character of the diffusion should change are indeed of that order. It would not appear that the longitudinal current is a factor which determines the instability in the Stellarator (the electric fields are 0·3–0·06 V/cm). Indeed, in this case the critical fields required for current instability to cause $1/H^2$ diffusion have a value $H^* \lessdot 10^4$ gauss, i.e. the change of régime occurs earlier than for Bohm diffusion. We note that, when the plasma temperature is decreased, expressions (45) and (47) predict that Bohm diffusion can change to $1/H^2$ diffusion earlier than this latter kind of diffusion would arise due to current instability. This is possibly the explanation of the fact that for a cold plasma ($T \approx 0{\cdot}1$ eV) carrying a sufficiently large longitudinal current (of the order of a few amperes) the rate of diffusion was observed by MOTLEY[19] to rise sharply.

The theory which we have given cannot, strictly speaking, be used when the Larmor radius of the particles becomes comparable with the transverse dimensions of the system, and the characteristic frequencies of the problem are of the order of $\omega_{Hi}$ (thus in the case when $r_{Hi} \approx r$ it is necessary, for instance, to take into account the finite size of the Larmor radius for the unperturbed motion). We do not therefore claim, for example, to explain the experiment of D'ANGELO and RYNN[20], where these parameters are near to this limit. It is, however, curious[20] that the diffusion of the caesium plasma occurs in accordance with classical theory. We should also mention that the present work concerns drift and current instabilities only for the conditions of a magnetized non-uniform plasma. Similar instabilities are, however, sometimes observed with no magnetic field present[21]. It may be expected that instabilities in a plasma which is described by equations (8)–(15) are not exhaustively covered by the considerations in the present work.

## 3. KINETIC THEORY OF THE STABILITY OF A NON-UNIFORM HIGH-TEMPERATURE PLASMA

In the previous section we considered the stability of a plasma, taking into account finite conductivity, on the basis of a two-fluid hydrodynamical plasma and we showed that instability connected with potential perturbations causes anomalous diffusion with, in particular, a diffusion coefficient of the order of that given by Bohm. If, in fact, the perturbations do not derive from a potential ($\omega > k_z V_A$) then the instability is stabilized. Assuming that $\omega \approx \omega_e(x)$ for the maximum diffusion coefficient and that $k_z$ is determined by (48) where $k_y \approx 2\pi/r$, the condition for this potential derivation can be rewritten in the form

$$\sigma_{\|} \frac{{V_{Ti}}^2}{c^2} \ll \omega_{Hi}.$$

With increasing plasma temperature the conductivity increases and this condition should eventually be violated. The plasma should then become more stable (or, more strictly, less unstable). However, it may be that under these circumstances we have gone outside the region of applicability of hydrodynamics, and that the problem of stability should be considered on the basis of collisionless kinetics, since the mean free path is increasing even faster than the electrical conductivity. We shall consider the stability of a collisionless plasma[7,22] in this section, and it will be convenient to consider separately the cases when the magnetic field lines are parallel or non-parallel.

### (i) *Instability of a non-uniform rarified plasma in a magnetic field having parallel lines of force*

As in the preceding sections, we shall concentrate the entire discussion on the example of a flat slab of plasma in a strong magnetic field $H_z$ ($H^2 \gg 8\pi nT \gg m_e H^2/m_i$), assuming for simplicity that only the density $n(x)$ is a function of the co-ordinate $x$, and that $T(x) = \text{const}$†. In the introduction, it has already been remarked that, in a

† We shall not be interested, in this review, in effects arising due to the presence of a gravitational field; they have previously been considered in detail[23].

high-temperature plasma, instability shows itself as the excitation of drift waves by resonant electrons. We shall now establish in more detail a qualitative analysis of the factors leading to instability.

In a uniform plasma the basic source of the imaginary part in the dispersion equation is the term of the type
$$\frac{eE_z}{m}\cdot\frac{\partial f}{\partial v_z}\bigg|_{v_z=\omega/k_z}$$
($f_{i,e}(\mathbf{v}, x)$ are the distribution functions of the ions and electrons in the unperturbed plasma). In a Maxwellian plasma it is responsible for the damping of the waves (the Landau damping). Similar terms should really also be retained for a slightly non-uniform plasma. However, there are additional factors, arising from the $(\mathbf{v}.\nabla)f$ term in the Boltzmann equation. For example, at low frequencies ($\omega \ll \omega_{Hi} = eH/m_i c$), when the motion of the ions and electrons across $H$ takes the form of a drift, the $(\mathbf{v}.\nabla)f$ term gives a contribution $cE_y/H \,.\, \partial f/\partial x$ to the 'electrical' drift. This term should naturally give a contribution to the imaginary part on account of the corresponding half residue $\left(\frac{cE_y}{H}\cdot\frac{\partial f}{\partial x}\bigg|_{v=\omega/k_z}\right)$. If $E_y \gg E_z$ [this occurs, for example, for irrotational perturbations, curl $\mathbf{E} = 0$, having a spatial dependence $\exp i(k_y y + k_z z)$, $(k_y \gg k_z)$], this term may (even for small gradients) exceed the Landau damping, and the plasma will be unstable.

In order to be sure of this, we shall estimate the work done on the particles of the plasma by the electric field of the wave:
$$\begin{aligned}\int \tilde{j}_z E_z \,\mathrm{d}r &= \int \mathrm{d}\mathbf{r}\, ev_z f_1 E_z\bigg|_{v_z=-\omega/k_z}\\ &\approx \int \mathrm{d}\mathbf{r}\,\frac{e\omega E_z}{k_z}\left(-\frac{eE_z}{m}\frac{\partial f_0}{\partial v_z}+\frac{eE_y}{H}\cdot\frac{\partial f_0}{\partial x}\right)_{v_z=-\omega/k_z}\\ &= \int \mathrm{d}\mathbf{r}\,\frac{e^2E_z{}^2\omega^2 f_0}{k_z{}^2T}\left[1-\frac{ck_yT}{eH_0\omega}\left(\frac{n_0'}{n_0}-\frac{T'}{2T}\right)\right]\bigg|_{v_z=-\omega/k_z},\end{aligned} \tag{51}$$
where
$$f_0 = \frac{n_0(x)m^{1/2}}{\sqrt{2\pi T}}\cdot\exp\left(\frac{-mv^2}{2T(x)}\right).$$
Here the integration is carried out over the localized region of the wave packet. From (51) it follows that the plasma will be unstable if the second factor in the integrand exceeds the term describing the Landau damping. The latter depends on the relationship between the spatial gradients of $n(x)$ and $T(x)$ and on the magnitude of the frequency. It is therefore clear that the plasma will be unstable for short drift waves $\lambda < r_i$, since for these the frequency $\omega$ is small compared with $\omega_n = (ck_yTn_0')/(eHn_0)$.

The order of magnitude of the instability growth rate may be determined from the expression for the work done on the particles by the electric field of the wave. By definition $\nu = 1/2w \,.\, \mathrm{d}w/\mathrm{d}t$ where $w$ is the energy of the wave packet. Noting that, for drift waves, there is a basic contribution to the wave energy coming from the energy contained in the density fluctuations
$$w = \int \frac{ne^2\varphi^2}{2T}\,\mathrm{d}\mathbf{r}$$

we obtain

$$\nu = \sqrt{\frac{\pi}{2}} \cdot \frac{\omega[\omega - \omega_n(1 - \eta/2)]}{|k_z| V_{Te}} . \tag{52}$$

Equation (52) is only true if curl $\mathbf{E} = 0$, this being so if $\omega \lessdot k_z V_A$. However, when the condition curl $\mathbf{E} = 0$ is not satisfied, it may be shown that the twisting magnetic field lines are damped in the absence of a temperature gradient [$T(x) = \text{const.}$], and are able to be set up only in the presence of a temperature gradient of definite sign ($\eta = \mathrm{d} \ln T/\mathrm{d} \ln n < 0$). This imposes limits on the magnitude of the phase velocity

$$V_{Ti} \ll \frac{\omega}{k_z} \lessdot V_A \tag{53}$$

where the first inequality has been obtained on the assumption that the Landau damping of the ions may be neglected. For a given wavelength $\lambda_\perp$ the maximum value of the growth rate $\nu_{\max} \approx \sqrt{(\pi m_e)/\beta m_i}\,(\omega - \omega_n)$ occurs precisely when $\omega/k_z \approx V_A$; it increases with decreasing $\lambda_\perp$. The growth rate is comparable to the frequency $\omega \approx (V_{Ti} n')/n$ [cf. (56)] for short wavelength perturbations $\lambda_\perp \approx r_e \beta^{-1/2}$, $\lambda_\parallel \approx n/n'$ $\beta^{-1/2}$.

The discussion of drift-wave excitation given here is dependent on the assumption that the perturbations have the form of a wave packet, since expression (51) is the 'localized' form for the law of the conservation of energy, for the wave plus particle system, within the localized bounds of a wave packet. For the case of a high-temperature plasma, a wave packet does succeed in forming because of the size of the packet $\Delta x \gtrsim \lambda_x$ being much smaller, for the wavelengths $\lambda_x < r_i$ considered, than the whole transparent region of the plasma, which is of the order of magnitude of the transverse dimension $R \approx n/n'$ of the system. Thus all perturbations may be represented as a packet, made from the waves:

$$\Psi = \Psi_0 \exp\left[ik_x(x)x + ik_y y + ik_z z + i\omega t\right] \tag{54}$$

where the spatial dependence of $\Psi$ in the direction of the gradient of $n(x)$ must be determined from the appropriate integro-differential equation for the magnitude of $\Psi(x)$,[24–26] which in approximation (54) leads to an algebraic, in the general case transcendental, equation for $k_x(x)$. Thus, in particular, for potential-derivable drift waves, the equation for $k_x(x)$ in a co-ordinate system where the unperturbed electric field $\mathbf{E}_0 = 0$ has the form[7]

$$-k^2\varphi = 4\pi \sum_j e_j^2 \Bigg[\frac{n}{T_j} - \frac{1}{T_j} \times \sum_{l=-\infty}^{+\infty} \int_{-\infty}^{\infty} \frac{\omega + \dfrac{k_\perp T_j}{m_j \omega_{Hj}} \cdot \dfrac{\mathrm{d}}{\mathrm{d}x}}{\omega + k_\parallel v_\parallel + l\omega_{Hj}} f_j^{(0)}(v_\parallel) F_l(k^2 r_j^2)\Bigg] \varphi \tag{55}$$

where $F_l(k^2r^2) = I_l(k^2r^2)\,\mathrm{e}^{-k^2r^2}$ (here $I_l$ is the Bessel function of order $l$ with imaginary argument);

$$f_j^{(0)}(v_\parallel) = n_0(x)\sqrt{\frac{m_j}{2\pi T_j}} \exp\left(-\frac{m_j v_\parallel^2}{2T_j}\right)$$

is the unperturbed particle distribution function for longitudinal velocities.

For low-frequency drift waves ($\omega \ll \omega_H$) only the term with $l = 0$ is retained in the sum over all $l$ in this equation, and the integrals are expanded using the conditions (53). Expression (55) then leads to the following equation ($T_e = T_i$, $k_\perp r_e \ll 1$):

$$2 - \frac{\omega + \omega_n(x)}{\omega} F(k_\perp{}^2 r_i{}^2) - i\sqrt{\frac{\pi}{2}} \cdot \frac{\omega - \omega_n(x)}{|k_z| V_{Te}} = 0 \tag{56}$$

from which it is seen that the frequency of the oscillations

$$\operatorname{Re} \omega \approx \frac{\omega_n(x) F}{2 - F} \tag{57}$$

as has already been mentioned, remains roughly constant for short waves $\lambda_\perp < r_i$, and the growth rate is given by expression (52).

So, as in the case of a uniform plasma, one may use an algebraic equation such as (55) for determining the instability growth rate. However, as has been indicated, the growth rate obtained describes the excitation of a wave packet only at a given point in space and it changes with movement of this point. According to the eikonal method, the motion of wave packets is described using equation:[16,27]

$$\frac{\partial n_k}{\partial t} - \left(\frac{\partial n_k}{\partial x} + \frac{\partial n_k}{\partial k_x} \cdot \frac{\partial k_x}{\partial x}\right)\frac{\partial \omega_k}{\partial k_x} = 2\nu_k n_k,$$

which expresses the conservation of the adiabatic invariant $n_k = w_k/\omega_k$. ($w_k$ is the spectral density of the oscillations; $\omega_k$ is the frequency; $\nu_k$ is the growth rate (or decay rate) of a wave packet in its resonant interaction with the particles.) The net effect of the growth or decay of the wave naturally depends on the net work done by the particles over all their trajectories through the wave packet. Thus, if a packet enters a region of damping before it has had time to grow sufficiently, it will decay completely.

A detailed examination shows that, for sufficiently long potential–derivable drift waves ($\lambda_\perp \geqslant r_i$), the allowed region of propagation is bounded on two sides, for the distribution of plasma particles shown in Fig. 2, and in all this region the electrons will excite a wave[26]. Therefore, the wave packet from long-wave perturbations, moving between the two points of inflection i.e. the points were $U(x) = 0$, will grow until non-linear effects become operative. On the other hand, short-wave perturbations are always able to fall in the region where there is very strong Landau damping of the ions. Nevertheless, due to the large growth rate $\nu \approx \omega$ these perturbations succeed in growing as far as the non-linear effects before they get into the region of damping.

In contrast to the instability associated with low-frequency drift waves ($\omega \ll \omega_H$), which exists for an arbitrarily small density gradient, the instability of drift waves having frequencies near to harmonics of the cyclotron frequency arises only if a certain critical value of density gradient is exceeded. We shall not delay here to consider the instability (arising from the same mechanism of resonant interaction between the particles and the wave) for which the dispersion equation is easily obtained from (53) by the substitution, in the denominator of the second term, of the frequency $\omega$ over $\omega - l\omega_H$. We shall merely observe that the maximum growth rate of the instability is given, as before, by the expression $\nu \sim (\omega - l\omega_H)(\omega - \omega_n)/|k_z| V_{Te}$ with $(\omega - l\omega_H)/|k_z| \approx V_A$, and the instability itself exists for gradients $r_i n'/n > 1/k_\perp r_i \sim \sqrt{m_e/m_i\beta}$ for $\nu \approx \omega - l\omega_H$.

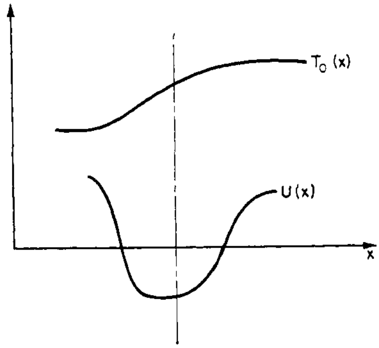


FIG. 2.—Possible region of localization ($U(x) < 0$) for long-wave oscillations.

Let us discuss in detail the instability of a hydrodynamic type[28]. For simplicity we shall assume[28] that $T_e = 0$, $T_i \neq 0$ and that the wave is propagated strictly perpendicular to the magnetic field ($k_z = 0$). Then, from (52) we obtain the dispersion equation for perturbations with frequencies $\omega \simeq -l\omega_H$,

$$1 + k^2\left(\mathrm{d}_i^2 + \frac{m_e}{m_i} r_i^2\right) + \frac{\omega_n^i}{\omega} = \frac{1}{\sqrt{2\pi}\, kr_i} \cdot \frac{\omega + \omega_n^i}{\omega + l\omega_H},$$

$$\mathrm{d}_i^2 = \frac{T_i}{4\pi ne^2}.$$

Here we shall replace the function $F(k_\perp^2 r_i^2)$ by its asymptotic expression $F \approx 1/\sqrt{2\pi (kr_i)^2}$ for $kr_i \gg 1$. Solving this quadratic equation for $\omega$, we obtain the growth rate and frequency of the perturbations

$$\omega = \frac{l\omega_H(1+\theta) + \omega_n^i - \dfrac{V_{Ti}}{\sqrt{2\pi}R}}{2\left(1 + \theta - \dfrac{1}{\sqrt{2\pi}\,kr_i}\right)} \pm \frac{\sqrt{\left[l\omega_H(1+\theta) + \omega_n^i - \dfrac{V_{Ti}}{\sqrt{2\pi}R}\right]^2 - 4\left(1 + \theta - \dfrac{1}{\sqrt{2\pi}\,kr_i}\right) l\omega_H \omega_n^i}}{2\left(1 + \theta - \dfrac{1}{\sqrt{2\pi}\,kr_i}\right)} \qquad (58)$$

where

$$R^{-1} \approx \frac{n'}{n}; \qquad \theta = k^2\left(\frac{m_e}{m_i} r_i^2 + \mathrm{d}_i^2\right).$$

It is then easily proved that the growth rate reaches its maximum value $\nu \approx 2\sqrt{(m_e kr_i)}/\sqrt{m_i} \times l\omega_H(1+\theta)^1$ for $l\omega_H(1+\theta) - \omega_n^i = -(V_{Ti})/\sqrt{2\pi}\, R$. Equating the frequency

$\omega \approx -\omega_n{}^i(1+\theta)^{-1}$ to its minimum value (in absolute magnitude) $\omega_H$ (for $l = 1$), we obtain the critical magnitude of density gradient at which instability sets in,

$$r_i R^{-1} \geqslant 2\left(\frac{m_e}{m_i}\right)^{1/2} \quad \text{for} \quad kr_i \approx \sqrt{\frac{m_i}{m_e}}. \tag{59}$$

The maximum growth rate for such short-wave perturbations, having a wavelength of the order of $\lambda \approx r_i\sqrt{m_e/m_i}$ equals

$$\nu \approx \left(\frac{m_e}{m_i}\right)^{1/4} l\omega_H \tag{60}$$

and the frequency intervals where the instability has $\nu \neq 0$ are

$$\Delta \operatorname{Re} \omega \approx \frac{V_{Ti}}{R} \approx \left(\frac{m_e}{m_i}\right)^{1/2} l\omega_H,$$

i.e. the instability is almost purely aperiodic.

To conclude this section we shall briefly consider the case when the collision frequency is sufficiently high that a hydrodynamic description ($\lambda_\parallel > l_e$) is justified for the electrons, whilst for the ions collisions are still negligible ($\nu > \nu_{i/i} = (\sqrt{m_e/m_i})\, \nu_{e/i}$)[29]. The motion of the electrons along the magnetic field is then described by the equation

$$-ik_z n T_0 - en_0 E_z - m_e n_0 V_{ez} \nu_e = 0. \tag{61}$$

As in the first section, we have retained the term $m_e n_0 V_{ez}\nu_e$ in the equation of motion of the electrons, corresponding to friction between the electrons and the ions. The physical mechanism of the instability therefore corresponds exactly to that considered in the introduction for the case of a low-temperature plasma.

Using the equation of continuity

$$i\omega n + \frac{eE_y}{H} n_0' + ik_z n_0 V_{ez} = 0$$

and the connexion (coming from the Boltzmann equation for the ions) between the density perturbation and that of the electric field $E = -\nabla\varphi$, for intermediate perturbation frequencies [$k_z V_{Ti} < \omega < k_z V_A$, cf. (52)]

$$n = \frac{e\varphi}{T} n_0\left[1 - F\left(1 + \frac{\omega_n}{\omega}\right)\right], \qquad \omega \ll \omega_H,$$

we obtain the frequency $\omega$ and the growth rate of the instability $\nu$

$$\omega = \frac{\omega_n F}{2 - F}; \qquad \nu \approx -\frac{\nu_e \omega_n(\omega_n - \omega)F}{k_z{}^2 V_{Te}{}^2(2-F)^2} \quad \text{for} \quad \nu \ll \omega. \tag{62}$$

As for the excitation of low-frequency drift waves ($\omega \ll \omega_H$) in a high-temperature plasma, described by the Boltzmann equations for the ions and the electrons, the instability considered is almost aperiodic for perturbations having certain wavelengths, the values of which may easily be found from (62) and from (53).

Summarizing the results of this section, it can be noted that a truly universal instability, occurring for any arbitrarily small density gradient, is observed only for low-frequency ($\omega \ll \omega_H$) potential perturbations in the frequency range $k_z V_{Ti} < \omega \lesssim k_z V_A$. Instability for perturbations having frequencies close to harmonics of the ion Larmor frequency, $l\omega_H$, may also exist only if the gradients exceed a certain critical value $r_i n'/n > (m_e/m_i)^{1/2}$ and, in contrast to the low-frequency modes, it is not associated with a resonant mechanism of excitation.

In this section the stability has been studied of perturbations relatively widely extended along the lines of force ($\lambda_{\|} > \omega/V_{Ti}$ or even $\lambda_{\|} = \infty$). In the following section the influence will be considered of a slight shear in the magnetic field, causing a violation of this necessary condition for the existence of instability.

(ii) *Stability of a non-uniform rarified plasma in a magnetic field having shear*

We shall take the lines of force as lying in the $(y, z)$ plane and making a small angle $\theta(x)$ with the $z$-axis, changing with $x$. Then the shear can lead to two stabilizing effects:

(a) Due to the fact that the phase velocity, along the lines of force, of a wave packet moving along the $x$-axis, $\omega/k_{\|}(x)$ ($k_{\|} = k_z + k_y \int^x (\mathrm{d}\theta/\mathrm{d}x)\,\mathrm{d}x$) decreases greatly and becomes comparable to the thermal velocity of the ions, $V_{Ti}$, the wave packet is subjected to a strong Landau damping by the ions. The condition for stabilization may be simply written in the form†

$$\int \frac{\mathrm{d}\theta}{\mathrm{d}x}\,\mathrm{d}x \geqslant \frac{\omega}{k_{\perp} V_{Ti}} \sim \frac{r_i}{R}. \tag{63}$$

(b) In the localized region of a wave packet one may always neglect the ion Landau damping over several wavelengths ($\omega/k_{\|} \gg V_{Ti}$), but due to the shear the region $\Delta X$ over which the plasma is transparent, being strongly dependent on the magnitude of $k_{\|}(x)$, becomes shorter than one wavelength $\lambda_x$ of the oscillation and the existence of such perturbations becomes impossible. This leads to stability with the condition

$$\Delta X \leqslant \lambda_x \tag{64}$$

(this stabilizing mechanism is entirely similar to the levelling of a potential well due to its narrowing, which occurs in quantum mechanics under similar conditions).

The first stabilizing effect acts on a universal instability for low-frequency ($\omega \ll \omega_H$) potential perturbations, in the frequency range $k_z V_{Ti} < \omega \leqslant k_z V_A$. The energy balance in the 'wave plus particles' system, taking into account Landau damping from the ions as described by the half-residue in the integrals containing the ion distribution function $f_i^{(0)}$ in (51), can be written in a form similar to (5),

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \int \mathrm{d}\mathbf{r}\, \frac{e^2 E_z^2 \omega^2 n_0 m_e^{1/2}}{k_z^2 \sqrt{2\pi T^3}} \left[1 - \frac{\omega_n}{\omega} + F\sqrt{\frac{m_e}{m_i}}\left(1 + \frac{\omega_n}{\omega}\right) \times \exp\left(-\frac{\omega^2}{2k_z^2 V_{Ti}^2}\right)\right].$$

It directly follows from this that the wave is damped for $\omega/k_z V_{Ti} \leqslant 1$. As is apparent from (63) the most difficult to stabilize are perturbations in which the possible region of localization is small, so that the wavelength $\lambda_{\|}(x)$ is unable to decrease strongly

† The stabilization of a drift instability was proposed by ROSENBLUTH[30]. A more detailed investigation of this question has been made by GALEEV[26].

within it. It may be shown that such small regions of localization occur near to the points of inflection $x_1$, where $k_x(x_1) = 0$, and have a width $\Delta x = r_i^{1/3}R^{2/3}$, so that from (63) we obtain a criterion for universal instability suppression[26]†

$$R\,\frac{\mathrm{d}\theta}{\mathrm{d}x} > \frac{1}{4}\left(\frac{r_i}{R}\right)^{2/3}. \tag{65}$$

This criterion is satisfied also for the instabilities considered at the end of the preceding section, since the motion of the ions is determined by the Boltzmann equation and their resonant interaction with the wave leads to the damping of the latter.

For perturbations not derivable from a potential ($\omega > k_z V_A$), in distinction to those which are, ion Landau damping may be neglected in any localized wave packet region. Therefore the stabilizing mechanism for these non-potential perturbations, which occur in the presence of a temperature gradient $\eta < 0$, results in a levelling described by condition (64), which in this case assumes the form[26]

$$R\,\frac{\mathrm{d}\theta}{\mathrm{d}x} > \sqrt{\beta\,|\eta|}. \tag{66}$$

(iii) *Discussion of non-linear effects*

The results of the linear stability theory of the equilibrium of a non-uniform rarified plasma in a strong magnetic field show that, at least in magnetic traps with a large Larmor radius ($r_i/R$ not too small), it is difficult to stabilize instabilities having wavelengths $r_e\beta^{-1/2}$ by means of field shear and they essentially remain universal. It is important to know to what extent the fluctuations, arising from this instability, impair the magnetic containment of the plasma.

The influence of the fluctuations of the electric and magnetic fields on the distribution of the particles is calculated, as in the low-temperature case, by a quasi-linear method[9]. In the most interesting case of an almost aperiodic instability ($\nu \approx \omega$) the splitting of the distribution function of the ions, which is carried out in this method, into rapidly oscillating and slowly varying parts, is not entirely rigorous, and the discussion takes on a qualitative character.

The limits of applicability of the method are somewhat wider for electrons, and are given by the expression $\nu \ll |k_z| V_{Te}$.

Let us consider, for simplicity, the case $T_i(x) = T_e(x) = \text{const}$. It is also convenient to restrict ourselves to the drift approximation in which the Boltzmann equation has the form

$$\frac{\partial f}{\partial t} + c\,\frac{\mathbf{E}\times\mathbf{H}}{H^2}\cdot\nabla f + v_z\,\frac{\mathbf{H}}{H}\cdot\nabla f + \frac{eE_z}{m}\,\frac{\partial f}{\partial v_z} = 0.$$

We shall assume fields of the form

$$E_y = \sum_{\mathbf{k}} E_{\mathbf{k}_y} \mathrm{e}^{i\omega_k t + i\mathbf{k}\cdot\mathbf{r}} + \text{complex conjugate}$$

† For small $\beta = 8\pi nT/H^2 < (r_i/R)^{2/3}$ and long wavelength perturbations $\lambda \gg r_i$, this criterion is somewhat relaxed for incidental reasons, and assumes the form[26]

$$R\,\frac{\mathrm{d}\theta}{\mathrm{d}x} > \left(\frac{r_i}{R}\right)^{1/2}\beta^{1/4}\left(\ln\sqrt{\frac{M_i}{m_e\beta}} - 1\right)^{1/} .$$

the connexion between **E** and **H** for instability (56) being given by the expressions:

$$E_{\mathbf{k}x} = \frac{k_x}{k_y} E_{\mathbf{k}y}; \qquad E_{\mathbf{k}z} = \left(1 - \frac{\omega(\omega + \omega_n)}{k_z^2 V_A^2}\right) E_{\mathbf{k}y};$$

$$H_{\mathbf{k}y} = -\frac{k_x}{k_y} H_{\mathbf{k}x}; \qquad H_{\mathbf{k}x} = \frac{c(\omega + \omega_n)}{k_z V_A^2} E_{\mathbf{k}y}.$$

After the normal procedure[8,9] we obtain the averaged equation for the slowly varying part

$$\frac{\partial f_0}{\partial t} = \sum_k \left(\frac{c}{H} \cdot \frac{\partial}{\partial x} + \frac{e}{m} \frac{k_z}{k_y} \frac{\partial}{\partial v_z}\right) \times \frac{\nu_{\mathbf{k}} |E_{\mathbf{k}y}|^2}{(\omega + k_z V_z)^2 + \nu^2} \left(1 - \frac{\omega^2 + \omega\omega_n}{k_z^2 V_A^2}\right) \times \left(\frac{c}{H} \cdot \frac{\partial}{\partial x} + \frac{e}{m} \cdot \frac{k_z}{k_y} \cdot \frac{\partial}{\partial v_z}\right) f_0 \equiv \mathrm{St}_{\mathrm{wave}}\{f\}. \tag{67}$$

Here the term

$$\frac{\partial}{\partial x}\left[\sum_k \left(\frac{cE_{ky}}{H}\right)^2 \frac{\nu_k}{(\omega + k_z V_z)^2 + \nu^2} \left(1 - \frac{\omega^2 + \omega\omega_n}{k_z^2 V_A^2}\right) \frac{\partial}{\partial x} f_0\right]$$

describes 'anomalous' diffusion. The cross term containing $\partial^2/\partial x \partial v_z$ also gives the particle flux along $x$. It may be shown that in the optimum case it leads to a contribution of the order of magnitude of that of the term with $\partial^2 f/\partial x^2$. The term with $\partial^2 f/\partial v_z^2$ is responsible for diffusion in velocity space. We shall call the entire right-hand side ($\mathrm{St}_{\mathrm{wave}}\{f\}$) the 'wave-particle Stoss term' ('collision' term). An accurate expression for this quasi-linear 'Stoss term', in the general case of an arbitrary ratio between the wavelength of the fluctuations and the Larmor radius, has been obtained[31] by integrating along the particle trajectories.

Short-wavelength perturbations ($kr_i \gg 1$) with a wavelength $\lambda \approx r_e \beta^{-1/2}$ will give a basic contribution to the anomalous diffusion. This comes about because it is just for these perturbations that the growth rate reaches its maximum at which $\nu \sim \omega$. We have so far assumed the velocity distribution of the electrons to approximate to Maxwellian. Actually however, as follows from the quasi-linear equation (67), just as for resonant electrons ($v_z \approx \omega/k_z$) the form of the distribution function may be distorted by the wave. Therefore all the expressions for the growth rate, assuming the existence of a Maxwellian velocity distribution for the electrons, are only correct in the case when electron–electron collisions, whilst being infrequent, nevertheless are able to establish a Maxwellian distribution. The formal significance of this is that in equation (67) it is necessary to keep in mind also the ordinary 'Stoss term' which takes into account the collisions between electrons ($\mathrm{St}_{\mathrm{particle}}\{f\}$), since $\mathrm{St}_{\mathrm{particle}}$ $\{f\}$ exceeds $\mathrm{St}_{\mathrm{wave}}\{f\}$. In the general case when the term $\mathrm{St}_{\mathrm{wave}}\{f\}$ is no longer small compared with the Coulomb term, one must determine the form of the electron distribution function for the velocity region $v_z \approx \omega/k_z < V_A$ from equation (67). The growth rate $\nu$ in the general case of an arbitrary distribution $f_e(v_z)$ has the form

$$\nu = \pi \frac{\omega T}{|k_z| e} \left(\frac{ek_z}{m} \cdot \frac{\partial f_e}{\partial v_z} - \frac{ck_y}{H} \cdot \frac{\partial f}{\partial x}\right). \tag{68}$$

We shall first of all deduce a general expression for the diffusion coefficient. For this we shall integrate equation (67) with respect to $v_z$. Using expression (68) for the growth rate, one may obtain (for $\nu \ll \omega$) the equation of diffusion

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial x}\left[\sum_{\mathbf{k}} \left(\frac{cE_{\perp y}}{H}\right)^2 \frac{\nu_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \cdot \frac{1}{kr_i}\right] \frac{\partial n}{\partial x}. \tag{69}$$

Evidently, for $\nu \approx \omega$, the expression for the diffusion coefficient

$$D = \sum_{\mathbf{k}} \left(\frac{cE_y}{H}\right)^2 \frac{\nu_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \cdot \frac{1}{kr_i} \tag{70}$$

is inexact and can only be used for order of magnitude estimates.

The most difficult stage in the solution of the problem consists in finding the turbulence spectrum $|E_{\mathbf{k}}|^2$. The quasi-stationary fluctuation spectrum which establishes itself after the development of instability is the result of the action of two factors: (1) due to the instability, energy is continuously fed into those fluctuation modes which are unstable ($\nu_{\mathbf{k}} > 0$); (2) due to the non-linear interaction between the modes, the energy is transferred through the spectrum into the region where $\nu_{\mathbf{k}} < 0$, and gets dissipated. In principle one may rigorously describe this process by means of the

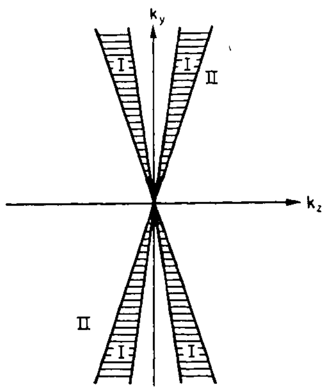


FIG. 3.—Diagram of the distribution of turbulence fluctuations in wave-vector space.

Boltzmann equation for the interacting waves only when $\nu \ll \omega$. Here we shall give a non-rigorous† estimate of the order of magnitude of the fluctuations, from the following graphical physical considerations. Let us consider two cases:

(a) $\mathrm{St}_{\text{particle}}\{f\} > \mathrm{St}_{\text{wave}}\{f\}$, when expression (52) may be used for the growth rate. If the turbulent 'background' is represented as the superposition of different harmonics with scale $\mathbf{k}$ then the process of establishing the spectrum in 'tongues' (Fig. 3) will occur in the following way. Instability excites waves with $\mathbf{k}$ lying inside region I. Non-linear interaction between such waves leads to the formation of fluctuations with $\mathbf{k}$ lying in region II, where strong damping due to the ions takes place. When the rate of both processes attains an equal order of magnitude, the quasi-stationary picture is

† In the most interesting case the unwieldy approach, using the Boltzmann equation for the waves, can hardly pretend to be highly accurate.

established. The chief non-linear term in the Boltzmann equation for the ions has the form $eE_{\perp}/m_i . \partial f/\partial v$. Comparing it with the linear term $\nu f \approx eE_{\perp}/m_i . \partial f/\partial v$ and putting $\partial f/\partial v \approx \mathbf{f}/V_{Ti}$, we obtain an estimate of the electric field amplitude

$$E_{\perp} \approx \frac{\bar{\nu} m_i V_{Ti}}{e} \tag{71}$$

where for $\bar{\nu}$ one evidently understands the maximum growth rate (which is attained for $kr_i \approx (m_i\beta/m_e)^{1/2}$. Now, for the diffusion coefficient we shall have

$$\begin{aligned} D_{(1)} &\approx \sum_k \left(\frac{cE_{ky}}{H}\right)^2 \frac{\nu_{\mathbf{k}}}{\omega_{\mathbf{k}}^2} \cdot \frac{1}{kr_i} \\ &\approx \frac{c^2 \overline{E_{\perp}}^2}{H^2} \cdot \frac{\bar{\nu}}{\bar{\omega}^2} \cdot \frac{1}{\overline{kr_i}} \approx \frac{cT}{eH} \cdot \frac{r_i}{R}\left(\frac{m_e}{m_i\beta}\right)^{1/2} \end{aligned} \tag{72}$$

where $R^{-1} \approx (1/n)\, \partial n/\partial x$. This estimate, although not pretending to give a numerically accurate coefficient, apparently reflects the physical dependence. It is interesting to observe that turbulence fluctuations having a characteristic dimension $\lambda \approx r_i$, in which a basic fraction of all the energy of the fluctuations is included, lead to a much smaller plasma diffusion coefficient. For an estimate of the distribution through the spectrum of the energy of such long wave fluctuations ($\lambda \gg r_e\, \beta^{-1/2}$) it may be considered that, owing to the smallness of the growth rate $\nu \ll \omega$, the phase shifts of separate fluctuations succeed in becoming disordered. Then in the random phase approximation the transfer of energy through the spectrum may be described by means of the Boltzmann equation for wave numbers $n_k = w_k/\omega_k$ with the collision integral being responsible for the non-linear interaction of the waves.[16,27] As a result we find that in the fluctuations with a characteristic scale length $\lambda \approx r_i$ there is contained an energy

$$w \approx \frac{ne^2}{2T} \sum_k \varphi_{\mathbf{k}}^2 \approx \frac{n_0 T_0}{200} \left(\frac{r_i}{R}\right)^2 \frac{\nu_{\mathbf{k}}}{\omega_{\mathbf{k}}} .$$

Substituting this value in (70) we obtain the diffusion coefficient

$$D \approx \frac{1}{100} \cdot \frac{cT}{eH} \cdot \frac{r_i}{R} \cdot \frac{m_e}{m_i\beta} .$$

(b) An interesting situation arises in the opposite limiting case $\mathrm{St}_{\mathrm{wave}}\{f\} \gg \mathrm{St}_{\mathrm{particle}}\{f\}$, which occurs at sufficiently high temperatures when Coulomb collisions become very infrequent. It turns out that in this case we are relieved of the most difficult stage, the consideration of the non-linear interaction of the waves. Indeed a reduced value of the growth rate $\nu$ already strongly enters into the diffusion coefficient

$$D \approx \overline{\frac{c^2 E_{\perp}^2}{H^2}} \cdot \frac{\bar{\nu}}{\bar{\omega}^2} \cdot \frac{1}{\overline{kr_i}} . \tag{73}$$

In order to find $\bar{\nu}$ one must know the form of the distribution function in the region of the resonance velocities $v_e \approx \nu/k_z \leqslant V_A$. Since $\mathrm{St}_{\mathrm{wave}}\{f\} \gg \mathrm{St}_{\mathrm{particle}}\{f\}$ we shall look for those solutions for $f$ in the resonance region, having the form $f = f_0 + f_1 + \ldots$ where the zero-order approximation $f_0$ satisfies the equation

$$\mathrm{St}_{\mathrm{wave}}\{f_0\} = 0. \tag{74}$$

The solution of this equation gives a smooth plateau in $(x, v_z)$ space, so that

$$\nu^0\{f\} = \pi \frac{\omega T}{|k_z|e}\left(\frac{ek_z}{m_e} \cdot \frac{\partial f_0}{\partial v_z} - \frac{ck_y}{H} \cdot \frac{\partial f_0}{\partial x}\right) = 0. \tag{75}$$

Finding $f_1$ from the equation

$$\mathrm{St}_{\mathrm{wave}}\{f^{(1)}\} = \mathrm{St}_{\mathrm{particle}}\{f_0\} = 0$$

and reckoning at the same time that the magnitude of $f_0$ itself (but not of its derivative with respect to $v_z$) differs little from Maxwellian, we obtain;

$$\nu^{(1)} = \nu_e \nu_M / \bar{D}_v(v_z) \tag{76}$$

where $\nu_M$ is the value of the growth rate for a Maxwellian distribution function, $\bar{D}_v = (e^2\bar{E}_z^2)/m^2\omega$. Now from (73) and (76) we obtain the coefficient of anomalous diffusion

$$D_{(2)} = \nu_e R^2 \sqrt{\frac{m_e}{m_i \beta}}\,. \tag{77}$$

The regions of applicability of the two limiting cases indicated may now be formulated thus. In the first case $D_{(1)} \gg D_{(2)}$ and vice versa for the second. For parameters $n \approx 10^{15}$ cm$^{-3}$, $H = 6.10^4$ gauss, the dividing boundary between the two cases lies in the region of $T \approx 10^4$ eV, i.e. for high temperatures it is necessary to use the formulae of the second case.

All these results are applicable if $\beta$ lies within the interval

$$\left(\frac{m_e}{m_i}\right)^{1/3} > \beta > \frac{m_e}{m_i}\,.$$

In the present section we have only discussed the most interesting case of plasma diffusion, due to the development of universal instability, taking place in any arbitrary density gradient. Aperiodic instability concerning perturbations with frequencies $\omega$ in the neighbourhood of harmonics of the ion cyclotron frequency $l\omega_H$[28] over a certain range of the plasma parameters $\beta$, $r_i$ and in large density gradients, may introduce a comparable contribution into the diffusion, but it does not substantially exceed that contribution from universal instability. Neither shall we dwell on the case, intermediate between hydrodynamics and the kinetic description of the plasma, since this would simply give a smooth transition between the values of diffusion coefficient in these two limiting cases.

Drift instabilities may be called universal in a certain sense. They remain, even in a plasma in which magnetohydrodynamic instabilities can be suppressed. However, as we have seen, it is possible to visualize a situation when the plasma is stable also with respect to the excitation of drift waves. In a 'cold' plasma, the presence of neutral gas is a stabilizing factor; it also leads to the phenomenon of the 'critical' field.

A second stabilizing factor is due to the circumstance that the drift waves are strongly 'extended' along the magnetic lines of force. In the case when the volume occupied by the plasma is limited along $H$, the plasma will be stable. Thus if one visualizes a plasma in the form of a column along $H$, then for stability it is necessary that the length of the column be less than about ten radii. This condition is not difficult to satisfy for the plasma in magnetic traps of the mirror type. It is more difficult to achieve it in traps of the toroidal type (in the Stellarator). In this case a marked twisting of the lines of force (shear, Section 3) may be necessary.

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