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Strong turbulence, self-organization and plasma confinement

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Abstract. This paper elucidates the close connections between hydrodynamic models of two-dimensional fluids and reduced models of plasma dynamics in the presence of a strong magnetic field. The key element is the similarity of the Coriolis force to the Lorentz force. The reduced plasma model, the Hasegawa–Mima equation, is equivalent to the two-dimensional ion vortex equation. The paper discusses the history of the Hasegawa–Mima model and that of a related reduced system called the Hasegawa–Wakatani model. The 2D fluid ↔ magnetized plasma analogy is exploited to argue that magnetized plasma turbulence exhibits a dual cascade, including an inverse cascade of energy. Generation of ordered mesoscopic flows in plasmas (akin to zonal jets) is also explained. The paper concludes with a brief explanation of the relevance of the quasi-2D dynamics to aspects of plasma confinement physics.

1 Introduction and historical review

Plasma is a rich medium for studying the physics of fluids, in that a large number of characteristic modes (or waves) are accessible because of plasma's collective electromagnetic and hydrodynamic properties. In addition to the collective modes, charged particles contribute to various forms of wave–particle interactions. These also play important roles in determining its dynamical properties. Magnetically confined plasma contains various forms of intrinsic free energy that may lead to excitation of these dynamics. Excited modes often behave nonlinearly through wave–wave or wave–particle interactions that broaden spectra. Their energy tends to cascade to lower frequency modes, leading to turbulent states, sometimes with self-organized structures. When frequencies of the characteristic modes are much higher than the rate of nonlinear frequency shifts or energy transfer rates, the turbulent dynamics can be treated by wave kinetic equations for wave action density (here, action is the ratio of wave energy density to its characteristic frequency), which is preserved in the phase space of wave number and position. Here the wave frequency and wave number play the role of energy and canonical momentum of the wave quantum, a consequence

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of the Wentzel–Kramers–Brillouin (WKB) approximation. The characteristic modes sometimes do not satisfy the golden rule of the wave action conservation, owing to strong nonlinear interactions in which the rate of wave energy cascades exceeds the characteristic frequencies. Then the turbulence characteristics become hydrodynamic in nature, in which the characteristic modes are masked and the turbulence spectra can be described only in wave number space. Here Kolmogorov's idea of inertial range spectra plays an important role. When this type of situation arises, plasma physicists call it a strong turbulent state. Strongly turbulent plasma sometimes exhibits non-trivial characteristics, e.g. when the system possesses global conservations in addition to energy and self-organizes into various interesting structures. Because of the rich characteristics of plasma such situations often arise (see examples in Hasegawa 1985). This paper describes one example where the self-organized state produced by drift wave instability, an instability intrinsic to inhomogeneous plasma, provides favorable self-organized state for confinement of the plasma energy.

Pressure gradients in magnetically confined plasmas induce various micro-instabilities with perpendicular scale sizes on the order of ion gyro radius ρ_i and frequencies ω_d of the order of $v_d/\rho_i \sim \omega_{ci}\rho_i/a$ where ω_{ci} is the ion cyclotron frequency, v_d is the diamagnetic drift speed and a is the perpendicular scale size of the plasma. Instabilities of this type do not stabilize easily because they are intrinsic to plasmas with inhomogeneity in plasma temperature and/or density. As a result, instabilities have often been called universal instability, or more specifically drift wave instability. Early theoretical research on this type of turbulence was centered on the so-called weak turbulence theory that allowed small frequency spread around the linear drift wave frequency (e.g. Sagdeev and Galeev, 1969). Pioneering experimental observations by Mazzucato (1976) by microwave scattering and by Surko and Slusher (1976) by laser scattering from the ATC Tokamak at Princeton Plasma Laboratory in early 1970s have revealed the existence of density fluctuations produced by these instabilities. However, contrary to theoretical expectations prior to this time, the observed density fluctuation had frequency spectra much broader than the drift wave frequency itself. As a result, the observed broad spectra were suspected to be caused by the sampling of the scattered signal integrated over the path of the ray of the diagnostic signal. To clarify the spectra, Slusher and Surko (1978) improved the scattering experiment by employing pinpoint dual laser scattering and showed that the broad spectrum is not caused by the effect of integrated path average but is intrinsic and local to the turbulence.

At Bell Laboratories where Surko and Slusher brought back the scattering data, Hasegawa and Mima (a postdoc who was working with Hasegawa), had the opportunity to discuss the experimental results directly with them. These experimental observations have required development of new theory of strong drift wave turbulence that should explain compressive plasma fluctuations with broad frequency spectra. The so-called strong turbulence theory existed prior to this time, using $\boldsymbol{E} \times \boldsymbol{B}$ nonlinearity. However, the associated plasma flow is incompressible, and thus is irrelevant to explain the observed density fluctuations (if the non-adiabatic electron response is negligible). Hasegawa and Mima (1977) recognized that the polarization drift is compressible and nonlinear since the total (not partial) time derivative of the electric field is inducing the drift and constructed a model quasi-two dimensional nonlinear equation for density fluctuations with the polarization drift of ions, together with the assumption of Boltzmann electron response. Note that the strong toroidal magnetic field tends to confine ion motions in the plane perpendicular to the magnetic field, while electrons with light mass are expected to move freely in the direction of the magnetic field, causing electron density to obey the Boltzmann distribution. The derivation of the Hasegawa–Mima equation is recalled in Section 2. Later this equation was rewritten in the real space variables (Hasegawa et al., 1979) and used to present the property of producing strong plasma turbulence even with the normalized amplitude of the density fluctuation much smaller than unity. In other words, nonlinear frequency shift becomes comparable to the linear drift frequency, ω_d, even at relatively small level of turbulence density fluctuations, $\tilde{n}/n_0 \approx \rho_i/a$, where n_0 is the background density.

It was later recognized, through discussions with the late Professor Tosiya Taniuti, that the model equation derived by Hasegawa and Mima may be identified as the ion vortex equation with convective $\boldsymbol{E} \times \boldsymbol{B}$ motion of vorticity. This recognition elucidated the relation between the drift wave turbulence and hydrodynamic turbulence. The identification also led Hasegawa et al. (1979) to adopt results of hydrodynamic turbulence developed by Kolmogorov (1941) and Kraichnan (1967), who introduced the concept of inertial range turbulence spectra for three- and two-dimensional fluids respectively. The inverse cascade, which had been predicted by Kraichnan for two-dimensional fluids turbulence, is a consequence of the existence of an additional conservative quantity called the enstrophy (squared vorticity). The Hasegawa–Mima equation does possess two conservation quantities, energy and potential enstrophy, a combination of enstrophy and density perturbation. The analogy between the plasma dynamics based on the Hasegawa–Mima equation and the two-dimensional hydrodynamics has led Hasegawa et al. (1979) to look into the spectral cascades of the drift wave turbulence based on combination of weak turbulence and strong turbulence concepts. The weak turbulence approach led to cascade of turbulence spectra to smaller frequencies, which led to smaller wave numbers of the spectra in the direction perpendicular to the density gradients but to larger wave numbers in the direction of the density gradient to $k_\perp \varrho_i \approx 1$. This concept lead to possible formation of zonal flows in the azimuthal direction in cylindrical plasma for the first time. On the other hand, the inverse cascade concept of the two-dimensional Navier–Stokes turbulence led to isotropic cascade and smaller wavenumbers. The choice between the weak turbulence and the hydrodynamic turbulence seemed to depend on the level or strength of the turbulence.

Recognition of the analogy of the drift wave turbulence and two-dimensional hydrodynamic turbulence also led Hasegawa et al. (1979) to relate the drift wave to the Rossby wave (Rossby et al., 1939) on the surface of the atmosphere of rotating planets. Here the gradient of the Coriolis parameter influences the density gradient in forming the wave frequency. More precise comparison between the Rossby wave and a drift wave in a plasma holds when a gradient of the magnetic field intensity, rather than the density, is introduced in the plasma. The mathematical structure of the Hasegawa–Mima equation, however, is quite analogous to that of the Charney equation (Charney, 1948) that describes the turbulent nature of the Rossby wave. The analogy is quite indicative of production of asymmetric property of vortices depending on whether they have a high pressure or low pressure core as well as possible excitation of azimuthal zonal flows in plasma similar to the one observed in the Jovian atmosphere (Williams, 1978; Hasegawa, 1983).

The model equation derived by Hasegawa and Mima successfully describes nonlinear evolution of the drift wave, but lacks a dissipation process that would drive (and dissipate) the drift wave instability. In other words, the linearized equation does not produce an unstable wave. In order to show the instability, either electron Landau damping or resistivity that provides phase lag in electron motion (in the direction of the ambient magnetic field) with respect to the potential field is required. Without a driving mechanism, the dynamical process that leads to strong turbulence cannot be properly studied.

Computer simulation is a powerful tool for analyzing the dynamical process that leads to full turbulence. However, to simulate the full development of the instability starting from the linear instability, one needs a computer sufficiently powerful to follow electron inertia as well as ion inertia that have a time difference of three orders of magnitude. Unlike hydrodynamic turbulence, the mass ratio problem in plasma particles has been a major obstacle in the numerical study of plasma turbulence (until recently). However, representing electron inertia by electron resistivity that can provide effective time lag in the electron response in the parallel motion can also drive drift wave instability. Based on this concept, Hasegawa, together with the late Professor Masahiro Wakatani of Kyoto University (Wakatani and Hasegawa, 1984), developed a simulation model that allows resistive growth of the drift wave and sink by ion viscosity to study turbulent spectra cascades based on the Hasegawa–Mima model. As shown in Section 3, the model equation now involves two dependent field variables, potential φ and number density n. The wave number spectrum obtained reveals two-dimensional inertial range spectrum of the enstrophy in large wave number regime of the type k^{-3}. This result indicates that the energy spectrum will have an inverse cascade nature and will form a self-organized structure. The collaborative works of Hasegawa and Wakatani continued and in 1987 (Hasegawa and Wakatani, 1987), we examined the self-organized state of the turbulence via inverse cascade of energy in a cylindrical plasma (Hasegawa, 1987). Wakatani suggested introducing magnetic curvature that can drive global hydrodynamic instability, as well. If the free energy source is large enough, the level of wave amplitude will become larger than the ratio of the drift wave frequency to the ion cyclotron frequency and strong turbulence is expected to result globally. This may be the case when the wave is generated by a combination of magnetic field curvature and a pressure gradient. The model equation again involves two dependent variables, the electrostatic potential, φ and the density fluctuation, n. With this model equation, we showed that the excited turbulence forms macroscopic (azimuthal) zonal flow. The structure of the zonal flow was explained in terms of the hydrodynamical self-organization concept with the help of the variational principle of minimizing the potential enstrophy with constraints of constant energy and fluid angular momentum. Furthermore, the macroscopic zonal flow produces a radial potential barrier, i.e. closed equipotential line in the azimuthal direction. Hasegawa and Wakatani claimed that the potential barrier would stop radial electron heat flow, aiding confinement of the plasma. The result was surprising, because plasma turbulence was shown to reduce radial transport for the first time, rather than enhance transport, unlike earlier predictions. It is now believed that the inhibition of radial particle transport by the azimuthal flow (zonal and mean) can account for the H-mode operation of tokamak plasmas (e.g. Biglari et al., 1990; Diamond et al., 2005). The derivation of the Hasegawa and Wakatani equation and the simulation result are introduced in Section 3. It was later recognized that the zonal flow excites the geodesic acoustic mode (GAM) in toroidal plasmas, which has a characteristic frequency that depends on the geometry of the device (e.g. Hamada et al., 2012). This has been helpful in experimental identification of the zonal flow.

For over a quarter of a century, the fundamental properties of plasma turbulence – the existence of strong turbulence, inverse cascade of turbulent energy and formation of zonal flow, as well as the reduction of the radial transport by the excitation of the zonal flow – have been verified by a large number of experiments, theory and simulations [see reviews by Diamond et al. (2005), Fujisawa (2009) and Kikuchi and Azumi (2015)]. It is now understood that the drift wave amplitude is regulated by zonal flows, the transport flux depends upon the drift wave amplitudes, and the transport flux decreases when the zonal flow amplitudes increase. Therefore, the transport flux depends upon the dissipation rate of the zonal flows (Fig. 1). The system is open to external energy input and energy sink. The quasi-equilibrium pressure profile is maintained by the injected low entropy energy. The increased system entropy is exhausted by the energy sink, which allows maintenance of the low entropy state of the system.

In this way, the energy and particle transport phenomena are closely related to the drift wave turbulence and the zonal flows. The existence of zonal flows is verified

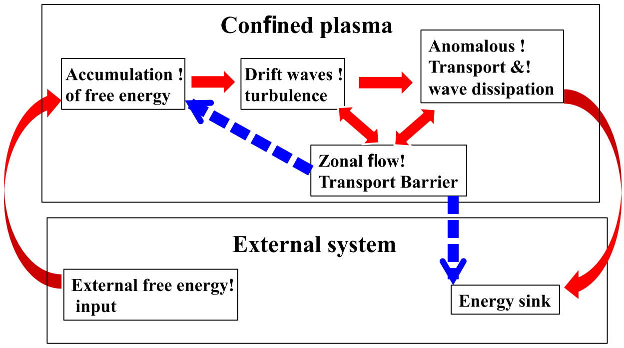

Fig. 1. Schematic of system of drift wave and zonal flow plasma turbulences and transport. Inhomogeneity of plasmas (density and temperature gradient) generates the drift wave that produces enhancement of transport and simultaneously zonal flow turbulence that reduces transport [see more detail in Fujisawa (2009)]. Note that the system is open to external energy input and energy sink. Increased system entropy is exhausted, which in return allows maintenance of a low entropy state of the system.

experimentally by observing, for example, the correlation of the potential fluctuations in the toroidal direction by the HIBP's (Heavy Ion Beam Probes) (Fujisawa, 2009). The experimental results show the characteristics of the zonal flows, which are (1) symmetric around the magnetic axis, (2) of finite radial wavelength, and (3) nonlinearly coupled with drift waves.

It may also be helpful to contrast weak versus strong turbulence theory in the context of their historical evolution. The frequency spectra of the turbulent fluctuations, and the implications of these for the temporal evolution of the system, have been common foci of studies of various types of plasma turbulence, including those of drift waves, Langmuir waves, whistler waves, Alfven waves, etc. In those systems, two nonlinear wave interaction processes are widely investigated, as they play important roles in the spectral evolution. These are mode–mode coupling and modulational instability (Taniuti and Washimi, 1968), which often trigger inverse cascades.

In the mode–mode coupling and/or wave particle scattering (nonlinear Landau damping), the total number of quanta (the total action) is conserved (the Manley–Rowe relation) for the non-zero frequency fluctuations and the spectra cascade into lower frequency modes. Through these processes, the turbulent fluctuation energy is transferred into long wavelength modes. As the final state of the inverse cascade, all the wave energy is expected to condensate into the minimum frequency mode of the system. These processes are common in various turbulences, such as drift wave turbulence (Hasegawa et al., 1979), Langmuir wave turbulence (Zakharov, 1972), kinetic Alfven wave (Hasegawa and Chen, 1975) turbulence, EMHD turbulence (Biskamp et al., 1996) and whistler wave turbulence (Galtier and Bhattacharjee, 2003).

The second process is the modulational instability, which generates long wavelength fluctuations, namely the low frequency modes. The modulational instability of drift wave turbulence has been investigated by Mima and Lee (1980), Diamond and Kim (1991), and Tynan et al. (2001). The coupling between drift waves and zero frequency modes, namely the axisymmetric mode, and zonal flow has been investigated by Rosenbluth and Hinton (1998), Diamond et al. (2000), and Nagashima et al. (2009). The modulational instability is driven by Reynolds stress in the drift wave turbulence. In drift wave turbulence theory, the shearing of drift waves by the zonal flows plays an important role in reducing the radial correlation of the

$\boldsymbol{E} \times \boldsymbol{B}$ vortices to decrease the fluctuation level and the transport (Diamond et al., 2000). These ideas are described in Section 4.

It is worthwhile to present in this account an alternative plasma confinement scheme where the self-organization is achieved by maximizing the global entropy of a plasma in a closed system (Hasegawa, 1987). The idea is based on the observed high beta stable plasma confinement in planetary magnetosphere where plasma confinement is achieved by distribution of plasma energy uniformly among magnetic flux tubes in the dipole magnetic field. In an ideal point dipole, the flux tube volume increases in proportion to r^4, where r is the distance from the point dipole core. A uniformly distributed energy among flux tubes presents a rather steep pressure profile $p \sim r^{-20/3}$, which can provide sufficiently large energy density near the core for fusion to take place. Such a confinement scheme will be presented in Section 5.

2 Hasegawa–Mima equation

When Mima was visiting Hasegawa at Bell Laboratories in 1976, Surko and Slusher (1976) brought back interesting laser scattering data from the tokamak at Princeton. The data supported those obtained by Mazzucato (1976) obtained using microwave scattering, showing a very broad frequency spectrum – one wider than the expected drift frequency itself. Hasegawa and Mima were intrigued by the data and immediately constructed a theory of drift wave turbulence to account for this unexpected result. In response to criticism suggesting that the broad spectra might have been due the path averaging of the scattering date, Slusher and Surko (1978) later performed a pinpoint scattering experiment by means of two lasers, demonstrating the validly of the earlier date.

We recognized that in order to account for the density fluctuation observed in those data, it was necessary to take compressible ion flow into account. Since $\boldsymbol{E} \times \boldsymbol{B}$ drift is incompressible, we recognized the importance of the polarization drift, which is intrinsically nonlinear because of the convective electric field. We then realized that this nonlinear polarization drift could best represent the nonlinearity needed to account for the observed broad spectrum and constructed the mass conservation equation, taking into account the ion polarization drift and Boltzmann electron response. The resulting nonlinear equation, which is now called the Hasegawa–Mima equation, is shown to properly account for the observed broad frequency of the density fluctuation. The nonlinear frequency shift caused by the nonlinear polarization drift can become comparable to the linear drift wave frequency itself, even at the level of density fluctuation relative to the background density much less than unity.

Thanks to a number of intensive discussions with the late Professor Taniuti, close connections were recognized between the Hasegawa–Mima equation and hydrodynamic equations for two-dimensional fluids as well as waves in atmospheres. The discussion further identified the Hasegawa–Mima equation as being nothing but the two-dimensional ion vortex equation.

2.1 Derivation of Hasegawa–Mima equation

Let us re-derive the Hasegawa–Mima equation based on the ion vortex concept. The ion vorticity $\mathbf{\Omega}$ due to the $\boldsymbol{E} \times \boldsymbol{B}$ drift can describe the Laplacian of the electrostatic potential, φ

$$\mathbf{\Omega} = (\nabla \times \mathbf{v}_\perp) \cdot \hat{\mathbf{z}} = \nabla \times \left(\frac{-\nabla\phi \times \hat{\mathbf{z}}}{B_0} \right) = \frac{\nabla_\perp^2 \phi}{B_0} \, \widehat{\mathbf{z}} \; . \tag{1}$$

Here, B_0 is the flux density of the ambient magnetic field and $\nabla_\perp^2$ is the Laplacian operator in the direction perpendicular to the magnetic field. The equation of ion vorticity can be constructed by taking the curl of the ion equation of motion in the direction normal to the ambient magnetic field. If the pressure and density gradients are parallel (no baroclinic effect), we obtain a conservation equation for the total ion vorticity,

$$\frac{d}{dt}(\Omega + \omega_{ci}) + (\Omega + \omega_{ci})\nabla \cdot \mathbf{v}_\perp = 0. \tag{2}$$

Here ω_{ci} is the ion cyclotron frequency and represents the magnetic vorticity and the time derivative is a Lagrangian (or convective) derivative to be explicitly shown in equation (4). The compressible ion flow is related to the ion density fluctuation, n, through the continuity equation,

$$\nabla \cdot \mathbf{v}_\perp = -\frac{d}{dt}\ln n = \frac{d}{dt}\left(\ln n_0 + \frac{e\phi}{T_e}\right). \tag{3}$$

The quasi-neutrality condition relates the ion density fluctuation to that of electrons, which will obey the Boltzmann equilibrium in this frequency range due to their inertia-less motions along the magnetic field. Even if ion dynamics are treated in a two-dimensional plane perpendicular to the direction of the magnetic field, electrons move freely in three dimensions. From equations (2) and (3), we can construct the equation for ion vorticity for its compressible flow,

$$\frac{d}{dt}\left(\ln\frac{\omega_{ci}}{n_0} + \frac{\Omega}{\omega_{ci}} + \frac{e\phi}{T_e}\right) \equiv \left(\frac{\partial}{\partial t} - \frac{\nabla\phi\times\,\widehat{\mathbf{z}}}{B_0}\cdot\nabla\right)\left(\ln\frac{\omega_{ci}}{n_0} + \frac{\Omega}{\omega_{ci}} + \frac{e\phi}{T_e}\right) = 0. \tag{4}$$

We note here that we made no assumption on the amplitude of the variables and thus equations (1) and (4) are exact and fully nonlinear. Equation (4) also shows that the time evolution of vorticity and potential becomes nonlinear due to the nonlinear convection of the ion vorticity, as is expected, while the nonlinear convection of the potential does not contribute to nonlinear responses since $\nabla\varphi \times \hat{\mathbf{z}}$ is orthogonal to $\nabla\varphi$. We further note that in equations (1) and (4), the wave amplitudes are represented by normalized ion vorticity and field potential, which by themselves, are small quantities. However, their space-time evolutions are balanced with the spatial scale of background magnetic field and density, which are also small since spatial variation of them are small in a distance of the ion Lamor radius. Therefore, the space-time evolution of the potential field becomes fully nonlinear if the normalized potential field, $e\varphi/T_e$ becomes on the order of the Larmor radius to the scale size of the plasma density variation, which is typically the size of the plasma minor radius. Thus, for these space-time scales, the evolution of the potential field becomes fully nonlinear and is given by,

$$\frac{\partial}{\partial t}(\nabla_\perp^2\phi - \phi) - [(\nabla\phi-\,\widehat{\mathbf{z}}\,)\cdot\nabla]\left[\nabla_\perp^2\phi - \frac{1}{\varepsilon}\ln\left(\frac{n_0}{\omega_{ci}}\right)\right] = 0. \tag{5}$$

Here the small parameter ε represents the normalized amplitude as well as the time as shown below,

$$\varepsilon\omega_{ci}t \equiv t, x/\rho_s \equiv x, \quad e\phi/T_e \equiv \varepsilon\phi, \quad \text{where} \quad \rho_s = \sqrt{T_e/m_i}/\omega_{ci}. \tag{6}$$

Equation (5) describes the fundamental property of electrostatic plasma turbulence in the drift wave frequency range in the absence of dissipations and is often referred to as the Hasegawa–Mima equation.

Inspection of equation (5) reveals several important properties of the potential field:

1. The time evolution scale is much smaller than the ion cyclotron frequency.

2. The (drift) wave nature is produced by inhomogeneity of plasma density and/or magnetic fields.

3. All the terms have comparable magnitudes if the scale of inhomogeneity within the distance of the ion Larmor radius is on the order of the normalized amplitude, ε, or less, and the equation becomes fully nonlinear.

4. Any potential function that depends only on the coordinate orthogonal to the direction of inhomogeneity is an exact solution.

Property #3 immediately accounts for the broad frequency spectra of drift wave turbulence observed by Mazzucato and by Surko and Slusher. This occurs when the drift wave becomes fully nonlinear – if the normalized amplitude becomes on the order of the ratio of ion Lamor radius over the scale length of the plasma density variation. In addition, property #4 indicates the appearance of zonal flow since a potential field that is any function of radius (if the background density and or magnetic field varies only in the direction of radius) is an exact solution.

2.2 Conservation laws, inverse cascades in kinetic and hydrodynamic regime

The model equation (5) that describes the drift wave turbulence demonstrates that drift wave becomes fully nonlinear even at a very low fluctuation level. In this section, we describe general properties of drift wave turbulence in hydrodynamic as well as in kinetic regimes based on this model equation.

2.2.1 Conservation laws

In the absence of inhomogeneity (the conservation laws in the presence of inhomogeneity will be discussed in Sect. 3), equation (5) has interesting conservation laws and consequences. First, it has energy conservation, which can be constructed by multiplying it with φ and integrating over the volume surrounded by a conductor wall,

$$\frac{\partial W}{\partial t} \equiv \frac{\partial}{\partial t}\int \left[(\nabla_\perp \phi)^2 + \phi^2\right] dV = 0. \tag{7}$$

Thus, the sum of the kinetic and potential energies is conserved. The interesting part of the conservation law is that there exists additional conservation. The additional conservation is the sum of the enstrophy, the squared vorticity, and the kinetic energy, sometimes called as potential enstrophy,

$$\frac{\partial U}{\partial t} \equiv \frac{\partial}{\partial t}\int \left[(\nabla_\perp \phi)^2 + (\nabla_\perp^2 \phi)^2\right] dV = 0. \tag{8}$$

The presence of conservation of the (potential) enstrophy, in addition to the energy, is familiar in two-dimensional hydrodynamics. As a result, interesting turbulence phenomena, such as dual cascades of turbulent spectra, is expected (see Kraichnan, 1967). We will discuss the physical mechanism and indication of the dual cascades in plasmas later in this section.

2.2.2 Boltzmann statistics

In the absence of dissipation, the Boltzmann statistics that are obtained by maximizing the entropy in such a system has a strange character because of the additional constraint that comes from the enstrophy conservation: the distribution function f that maximizes entropy with constraints of conservation of energy and enstrophy is obtained by variational principle,

$$\delta\left(\int f \ln f dV - \lambda_1 W - \lambda_2 U\right) = 0. \tag{9}$$

Here λs are Lagrange multipliers, and the resultant distribution function is given by

$$f \sim \exp(-\lambda_1 W - \lambda_2 U). \tag{10}$$

In the wave number space, since $U_{\mathbf{k}} = k^2 W_{\mathbf{k}}$, the average energy per mode becomes, from equation (10),

$$\langle W_k \rangle = \frac{1}{\lambda_1 + k^2 \lambda_2}. \tag{11}$$

As pointed out by Onsager (1949), this result not only violates the equipartition law but also could result in a negative temperature if the product of λ_1 and λ_2 is negative. Thus, modal statistics can be strange, indicating consequences such as unexpected states.

2.2.3 Self-organized state in hydrodynamic regime

In the presence of dissipation, turbulence described by the Hasegawa–Mima equation for a homogeneous plasma may lead to a dual cascade of the spectrum as a result of the two conserved quantities, the energy and the enstrophy, in a way similar to the two dimensional Navier–Stokes turbulence, as described by Kraichnan (1967). The inertial range Kolmogorov spectra (Kolmogorov, 1941) can be defined dually, one for energy and the other for enstrophy. Kraichnan argues that when a two-dimensional fluid is excited at a wave number $\mathbf{k_s}$, the enstrophy spectrum would cascade to large wave numbers, because it will be dissipated by viscosity, owing to higher order dependence of wavenumber $\mathbf{k}$ than the energy. This will lead to inertial range spectrum of enstrophy, where the omnidirectional energy spectrum will be given by k^{-3}. Kraichnan showed that in this spectrum range there is no energy cascade. On the other hand, the omnidirectional Kolmogorov spectrum given by $k^{-5/3}$, applies to the range of wave numbers that give energy cascade, which applied to wave numbers smaller than $\mathbf{k_s}$ in two-dimensional fluids. Here there is no enstrophy cascade. In hydrodynamic turbulence, the energy cascades to small wave numbers and condensates to the minimum wave number that the system allows, while the enstrophy cascades to large wavenumbers where it is dissipated by viscosity. This is a process of self-organization of two-dimensional turbulence. In homogeneous plasma, the self-organized state can simply be described by variation of minimizing the enstrophy with the constraint of constant energy, Hasegawa (1985),

$$\delta\left(\int U - \lambda W\right) = 0. \tag{12}$$

With the help of the definition of the energy and the enstrophy in equations (7), (9) and (12) gives the eigenvalue equation for φ,

$$\nabla^2\phi + \lambda\phi = 0. \tag{13}$$

This equation may be solved for a given boundary condition and the smallest eigenvalue gives the self-organized state.

2.2.4 Weak turbulence theory

If the normalized amplitude of the wave is much smaller than the ratio of the drift wave frequency to the ion cyclotron frequency, the nonlinear term in the Hasegawa–Mima equation may be treated as perturbation and the evolution of the turbulent spectrum may be studied using the weak turbulence theory. We take three waves with wave numbers $\mathbf{k}_1, \mathbf{k}_2$, and $\mathbf{k}_3$, such that

$$k_1^2 \leq k_2^2 \leq k_3^2.$$

The linearized Hasegawa–Mima equation gives the drift wave frequency as

$$\omega_{\mathbf{k}} = \frac{(\mathbf{k}\times\ \widehat{\mathbf{z}})\cdot\nabla\ln(n_0/\omega_{ci})}{\varepsilon(1+k^2)}, \tag{14}$$

and the nonlinear term can be treated as a perturbation to produce coupling of the three waves (Hasegawa et al., 1979). In the three-wave couplings, the number of wave quanta may be defined by

$$N_p = (1+k_p^2)|\varphi_p|^2/|k_q^2 - k_r^2|, k_p^2 \neq k_r^2, \tag{15}$$

where φ_p is the Fourier amplitude of the pth mode. The wave quanta are conserved such that,

$$N_3 - N_1 = \text{const.}, \quad N_1 + N_2 = \text{const.}, \quad N_2 + N_3 = \text{const.}$$

This result shows that a loss of one quantum of the wave with wave number $\mathbf{k}_2$ appears as a gain in one quantum of the wave with the wave numbers given by $\mathbf{k}_1$ or $\mathbf{k}_3$. This indicates a dual cascade of energy to smaller and larger wave numbers. A clearer picture that is consistent with the well-known weak turbulence result can be obtained by requiring resonant three wave interactions such that the wave matching condition is satisfied. This is possible in small wave number region such that,

$$k_p^2 - k_q^2 = k_{qy}/\omega_q - k_{ry}/\omega_r = \omega_p M, \tag{16}$$

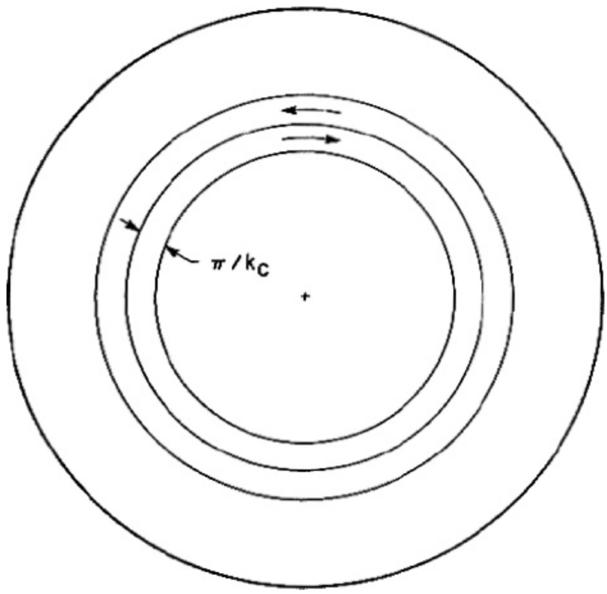

Fig. 2. Zonal flow pattern predicted from the weak turbulence theory.

where k_y is the wave number in the direction of $\widehat{\mathbf{z}} \times \nabla \ln n_0$ and

$$M = \frac{\omega_p(k_{ry} - k_{qy}) + \omega_q(k_{py} - k_{ry}) + \omega_r(k_{qy} - k_{py})}{3\omega_p\omega_q\omega_r}. \tag{17}$$

In this regime, the wave energy decays into smaller frequencies since the wave quanta defined by

$$N_k = W_k/\hbar\omega_k, \tag{18}$$

is conserved. Since the drift wave frequency is proportional to k_y, this result indicates that the wave energy decays to that with smaller k_y, keeping k_x more or less constant at $1/\rho_s$ ($\equiv k_c$). In cylindrical plasma, this indicates that the wave energy cascades to a mode having zero azimuthal wave numbers as shown in Figure 2, with radial wave number being kept at k_c.

The theory presented by Hasegawa et al. (1979) is the first prediction of the formation of the zonal flow. It should be noticed, however, that at the limit of zero-frequency the wave-quantum concept of weak turbulence theory breaks down, thus the introduction of hydrodynamical turbulence is unavoidable.

3 Hasegawa–Wakatani model, prediction of the zonal flow in hydrodynamic regime and plasma confinement

The model equation derived by Hasegawa and Mima successfully describes nonlinear evolution of the drift wave but lacks a dissipation process that drives the drift wave instability. Representing electron inertia by electron resistivity that can provide effective time lag in the electron parallel motion can also drive drift wave instability. Based on this concept, Hasegawa together with Wakatani (Hasegawa and Wakatani, 1983), developed a simulation model that allows resistive growth of the drift wave and sink by ion viscosity and studied turbulent spectra cascade based on the Hasegawa–Mima model. The model equation involves two dependent variables, potential φ and number density n. The wave number spectrum obtained has revealed two-dimensional inertial range spectrum of the enstrophy in large wave number regime of the type k^{-3}. This result indicates that the energy spectrum will have inverse cascade nature and will form a self-organized structure. The collaborative works of Hasegawa with Wakatani continued and in 1987 (Hasegawa and Wakatani, 1987), we decided to look into the self-organized state of the turbulence via inverse cascade of energy in a cylindrical plasma. Wakatani suggested introducing magnetic curvature that could drive global hydrodynamic instability, as well. The model equation again involves two dependent variables, the electrostatic potential, ϕ, and the density fluctuation, n. In a cylindrical geometry, the equation of vorticity reads,

$$\frac{\rho_s^2}{a^2}\frac{d}{dt}\nabla_\perp^2\phi = (\nabla \ln n \times \nabla\Delta)\cdot \widehat{\mathbf{z}} + \frac{\omega_{ce}}{\upsilon_{ei}}\left(\frac{a}{R}\right)^2 \nabla_{||}^2(\ln n - \phi) + \frac{\mu}{\omega_{ci}a^2}\nabla_\perp^4\phi. \tag{19}$$

Here Δ represents the curvature of the toroidal magnetic field, ω_{ce} and υ_{ei} are electron cyclotron frequency and electron–ion collision frequency, a and R are minor and major radia of the toroidal container, and μ represents the ion viscosity. The equation of continuity is also modified due to the curvature and reads,

$$\frac{d}{dt}\ln n = (\nabla \ln n \times \nabla\Delta)\cdot \widehat{\mathbf{z}} + \frac{\omega_{ce}}{\upsilon_{ei}}\left(\frac{a}{R}\right)^2 \nabla_{||}^2(\ln n - \phi). \tag{20}$$

Equations (19) and (20) form a closed set of nonlinear equations that describe evolution of the potential and density fields. Even in the presence of the curvature term, it can be shown that in the in viscid limit, the total energy, W and potential enstrophy, U are conserved;

$$W = \frac{1}{2}\int \left[(\ln n)^2 + \frac{\rho_s^2}{a^2}(\nabla_\perp\phi)^2\right] dV, \tag{21}$$

and

$$U = \frac{1}{2}\int \left[\frac{\rho_s^2}{a^2}\nabla_\perp^2\phi - \ln n\right]^2 dV. \tag{22}$$

3.1 Result of simulation and discovery of the zonal flow

The conservation of W and U indicates that the existence of inverse cascade of energy spectrum as discussed earlier. In the case of toroidal plasma, in addition to these two conservations, the total angular momentum in the poloidal direction (which should be zero) should also be conserved. Thus, the self-organized state should be obtained by minimizing the potential enstrophy (22), with constrains of conservation of energy (21) as well as the total toroidal angular momentum. The minimization of the potential enstrophy with these constraints will lead to a self-organized structure in which the minimum eigenvalue having a node between the center and the wall will give the self-organized potential field. As a result, at self-organization, the radial electric field reverses its sign at some radial position. This will lead to generation of global azimuthal flow of the plasma that will have a shear. Equations (19) and (20) trace the evolution of the turbulence field excited by the magnetic field curvature, together

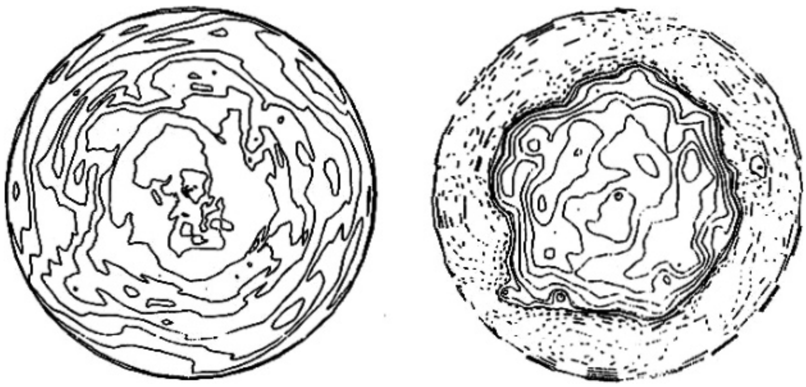

Fig. 3. Equipotential contours of simulation results obtained from equations (19) and (20). The solid (dotted) lines are for positive (negative) equipotential. The curvature-driven plasma turbulence produces a self-organized state in which a global equipotential that a $\varphi = 0$ closed surface near outer edge of the plasma. We note here that positive potential regions near the core region designated by solid lines have large structures while negative potential regions designated by dotted lines near the edge have small structures.

with given density gradient. Results of the simulation are shown in Figure 3. Here the left figure is the initial potential profile (stream function), which is randomly given, and the right figure shows the potential profile at a final time. Here solid (dotted) lines show positive (negative) equipotential lines.

As illustrated in Figure 3b, the self-organized state produces an equipotential line that has a closed node at some outer radial position indicating the formation of a global azimuthal shear flow. In Figure 4, we show the evolution of the radial potential profile in the simulation. Here, the solid line indicates the theoretically predicted self-organized state based on the minimization of the potential enstrophy with constraints of both the total energy and the global angular momentum. This figure shows that the self-organized state produces a global radial electric field that reverses its sign. This electric field creates an $\boldsymbol{E} \times \boldsymbol{B}$ rotation of the plasma in the azimuthal direction that changes direction at some radius, indicating a generation of an azimuthal shear flow. This is the first observation of a zonal flow generation in turbulent plasma. The zonal flow structure is global rather than microscopic, unlike that indicated by kinetic theory (Fig. 1). It may be due to the fact that the instability here is driven by global magnetic field curvature that is hydrodynamic.

3.2 What are the effects of zonal flow on plasma transport?

Zonal flows have been observed to inhibit transport of vortex eddies across the flows. One well-known example is the atmospheric dynamics of the planet Jupiter. As has been demonstrated by Hasegawa et al. (1979), the Hasegawa–Mima equation has a structure mathematically identical to that which describes the horizontal motion of planet atmosphere with gravity and gradient of Coriolis force. On a planet surface, the latitudinal direction in which the Coriolis parameter varies corresponds to the radial direction in cylindrical plasma, the longitudinal direction corresponds to the azimuthal direction and the vertical direction to the axial direction. Figure 5 is a well-known photograph of the Jovian atmosphere.

Existence of longitudinal zonal flow is clearly visible. The zonal flows inhibit convection of atmospheric vortices in the latitudinal direction across the zonal flows.

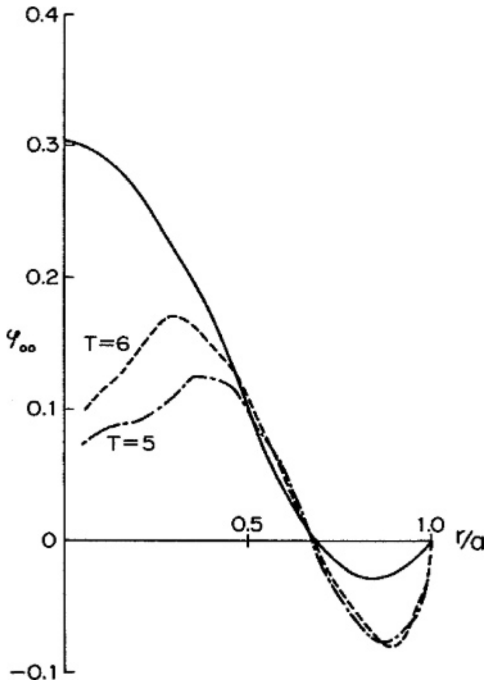

Fig. 4. Temporal variations of the radial profile of axis symmetric potential profile are shown for different time. The solid line is the theoretically predicted potential profile obtained from the self-organization theory.

Fig. 5. A profile of atmospheric motion on Jovian surface. Existence of longitudinal zonal flows is clearly visible. The zonal flows are seen to inhibit convection of atmospheric vortices in the latitudinal direction crossing the flow.

For example, the red spot seen on the right lower side of the picture is found to be stably trapped over several centuries.

Comparison of the mathematical structure of the model equation that describes the plasma turbulence in the magnetic field to that of the planetary atmospheric motion indicates that a similar turbulent dynamic may also operate in plasmas.

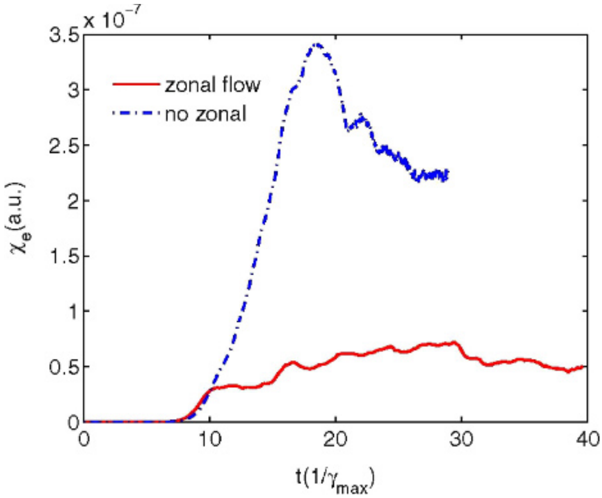

Fig. 6. Simulation result obtained by Xiao et al. (2010). The figure shows time evolution of electron heat transports. The solid line is the heat transport obtained in the simulation where self-generated zonal flow is present, while the dotted lines is the heat transport in which the zonal flows are artificially removed.

The simulation result obtained by Hasegawa (1987) indicates that convection of the plasma vortices seems to be inhibited across the equipotential surface of $\varphi = 0$. Because electrons are expected to move on the constant φ surface, they cannot move across the equipotential surface. Furthermore, if vortices cannot convect across this surface, zonal flows are expected to inhibit ion motion also in the radial direction. Interestingly, this indicates that instability which excites a self-organized state of plasma may inhibit anomalous plasma transport in the radial direction. This contradicts the prediction of anomalous diffusion. The role of zonal flow in good confinement phase of tokamak plasmas called H-mode, is now believed to be related to the appearance of the zonal flow. A recent simulation result obtained by Xiao et al. (2010) shows the reduction of the electron heat conduction in the presence of zonal flows, as shown in Figure 6.

The figure shows time evolution of electron heat transport. The solid line is the heat transport obtained in the simulation where the self-generated zonal flow is present, while the dotted line is the heat transport in which the zonal flows are artificially removed.

Although, at present, the existence of zonal flows and their possible role in the reduction of anomalous transport have been confirmed in various experiments (e.g. Fujisawa, 2009) and in theoretical studies (e.g. Kikuchi and Azumi, 2015), the stability of the zonal flow and its influence on plasma transport are still being actively pursued.

3.3 Indication of asymmetric properties of vortices having positive and negative core charges

As is well known in the atmosphere of rotating planets, two types of vortices – one with high pressure in the core and the other with low pressure in the core – have different nonlinear properties. Vortices that have a low pressure core tend to increase in intensity as they shrink and grow to typhoon strength, while those with high pressure core tend to expand their scale. In plasmas, a vortex with positive (or negative)

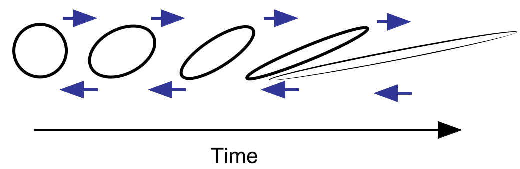

Fig. 7. Generation of larger wave number spectra due to shearing of vortices.

charge in the core, which corresponds to a high (or low) pressure vortex in the atmosphere, expands (or shrinks) its size because the centrifugal force (i.e., polarization drift) on the ions rotating around the core tends to weaken (or strengthen) the core charge. This indicates that vortices with a positive (or negative) core charge tend to increase (or decrease) their size over time, producing asymmetric evolution of these two types of vortices. As a result, the positive (negative) charge vortex shifts its spectrum to smaller (or larger) wavelengths. Consequently, randomly distributed vortices with positive and negative core charges will cascade its spectra to both large and small wave numbers depending on the signs of the core charge. This may be interpreted as the physical process of the dual cascade that takes place as a result of two conserved quantities, energy and enstrophy. These are shown in equations (7) and (9) as well as in equations (21) and (22). Furthermore, the positive (or negative) charge vortices carry energy (enstrophy) spectrum to long (or short) wavelengths since the enstrophy spectrum has larger wavenumber dependence than energy. As a matter of fact, close observation of the simulation result of Hasegawa and Wakatani shown in Figure 3 reveals the coarser (or finer) structures in positive (negative) potential region near the core (or edge). These differences will have effects on transport; the so-called convective cell dominates in the regions of positive charge vortices while turbulent or microscopic transport will dominate in the negative charge vortices.

4 Nonlinear mode coupling between drift waves and zonal flows: a predator–prey model

4.1 Modulational instability as a generation mechanism of zonal flows

Zonal flows may be generated in the drift wave turbulence by the process relating to the modulation of drift wave amplitudes. This process has a positive feedback loop when the zonal flow amplitude is low. However, when the zonal flow amplitudes are higher, the shear flows convert lower mode number drift waves to higher mode number waves through the shearing of drift waves, as illustrated in Figure 7.

The wave kinetic equations for the coupling of short wavelength drift waves and the long wavelength fluctuations (long wavelength drift waves and/or zonal flows) have been studied by Mima and Lee (1980) in the modulational instability of long wavelength drift wave, and in the modulational instability of zonal flow (Tynan et al., 2001; Diamond and Kim, 1991). The nonlinear coupling between short and long wavelength modes is due to the Reynolds stress, which is produced by the nonlinear polarization drift. The radial (in x direction) and azimuthal momentum equation (in y direction) are given as follows:

$$m_i\left[\frac{\partial}{\partial t}n_i\boldsymbol{v}_i+\nabla\cdot(n_i\boldsymbol{v}_i\boldsymbol{v}_i)\right]=en_i(\boldsymbol{E}+\boldsymbol{v}_i\times\hat{\mathbf{z}}B_0)-\nabla P_i. \tag{23}$$

The slowly varying Reynolds stress force per ion in the radial direction (x direction) is obtained by taking the time average of the second term of the right hand side of equation (23) for the incompressible ion fluid,

$$\boldsymbol{F} = -T_e(\hat{\boldsymbol{x}}k_\perp^2\rho_s^2\frac{\partial}{\partial x}\left|\varphi_k\right|^2 - 2\hat{\boldsymbol{y}}\; k_x k_y \rho_s^2 \frac{\partial}{\partial x}\left|\varphi_k\right|^2), \tag{24}$$

where c_s is the ion sound velocity, k_x and k_y are the radial (in x direction) and the azimuthal (in y direction) wave numbers respectively, and

$$\frac{e\phi}{T_e} = \varphi_k(x,t)e^{ik_x x + k_y y - i\omega t} + c.c. \tag{25}$$

Here $\varphi_k(x,t)$ is the normalized amplitude of the electrostatic potential fluctuation of a drift wave, which varies slowly in x and t. The Reynolds stress: equation (24) excites the low frequency drift flow: $\boldsymbol{V}_{ZF}$ as follows Hasegawa and Mima (1978) and Diamond et al. (2005),

$$\boldsymbol{V}_{ZF} = \frac{\vec{\boldsymbol{F}} \times \hat{\mathbf{z}}}{eB} + \boldsymbol{V}_{E\times B} = \rho_s c_s\left(\hat{\boldsymbol{y}}k_y^2\rho_s^2\frac{\partial}{\partial x}\left|\varphi_k\right|^2 + 2\hat{\boldsymbol{x}}k_x k_y\rho_s^2\frac{\partial}{\partial x}\left|\varphi_k\right|^2 + \hat{\boldsymbol{z}}\times\nabla\Phi_{ZF}\right). \tag{26}$$

Here Φ_{ZF} is the normalized electrostatic potential associated with a zonal flow. When the drift wave amplitude and the zonal flow amplitude are time dependent, then the polarization drift associated with the drift of equation (26) becomes,

$$\boldsymbol{V}_{ZFp} = -\hat{\boldsymbol{x}}c_s\frac{\partial}{\omega_{ci}\partial t}\left(k_y^2\rho_s^2\frac{\partial}{\partial x}\left|\varphi_k\right|^2 + \rho_s\frac{\partial\Phi_{ZF}}{\partial x}\right). \tag{27}$$

The divergence of equations (26) and (27) should be zero if the ion density fluctuation associated with the zonal flow is negligible. Then the following equation for the zonal flow excitation is obtained (Diamond et al., 2005).

$$\frac{\partial}{\partial t}\frac{\partial^2\,\Phi_{ZF}}{\partial x^2} = 2k_x k_y\rho_s c_s\frac{\partial^2}{\partial x^2}\left|\varphi_k\right|^2 - \gamma_D\frac{\partial^2\,\Phi_{ZF}}{\partial x^2}. \tag{28}$$

This indicates that zonal flows may be generated by the Reynolds stress, which is a consequence of the nonlinear polarization drift. The last term of equation (28) is introduced here to take into account the viscous dumping of the shear flow.

The coupling between low frequency modes and the drift waves is induced by the nonlinear frequency shift: $\Delta\omega_{NL}$ included in $\boldsymbol{k}\cdot\boldsymbol{V}_{ZF}$. The nonlinear frequency shift is dominated by the slowly varying potential fluctuations of Φ_{ZF}, namely the zonal flows that are induced by the Reynolds stress. This frequency shift couples two drift waves to induce the modulational instability of drift waves. The original drift instability will be suppressed by the modulation of the drift wave.

For the drift wave turbulence, many modes are coupled simultaneously and the equation (28) is rewritten as,

$$\frac{\partial}{\partial t}V_{ZF}^{'} = 2\sum_{k_x,\,k_y} k_x k_y\rho_s c_s\frac{\partial^2}{\partial x^2}\left|\varphi_k\right|^2 - \gamma_D V_{ZF}^{'}, \tag{29}$$

where $V_{ZF}^{'} = \frac{\partial V_{ZF}}{\partial x} = \frac{\partial^2\,\Phi_{ZF}}{\partial x^2}$ is the vorticity.

Let us introduce the action for the drift waves by $N(k,x,t) = \varepsilon(k,x,t)/\omega_k$, where $\varepsilon(k,x,t) = (1+k^2)\left|\varphi_k\right|^2$ is the normalized drift wave energy, wave number k is normalized by ρ_s, and $\omega_k = \frac{k_y v_d}{\omega_{ci}(1+k^2)}$ is the normalized drift wave frequency. The equation for the slowly varying drift wave amplitude introduced by equation (25) in the zonal flows is derived from the Hasegawa–Mima equation as follows:

$$\frac{1}{1+k^2}\frac{\vartheta}{\vartheta t}(1+k^2)^2\ \left|\varphi_k(x,t)\right|^2 - 2\omega_k k_x \frac{\partial}{\partial x}\left|\varphi_k(x,t)\right|^2 - \frac{\partial k_y V_{ZF}}{\partial x}\frac{\partial}{\partial k_x}(1+k^2)\left|\varphi_k(x,t)\right|^2 = 0.$$

Here, as before, the time t is normalized by ω_{ci} and x is normalized by ρ_s.

Since $N_k = (1+k^2)\ \left|\varphi_k(x,t)\right|^2/\omega_k$, the modulation of N_k is described by the wave kinetic equation in the zonal flows as follows,

$$\frac{\partial N_k}{\partial t} + v_{gx}\frac{\partial N_k}{\eth x} - \frac{\partial \omega_k}{\partial x}\frac{\partial N_k}{\partial k} = \gamma_g N_k. \tag{30}$$

Here $v_{gx} = \frac{\partial \omega_k}{\partial k_x}$ is the group velocity and $\gamma_g/2$ is the growth rate of the drift wave. Equation (30) relates the modulation of the drift wave amplitude: $\delta N = N - \langle N\rangle$ to the shear flow as follows,

$$\frac{\partial \delta N}{\partial t} + v_g\frac{\partial \delta N}{\eth x} - \frac{\partial V_{ZF}}{\partial x}\frac{\partial\langle N\rangle}{\partial k} = \gamma_g \delta N. \tag{31}$$

Equations (29) and (31) describe the coupling between the zonal flow and the drift wave to find the growth of the modulation.

4.2 Predator–prey

The coupling between low frequency (or zero frequency modes, i.e. zonal flows) and drift waves are analogous to "predator–prey model" (Hoppensteadt, 2006). Diamond et al. (2005) claim that "drift waves excite zonal flows, while zonal flows suppress drift waves. The degree of excitation or suppression depends upon the amplitudes of the drift waves and the zonal flows". The predator–prey model can be described by equations (28) and (31). Namely, the drift wave spectral intensity evolves according to the quasi-linear diffusion equation in the wave number space as follows (Diamond et al., 2005),

$$\frac{\partial\langle N_k\rangle}{\partial t} - \frac{\partial}{\partial k_x}D_k\frac{\partial\langle N_k\rangle}{\partial k_x} = \gamma_k\langle N_k\rangle - \frac{\Delta\omega_k}{N_0}\langle N_k\rangle^2 \tag{32}$$

$$\frac{\partial\left|\phi_{ZFq}\right|^2}{\partial t} = \Gamma_q\left[\frac{\partial\langle N_k\rangle}{\partial k_x}\right]\left|\phi_{ZFq}\right|^2 - \gamma_d\left|\phi_{ZFq}\right|^2 - \gamma_{NL}\left[\left|\phi_{ZFq}\right|^2\right]\left|\phi_{ZFq}\right|^2. \tag{33}$$

Note here that all the quantities are normalized as in Section 4.1. The second term of the left hand side of equation (32) is the diffusion of drift waves in the wave number space by the zonal flow-drift wave scattering. The diffusion coefficient, D_k, proportional to the zonal flow amplitude: $\left|\phi_{ZFq}\right|^2$. The last term of the right hand side of equation (32) represents the nonlinear self-interaction, which is proportional to $\langle N_k \rangle^2$. The second term of the right hand side of equation (32) is the nonlinear wave scattering. Equations (32) and (33) are further reduced to the rate equation by assuming $\sum_k \frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r} \langle N_k \rangle \approx -\alpha \sum_q |\phi_{ZFq}|^2 \sum_k \langle N_k \rangle$, as

$$\frac{\partial}{\partial t}\langle \overline{N} \rangle = \gamma_L \langle \overline{N} \rangle - \gamma_{NL} \langle \overline{N} \rangle^2 - \alpha \langle U^2 \rangle \langle \overline{N} \rangle \tag{34}$$

$$\frac{\partial}{\partial t}\langle U^2 \rangle = -\gamma_{\text{damp}} \langle U^2 \rangle + \alpha \langle U^2 \rangle \langle \overline{N} \rangle. \tag{35}$$

Here, drift wave fluctuation: $\langle \overline{N} \rangle = \sum_k \langle N_k \rangle \leftrightarrow$ Prey and zonal flow: $\langle U^2 \rangle = \sum_q |\phi_{ZFq}|^2 \leftrightarrow$ Predator.

Equations (34) and (35) have two stationary solutions. One has no zonal flows: $\langle U^2 \rangle = 0$, then

$$\langle \overline{N} \rangle = \frac{\gamma_L}{\gamma_{NL}}.$$

This means that the drift wave instability is saturated through the mode coupling among drift waves. The other stationary solution has the finite amplitude of zonal flows, which is,

$$\langle \overline{N} \rangle = \gamma_{\text{damp}}$$

$$\langle U^2 \rangle = \frac{\alpha \gamma_L - \gamma_{NL} \gamma_{\text{damp}}}{\alpha^2}.$$

It is known that the model equations include the limit cycle, bifurcations and chaotic transitions between high and low zonal flow amplitude. These behaviors may be relevant to transition phenomena observed in tokamaks.

We note here, however, that the mode coupling theory based on the weak turbulence concept (presented in this section) applies to the weakly nonlinear state, while experimentally observed turbulence of the drift wave frequency range by Slusher and Surko (1978) has spectral width much larger than the drift wave frequency itself. Thus, the weak turbulence theory – while it provides a physical view at an early stage of turbulence – may not be applicable at a stage of fully developed turbulence. There, the hydrodynamic concept of turbulence, such as inertial range spectra and self-organized structure, becomes the only reliable theoretical approach. In this case, the dynamical process may be found only by computer simulations.

5 Plasma confinement in a dipole magnetic field

While plasma confinement based on the self-organized state of turbulence described in Sections 2 and 3 may be applicable to an open system with continuous energy injection and exhaust, there exists a plasma container of a closed system in which the self-organized state is obtained by maximization of the entropy of the system. Hasegawa (1987) presented one such scheme based on a dipole magnetic field.

In an axisymmetric system, the plasma pressure, p, is given by a function of the magnetic flux function ψ. Thus, if the energy content is uniformly distributed among all the flux tubes, plasma is absolutely stable. In a cylindrical plasma where the axial magnetic field is uniform, the stable equilibrium is achieved only when the plasma pressure is uniformly distributed. Rather, for a plasma pressure to peak near the axis, the curvature of the magnetic line of force should be convex toward the plasma for the plasma to be stable with respect to interchange of flux tube. A mirror machine with the Joffe bar is one such example. Magnetic field configuration with a concave structure is often called a bad curvature, because the plasma becomes hydromagnetically unstable. Thus, it was a surprise for many laboratory plasma physicists to find magnetospheric plasmas of various planets. NASA's interplanetary missions have demonstrated stable plasma confinement, having plasma beta often exceeding unity in spite of their bad curvatures of dipole magnetic fields. Hasegawa recognized that the flux tube volume, $V = \int dl/\boldsymbol{B}$, in a dipole field increases radially outward in proportion to r^4, the energy content per flux tube, pV^γ, $(\gamma = 5/3)$ becomes uniform if the pressure varies as $r^{-20/3}$. Thus, if the pressure as a function of radius from the center of a point dipole varies slower than $r^{-20/3}$, plasma in a dipole magnetic field is absolutely stable, in spite of its bad curvature. This observation indicates that if a plasma and/or its energy is injected into a dipole magnetic field, plasma will diffuse inward with a help of low frequency fluctuations (that may destroy the third adiabatic invariant, the flux function), toward the core region until it reaches the maximum entropy state where the plasma energy content is uniformly distributed among flux tubes. At this point, the plasma pressure achieves coordinate space distribution of $r^{-20/3}$, which may be sharp enough for fusion ignition near the core.

We can also describe the confinement scheme in terms of an action-angle variable in particle phase space. In a dipole magnetic field where the magnetic field is axis symmetric, the magnetic field may be described by the flux function, $\psi(r, \theta)$, in the spherical coordinate system where the appropriate action variables of the guiding center particle motion are the magnetic moment, μ, the second adiabatic invariant, J, associated with the bounce motion and the third invariant, the flux function, ψ. Since the Hamiltonian of the motion has no dependency on angle variables associated with these actions, the stationary phase space distribution may be given by a function of these actions, $f(\mu, J, \psi)$. The flux function at the particle position is the only coordinate space variable, thus the phase space distribution, $f(\mu, J)$, that has no dependence on ψ represents a maximum entropy state of the plasma in these coordinate systems. A plasma described by the distribution function is absolutely stable with respect to any instabilities that originate from coordinate space inhomogeneity such as drift wave and/or interchange, provided that μ and J remain invariants of motion. What is interesting is that fluid variables such as density, pressure and temperature are rather steep functions of the real coordinate space, r, because the phase space volume of the flux tube, increases as r^4 in a dipole field, such that $n \sim r^{-4}$, $P \sim r^{-20/3}$, $T \sim r^{-8/3}$.

The plasma with this profile is absolutely stable provided a proper pressure confinement scheme is provided at the outer edge where the pressure is sufficiently low.

The proposed trap based on the dipole magnetic field structure has been built at the University of Tokyo as well as MIT-Columbia where the magnetic field is produced by a floating super conducting ring current. Both of the devices have demonstrated absolutely stable plasma confinement in these devices (Kawazura et al., 2015; Davis et al., 2014). A plasma container using a dipole magnetic field is suitable for an advanced fusion reactor because of its capacity to stably confine high-beta plasma.

6 Concluding remarks

The theoretical picture of plasma confinement that is described in Sections 2–4 requires excitation of zonal flow via turbulent self-organization. For this idea to be applied to the design of fusion devices, additional fundamental knowledge is necessary to make the stationary operation possible. Let us recall that the inertial range spectrum of turbulent energy that was first described by Kolmogorov in three-dimensional fluids is a driven-dissipative spectrum where fluid viscosity provides necessary dissipation. The inertial range spectrum is established by the balance of the drive and the viscous dissipation. This means that Kolmogorov's system is an open system, energy-wise, where continuous energy injection at a wave number as well as an energy sink at large wave numbers is conjectured. Similarly, for the process described in this manuscript to operate stably, it requires proper energy injection as well as energy dissipation in order for the zonal flow to be maintained to confine the plasma. We must assume the system to be open, energy-wise. The zonal flow required for the plasma confinement cannot be maintained without proper energy input *and* appropriate energy dissipation. Unlike a two-dimensional fluid where no energy sink exists at long wavelengths – whereby stationary energy spectrum may be difficult to establish – plasma can provide energy dissipation processes at long wavelengths by means of low frequency RF out-coupling and/or by thermal flow to the wall. Without dissipation, the zonal flow that represents a condensed state of turbulent energy will collapse. The predator–prey model presented by Diamond et al. (2005, 2011) presents one way to avoid the collapse. Here, we propose an alternative process to achieve stationary state. In a standard tokamak operation, there exists continuous energy input in the form of neutral beam, RF and/or current drive and the like. Thus, the concept of plasma confinement by means of zonal flow driven by turbulent self-organization in the presence of continuous energy input requires a perception of plasma as an open system, energy-wise. Here, maintenance of a proper pressure profile is required for ignition but not necessarily the confinement of injected energy as a whole. As a result, the Lawson criterion that requires certain confinement time of plasma energy density for fusion ignition should be replaced by another criterion in such an open system.

For a stationary fusion fuel burning to take place in the present system, the injected power for the heating and current drive should escape at approximately the same rate via RF and/or thermal flow. Here, the injected power is used to heat the plasma *as well as* to maintain the proper pressure profile, thus work is done to the plasma even if the injected power is lost from the plasma at the same rate. Here, the role of the injected energy is to provide low entropy, or high quality of energy to the system needed to maintain proper heat and pressure profile. In this regard, a new scenario of plasma confinement is required to establish the confinement of low entropy state, rather than energy. Plasma with a desired pressure profile (as a function of magnetic flux function) has a low entropy state spatially as compared with plasma with uniform pressure profile in flux function.

Ignition and burning of fusion fuel can still be achieved in such a system, even though the energy is not confined as assumed in a classical fusion system. If a fusion device is to depend on the turbulent sustained zonal flow that requires a continuous cascade of energy spectra to lower wave numbers and a sink of cascaded energy to maintain stationary structure, the relevant new parameter for ignition is the time scale of maintenance of such a structure or a low entropy state, rather than that of energy confinement. Here, the new Lawson criterion should be given for the time of maintenance of the low entropy state or a state of geographically localized plasma pressure by means of the turbulence-driven zonal flow. The continuously injected energy is needed to sustain this low entropy state, and thus the energy source should have a low entropy property – or the injected energy should be of a high quality energy.

In fact, high frequency monochromatic RF or particle beam used for the plasma heating and current drive do satisfy this requirement. In contrast, the exhausted energy – either in a form of low frequency RF having a wide spectrum or in a form of particle diffusion – has much higher entropy than that of the injected energy. Thus, work is done by the input energy whose low entropy state is used to maintain the plasma profile and heating and the increased entropy is exhausted out of the system. In exchange, plasma can maintain the low entropy state. The energy cycle of the system is analogous to a biological system (Schrödinger, 1944) where life processes are maintained by consumption of food, which provides low entropy energy or negentropy and exhausting increased entropy out of the body.

In Section 5, an alternative plasma confinement scheme is presented in this regard, where the plasma is contained in a closed system and has a maximum entropy state. The maximum entropy state is provided by a uniform distribution of plasma energy among all the flux tubes and yet has a steep pressure profile in the real coordinate. The scheme utilizes a dipole magnetic field generated by a superconducting floating ring current. The idea was provided by Hasegawa and experimental verifications were made by Kawazura et al. (2015) and Davis et al. (2014). A plasma container using the dipole magnetic field is suitable for advanced fuel fusions because of its nature of high-beta plasma confinement.

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