PHYSICS 210B : NONEQUILIBRIUM STATISTICAL PHYSICS
HW ASSIGNMENT #4 SOLUTIONS

(1) Evaluate, for general α, the averages of the following stochastic integrals:

$$\int\limits_0^t dW(s)\, W(s)\, s \qquad , \qquad \int\limits_0^t dW(s)\, W^3(s)\, e^{-\lambda s} \qquad , \qquad \int\limits_0^t dW(s)\, W^{2k+1}(s) \quad .$$

Solution:

We evaluate the general stochastic integral,

$$\begin{aligned} I_k[\phi] &= \int\limits_0^t dW(s)\, W^{2k+1}(s)\, \phi(s) \\ &= \sum_{j=0}^{N-1} \Big[(1-\alpha)\, W_j^{2k+1}\, \phi_j + \alpha\, W_{j+1}^{2k+1}\, \phi_{j+1}\Big] \big(W_{j+1} - W_j\big) \quad . \end{aligned}$$

Therefore,

$$\begin{aligned} \big\langle I_k[\phi] \big\rangle &= \alpha \sum_{j=0}^{N-1} \phi_{j+1} \Big(\langle W_{j+1}^{2k+2} \rangle - \langle W_j\, W_{j+1}^{2k+1} \rangle \Big) \\ &= \frac{(2k+2)!}{2^{k+1}(k+1)!} \cdot \alpha \int\limits_0^t ds\; s^k\, \phi(s) \quad . \end{aligned}$$

Thus,

$$\begin{aligned} \bigg\langle \int\limits_0^t dW(s)\, W(s)\, s \bigg\rangle &= \tfrac{1}{2}\, \alpha t^2 \\ \bigg\langle \int\limits_0^t dW(s)\, W^3(s)\, e^{-\lambda s} \bigg\rangle &= 3\alpha \int\limits_0^t ds\; s\, e^{-\lambda s} = 3\alpha \bigg[\frac{1}{\lambda^2}\big(1 - e^{-\lambda t}\big) - \frac{t}{\lambda}\, e^{-\lambda t} \bigg] \\ \bigg\langle \int\limits_0^t dW(s)\, W^{2k+1}(s) \bigg\rangle &= \frac{(2k+2)!}{2^{k+1}(k+1)!} \cdot \frac{\alpha\, t^{k+1}}{k+1} \quad . \end{aligned}$$

Note that in the limit $\lambda \to 0$ the second integral yields $\frac{3}{2}\alpha t^2$, which agrees with the results from the third expression.

(2) Derive Eqn. 3.107 of the lecture notes.

Solution:

Starting from

$$
\begin{aligned}
du &= -\beta\, u\, dt + \beta\, dW(t) \\
\frac{dz}{dt} &= i\nu\, z + i\lambda\, u(t)\, z \ ,
\end{aligned}
$$

we obtained the solution

$$
z(t) = z(0)\, \exp\left\{ i\nu t + \frac{i\lambda}{\beta}\, u(0)\, \big(1 - e^{-\beta t}\big) + i\lambda \int\limits_0^t dW(s)\, \Big(1 - e^{-\beta(t-s)}\Big) \right\} .
$$

We wish to compute the quantity $Y(s) = \lim_{t\to\infty} \big\langle z(t+s)\, z^*(t) \big\rangle$. We therefore have

$$
\big\langle z(t+s)\, z^*(t) \big\rangle = |z(0)|^2\, e^{i\nu s}\, \exp\left\{ \frac{i\lambda}{\beta}\, u(0)\, e^{-\beta t} \big(e^{-\beta s} - 1\big) \right\} \times
$$

$$
\exp\left\{ -\frac{\lambda^2}{2} \left\langle \left(\int\limits_0^t dW(\sigma)\, \Big[1 - e^{-\beta(t-\sigma)}\Big] - \int\limits_0^{t+s} dW(\sigma)\, \Big[1 - e^{-\beta(t+s-\sigma)}\Big] \right)^2 \right\rangle \right\}
$$

We now invoke the result of Eqn. 3.33,

$$
\left\langle \int\limits_0^t dW(s)\, F(s) \int\limits_0^{t'} dW(s')\, G(s') \right\rangle = \int\limits_0^{\tilde t} ds\, F(s)\, G(s) \ ,
$$

where $\tilde t = \mathsf{min}(t, t')$, to obtain

$$
\begin{aligned}
&\left\langle \left(\int\limits_0^t dW(\sigma)\, \Big[1 - e^{-\beta(t-\sigma)}\Big] - \int\limits_0^{t+s} dW(\sigma)\, \Big[1 - e^{-\beta(t+s-\sigma)}\Big] \right)^2 \right\rangle \\
&\qquad = t - \frac{2}{\beta}\big(1 - e^{-\beta t}\big) + \frac{1}{2\beta}\big(1 - e^{-2\beta t}\big) + (t+s) - \frac{2}{\beta}\big(1 - e^{-\beta(t+s)}\big) + \frac{1}{2\beta}\big(1 - e^{-2\beta(t+s)}\big) \\
&\qquad\qquad - 2t + \frac{2}{\beta}\big(1 - e^{-\beta t}\big) + \frac{2}{\beta}\big(1 - e^{-\beta t}\big)\, e^{-\beta s} - \frac{2}{2\beta}\big(1 - e^{-2\beta t}\big)\, e^{-\beta s} \\
&\qquad \underset{t\to\infty}{=} s - \frac{1}{\beta}\big(1 - e^{-\beta s}\big) \quad ,
\end{aligned}
$$

where we have assumed $s > 0$. For $s < 0$, it is clear that we must replace s with $|s|$. The final result is

$$
Y(s) = \lim_{t\to\infty} \big\langle z(t+s)\, z^*(t) \big\rangle = |z(0)|^2\, \exp\left\{ i\nu s - \tfrac{1}{2}\lambda^2\, |s| + \frac{\lambda^2}{2\beta}\, \Big(1 - e^{-\beta|s|}\Big) \right\} .
$$

(3) For the colored noise example in §3.5.3 of the notes, compute numerically $\hat{Y}(\omega)$ and plot your results as a function of $\omega - \nu$. Set $\lambda \equiv 1$ and plot your results for a representative set of different values of the parameter β.

Solution:

We may derive an expansion for $\hat{Y}(\omega)$ as follows. First, for convenience we set $|z(0)|^2 = 1$. Then we have

$$\begin{aligned} Y(s) &= \exp\left\{ i\nu s - \tfrac{1}{2}\lambda^2\,|s| + \frac{\lambda^2}{2\beta}\left(1 - e^{-\beta|s|}\right) \right\} \\ &= e^{i\nu s} e^{-\lambda^2|s|/2}\, e^{\lambda^2/2\beta} \sum_{n=0}^{\infty} \frac{1}{n!}\bigg(-\frac{\lambda^2}{2\beta}\bigg)^n e^{-n\beta|s|} \end{aligned}$$

Taking the Fourier transform, we have

$$\hat{Y}(\omega) = e^{\lambda^2/2\beta} \sum_{n=0}^{\infty} \frac{1}{n!}\bigg(-\frac{\lambda^2}{2\beta}\bigg)^n \frac{2\,(n\beta + \frac{1}{2}\lambda^2)}{(\omega-\nu)^2 + (n\beta + \frac{1}{2}\lambda^2)^2}\ .$$

Define the parameter $\varepsilon \equiv \lambda^2/2\beta$, and define rescaled frequencies $\bar{\omega} \equiv \omega/\beta$ and $\bar{\nu} \equiv \nu/\beta$. Then $\hat{Y}(\omega) = \beta^{-1}\hat{\mathcal{Y}}_\varepsilon(\delta)$, where $\delta = \bar{\omega} - \bar{\nu}$ and

$$\begin{aligned} \hat{\mathcal{Y}}_\varepsilon(\delta) &= 2\exp(\varepsilon) \sum_{n=0}^{\infty} \frac{(-\varepsilon)^n}{n!}\, \frac{n+\varepsilon}{\delta^2 + (n+\varepsilon)^2} \\ &= 2\int_0^\infty \! d\tau\, \cos(\delta\tau)\, \exp\Big\{-\varepsilon\left(e^{-\tau} - 1 + \tau\right)\Big\} \quad . \end{aligned}$$

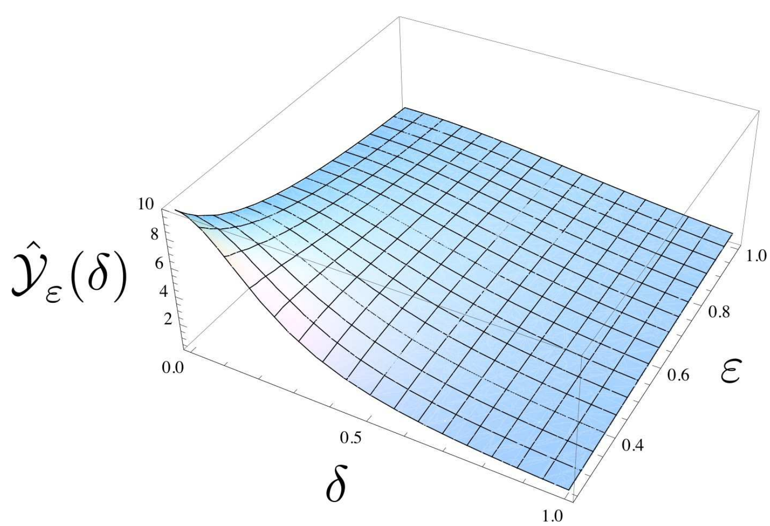

Figure 1: The integral $\hat{\mathcal{Y}}_\varepsilon(\delta)$ from problem (3).

Note that $f(\tau) = e^{-\tau} - 1 + \tau$ is nonnegative and monotonically increasing for $\tau \ge 0$, with $f(0) = 0$. For $\varepsilon \ll 1$, we can expand $f(\tau) = \frac{1}{2}\tau^2 + \mathcal{O}(\tau^3)$ and obtain

$$\hat{\mathcal{Y}}_\varepsilon(\delta) \simeq \sqrt{\frac{2\pi}{\varepsilon}}\, e^{-\delta^2/2\varepsilon} \qquad (\varepsilon \to 0)\ .$$

We evaluate numerically via `Mathematica`, *viz.*

```
Y[x_, a_] := NIntegrate[ 2 Cos[x * y] Exp[ −a ( Exp[−y] − 1 + y ) ], {y, 0, Infinity}]
Plot3D[ Y[x, a], {x, 0, 1}, {a, 0.25, 1}, PlotRange → Full]
```

The resulting plot is shown in Fig. 1.