**PHYSICS 210B : NONEQUILIBRIUM STATISTICAL PHYSICS**
**HW ASSIGNMENT #3 SOLUTIONS**

**(1)** Show that for time scales sufficiently greater than $\gamma^{-1}$ that the solution $x(t)$ to the Langevin equation $\ddot{x}+\gamma\dot{x} = \eta(t)$ describes a Markov process. You will have to construct the matrix $M$ defined in eqns. 2.213 and 2.214 of the lecture notes. You should assume that the random force $\eta(t)$ is distributed as a Gaussian, with $\langle\eta(s)\rangle = 0$ and $\langle\eta(s)\,\eta(s')\rangle = \Gamma\,\delta(s-s')$.

Solution:

The probability distribution is

$$P(x_1,t_1\,;\,\ldots\,;\,x_N,t_N) = \text{det}^{-1/2}(2\pi M)\,\exp\bigg\{-\tfrac{1}{2}\sum_{j,j'=1}^{N} M^{-1}_{jj'}\,x_j\,x_{j'}\bigg\}\,,$$

where

$$M(t,t') = \int\limits_0^t\!ds\int\limits_0^{t'}\!ds'\,G(s-s')\,K(t-s)\,K(t'-s')\,,$$

and $K(s) = (1-e^{-\gamma s})/\gamma$. Thus,

$$\begin{aligned} M(t,t') &= \frac{\Gamma}{\gamma^2}\int\limits_0^{t_{\text{min}}}\!ds\,(1-e^{-\gamma(t-s)})(1-e^{-\gamma(t'-s)}) \\ &= \frac{\Gamma}{\gamma^2}\left\{t_{\text{min}} - \frac{1}{\gamma} + \frac{1}{\gamma}\left(e^{-\gamma t} + e^{-\gamma t'}\right) - \frac{1}{2\gamma}\left(e^{-\gamma|t-t'|} + e^{-\gamma(t+t')}\right)\right\}. \end{aligned}$$

In the limit where $t$, $t'$, and $|t-t'|$ are all large compared to $\gamma^{-1}$, we have $M(t,t') = 2D\,\text{min}(t,t')$, where the diffusions constant is $D = \Gamma/2\gamma^2$. Thus,

$$M = 2D\begin{pmatrix} t_1 & t_2 & t_3 & t_4 & t_5 & \cdots & t_N \\ t_2 & t_2 & t_3 & t_4 & t_5 & \cdots & t_N \\ t_3 & t_3 & t_3 & t_4 & t_5 & \cdots & t_N \\ t_4 & t_4 & t_4 & t_4 & t_5 & \cdots & t_N \\ t_5 & t_5 & t_5 & t_5 & t_5 & \cdots & t_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_N & t_N & t_N & t_N & t_N & \cdots & t_N \end{pmatrix}\,,$$

since $t_1 \geq t_2 \geq \cdots \geq t_N$. To find the determinant of $M$, subtract row 2 from row 1, then subtract row 3 from row 2, *etc.* The result is

$$\widetilde{M} = 2D\begin{pmatrix} t_1-t_2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ t_2-t_3 & t_2-t_3 & 0 & 0 & 0 & \cdots & 0 \\ t_3-t_4 & t_3-t_4 & t_3-t_4 & 0 & 0 & \cdots & 0 \\ t_4-t_5 & t_4-t_5 & t_4-t_5 & t_4-t_5 & 0 & \cdots & 0 \\ t_5-t_6 & t_5-t_6 & t_5-t_6 & t_5-t_6 & t_5-t_6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_N & t_N & t_N & t_N & t_N & \cdots & t_N \end{pmatrix}\,.$$

Note that the last row is unchanged, since there is no row $N+1$ to subtract from it Since $\widetilde{M}$ is obtained from $M$ by consecutive row additions, we have

$$\det M = \det \widetilde{M} = (2D)^N (t_1 - t_2)(t_2 - t_3) \cdots (t_{N-1} - t_N)\, t_N \quad .$$

The inverse is

$$M^{-1} = \frac{1}{2D} \begin{pmatrix} \frac{1}{t_1-t_2} & -\frac{1}{t_1-t_2} & 0 & \cdots & & & & 0 \\ & & & & & & & \vdots \\ -\frac{1}{t_1-t_2} & \frac{t_1-t_3}{(t_1-t_2)(t_2-t_3)} & -\frac{1}{t_2-t_3} & 0 & \cdots & & & \\ 0 & \vdots & \ddots & & & & & \\ \vdots & 0 & -\frac{1}{t_{n-1}-t_n} & \frac{t_{n-1}-t_{n+1}}{(t_{n-1}-t_n)(t_n-t_{n+1})} & -\frac{1}{t_n-t_{n+1}} & 0 & \cdots & \\ & & & \ddots & & & & \\ 0 & \cdots & & & \cdots & 0 & -\frac{1}{t_{N-1}-t_N} & \frac{t_{N-1}}{t_N}\,\frac{1}{t_{N-1}-t_N} \end{pmatrix} .$$

This yields the general result

$$\sum_{j,j'=1}^{N} M^{-1}_{j,j'}(t_1, \ldots, t_N)\, x_j\, x_{j'} = \sum_{j=1}^{N} \left( \frac{1}{t_{j-1} - t_j} + \frac{1}{t_j - t_{j+1}} \right) x_j^2 - \frac{2}{t_j - t_{j+1}}\, x_j\, x_{j+1} \quad ,$$

where $t_0 \equiv \infty$ and $t_{N+1} \equiv 0$. Now consider the conditional probability density

$$\begin{aligned} P(x_1, t_1 \,|\, x_2, t_2\,;\, \ldots\,;\, x_N, t_N) &= \frac{P(x_1, t_1\,;\, \ldots\,;\, x_N, t_N)}{P(x_2, t_2\,;\, \ldots\,;\, x_N, t_N)} \\ &= \frac{\det^{1/2} 2\pi M(t_2, \ldots, t_N)}{\det^{1/2} 2\pi M(t_1, \ldots, t_N)}\, \frac{\exp\left\{ -\frac{1}{2} \sum_{j,j'=1}^{N} M^{-1}_{jj'}(t_1, \ldots, t_N)\, x_j\, x_{j'} \right\}}{\exp\left\{ -\frac{1}{2} \sum_{k,k'=2}^{N} M^{-1}_{kk'}(t_2, \ldots, t_N)\, x_k\, x_{k'} \right\}} \end{aligned}$$

We have

$$\begin{aligned} \sum_{j,j'=1}^{N} M^{-1}_{jj'}(t_1, \ldots, t_N)\, x_j\, x_{j'} &= \left( \frac{1}{t_0 - t_1} + \frac{1}{t_1 - t_2} \right) x_1^2 - \frac{2}{t_1 - t_2}\, x_1\, x_2 + \left( \frac{1}{t_1 - t_2} + \frac{1}{t_2 - t_3} \right) x_2^2 + \ldots \\ \sum_{k,k'=2}^{N} M^{-1}_{kk'}(t_2, \ldots, t_N)\, x_k\, x_{k'} &= \left( \frac{1}{t_0 - t_2} + \frac{1}{t_2 - t_3} \right) x_2^2 + \ldots \end{aligned}$$

Subtracting, and evaluating the ratio to get the conditional probability density, we find

$$P(x_1, t_1 \,|\, x_2, t_2\,;\, \ldots\,;\, x_N, t_N) = \frac{1}{\sqrt{4\pi D (t_1 - t_2)}}\, e^{-(x_1 - x_2)^2 / 4D(t_1 - t_2)} \quad ,$$

which depends only on $\{x_1, t_1, x_2, t_2\}$, *i.e.* on the current and most recent data, and not on any data before the time $t_2$. Note the normalization:

$$\int\limits_{-\infty}^{\infty} dx_1\, P(x_1, t_1 \,|\, x_2, t_2\,; \,\ldots\,; x_N, t_N) = 1 \quad .$$

**(2)** Provide the missing steps in the solution of the Ornstein-Uhlenbeck process described in §2.4.3 of the lecture notes. Show that applying the method of characteristics (see appendix IV) to Eqn. 2.61 leads to the solution in Eqn. 2.62.

Solution:

We solve

$$\frac{\partial \hat{P}}{\partial t} + \beta k\, \frac{\partial \hat{P}}{\partial k} = -Dk^2 \hat{P} \tag{1}$$

using the method of characteristics, writing $t = t_\zeta(s)$ and $k = k_\zeta(s)$, where $s$ parameterizes the curve $\big(t_\zeta(s), k_\zeta(s)\big)$, and $\zeta$ parameterizes the initial conditions, which are $t(s=0)=0$ and $k(s=0)=\zeta$. The above PDE in two variables is then equivalent to the coupled system

$$\frac{dt}{ds} = 1 \qquad , \qquad \frac{dk}{ds} = \beta k \qquad , \qquad \frac{d\hat{P}}{ds} = -Dk^2 \hat{P} \; .$$

Solving, we have

$$t_\zeta = s \qquad , \qquad k_\zeta = \zeta\, e^{\beta s} \qquad , \qquad \frac{d\hat{P}}{ds} = -D\,\zeta^2\, e^{2\beta s}\, \hat{P} \; ,$$

and therefore

$$\hat{P}(s,\zeta) = f(\zeta)\, \exp\left\{ -\frac{D\zeta^2}{2\beta}\big(e^{2\beta s} - 1\big) \right\} .$$

We now identify $f(\zeta) = \hat{P}(k\, e^{-\beta t}, t=0)$, hence

$$\hat{P}(k,t) = \exp\left\{ -\frac{D}{2\beta}\big(1 - e^{-2\beta t}\big) k^2 \right\} \hat{P}(k,0) \; .$$

**(3)** Consider a discrete one-dimensional random walk where the probability to take a step of length 1 in either direction is $\frac{1}{2}p$ and the probability to take a step of length 2 in either direction is $\frac{1}{2}(1-p)$. Define the generating function

$$\hat{P}(k,t) = \sum_{n=-\infty}^{\infty} P_n(t)\, e^{-ikn} \; ,$$

where $P_n(t)$ is the probability to be at position $n$ at time $t$, with $P_n(0) = \delta_{n,0}$. Solve for $\hat{P}(k,t)$ and provide an expression for $P_n(t)$. Evaluate $\sum_n n^2\, P_n(t)$.

Solution:

We have the master equation

$$\frac{dP_n}{dt} = \tfrac{1}{2}(1-p)\,P_{n+2} + \tfrac{1}{2}p\,P_{n+1} + \tfrac{1}{2}p\,P_{n-1} + \tfrac{1}{2}(1-p)\,P_{n-2} - P_n \;.$$

Upon Fourier transforming,

$$\frac{d\hat{P}(k,t)}{dt} = \Big[(1-p)\cos(2k) + p\cos(k) - 1\Big]\,\hat{P}(k,t) \;,$$

with the solution

$$\hat{P}(k,t) = e^{-\lambda(k)\,t}\,\hat{P}(k,0) \;,$$

where

$$\lambda(k) = 1 - p\cos(k) - (1-p)\cos(2k) \;.$$

One then has

$$P_n(t) = \int\limits_{-\pi}^{\pi} \frac{dk}{2\pi}\,e^{ikn}\,\hat{P}(k,t) \;.$$

The average of $n^2$ is given by

$$\langle n^2 \rangle_t = -\frac{\partial^2 \hat{P}(k,t)}{\partial k^2}\bigg|_{k=0} = \Big[\lambda''(0)\,t - \lambda'(0)^2\,t^2\Big] = (4-3p)\,t \;.$$

Note that $\hat{P}(0,t) = 1$ for all $t$ by normalization.

**(4)** Numerically simulate the one-dimensional Wiener and Cauchy processes discussed in §2.6.1 of the lecture notes, and produce a figure similar to Fig. 2.3.

*Hint*: To generate normal (Gaussian) deviates with a distribution $p(y) = (4\pi D\varepsilon)^{-1/2}\exp(-y^2/4D\varepsilon)$, we must invert the relation

$$x(y) = \frac{1}{\sqrt{4\pi D\varepsilon}} \int\limits_{-\infty}^{y} ds\; e^{-s^2/4D\varepsilon} = \tfrac{1}{2} + \tfrac{1}{2}\,\mathsf{erf}\bigg(\frac{y}{\sqrt{4D\varepsilon}}\bigg) \quad .$$

This is somewhat unpleasant. A slicker approach is to use the *Box-Muller method*, which you can read about on Wikipedia.

Solution:

Most computing languages come with a random number generating function which produces uniform deviates on the interval $x \in [0,1]$. Suppose we have a prescribed function $y(x)$. If $x$ is distributed uniformly on $[0,1]$, how is $y$ distributed? Clearly

$$\big|p(y)\,dy\big| = \big|p(x)\,dx\big| \qquad \Rightarrow \qquad p(y) = \left|\frac{dx}{dy}\right| p(x) \quad ,$$

where for the uniform distribution on the unit interval we have $p(x) = \Theta(x)\,\Theta(1-x)$. For example, if $y = -\ln x$, then $y \in [0, \infty]$ and $p(y) = e^{-y}$ which is to say $y$ is exponentially distributed. Now suppose we want to specify $p(y)$. We have

$$\frac{dx}{dy} = p(y) \qquad \Rightarrow \qquad x = F(y) = \int\limits_{y_0}^{y} ds\; p(s) \quad ,$$

where $y_0$ is the minimum value that $y$ takes. Therefore, $y = F^{-1}(x)$, where $F^{-1}$ is the inverse function.

To generate normal (Gaussian) deviates with a distribution $p(y) = (4\pi D\varepsilon)^{-1/2} \exp(-y^2/4D\varepsilon)$, we have

$$x = F(y) = \frac{1}{\sqrt{4\pi D\varepsilon}} \int\limits_{-\infty}^{y} ds\; e^{-s^2/4D\varepsilon} = \tfrac{1}{2} + \tfrac{1}{2}\,\mathrm{erf}\bigg(\frac{y}{\sqrt{4D\varepsilon}}\bigg) \quad .$$

We now have to invert the error function, which is slightly unpleasant.

A slicker approach is to use the *Box-Muller* method, which used a two-dimensional version of the above transformation,

$$p(y_1, y_2) = p(x_1, x_2) \left| \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \right| \quad .$$

This has an obvious generalization to higher dimensions. The transformation factor is the Jacobian determinant. Now let $x_1$ and $x_2$ each be uniformly distributed on $[0, 1]$, and let

$$x_1 = \exp\bigg(-\frac{y_1^2 + y_2^2}{4D\varepsilon}\bigg) \qquad\qquad y_1 = \sqrt{-4D\varepsilon \ln x_1}\;\cos(2\pi x_2)$$

$$x_2 = \frac{1}{2\pi}\,\tan^{-1}(y_2/y_1) \qquad\qquad y_2 = \sqrt{-4D\varepsilon \ln x_1}\;\sin(2\pi x_2)$$

Then

$$\frac{\partial x_1}{\partial y_1} = -\frac{y_1\, x_1}{2D\varepsilon} \qquad\qquad \frac{\partial x_2}{\partial y_1} = -\frac{1}{2\pi}\,\frac{y_2}{y_1^2 + y_2^2}$$

$$\frac{\partial x_1}{\partial y_2} = -\frac{y_2\, x_1}{2D\varepsilon} \qquad\qquad \frac{\partial x_2}{\partial y_2} = \frac{1}{2\pi}\,\frac{y_1}{y_1^2 + y_2^2}$$

and therefore the Jacobian determinant is

$$J = \left| \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \right| = \frac{1}{4\pi D\varepsilon}\, e^{-(y_1^2 + y_2^2)/4D\varepsilon} = \frac{e^{-y_1^2/4D\varepsilon}}{\sqrt{4\pi D\varepsilon}} \cdot \frac{e^{-y_2^2/4D\varepsilon}}{\sqrt{4\pi D\varepsilon}} \quad ,$$

which says that $y_1$ and $y_2$ are each independently distributed according to the normal distribution $p(y) = (4\pi D\varepsilon)^{-1/2} \exp(-y^2/4D\varepsilon)$. Nifty!

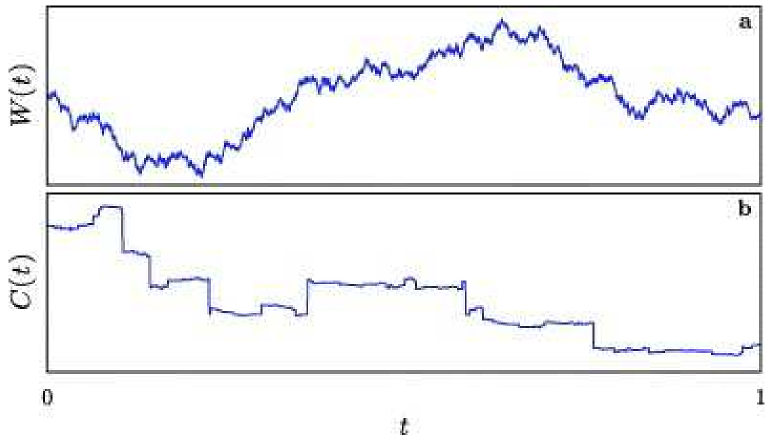


Figure 1: (a) Wiener process sample path $W(t)$. (b) Cauchy process sample path $C(t)$. From K. Jacobs and D. A. Steck, *New J. Phys.* **13**, 013016 (2011).

For the Cauchy distribution, with

$$p(y) = \frac{1}{\pi} \cdot \frac{\varepsilon}{y^2 + \varepsilon^2} \quad ,$$

we have

$$F(y) = \frac{1}{\pi} \int\limits_{-\infty}^{y} ds \, \frac{\varepsilon}{s^2 + \varepsilon^2} = \tfrac{1}{2} + \tfrac{1}{\pi} \tan^{-1}(y/\varepsilon) \quad ,$$

and therefore

$$y = F^{-1}(x) = \varepsilon \tan\left(\pi x - \tfrac{\pi}{2}\right) \quad .$$

**(5)** Due to quantum coherence effects in the backscattering from impurities, one-dimensional wires don't obey Ohm's law in the limit where the 'inelastic mean free path' is greater than the sample dimensions, which you may assume here. Rather, let $\mathcal{R}(L) = e^2 R(L)/h$ be the dimensionless resistance of a quantum wire of length $L$, in units of $h/e^2 = 25.813\,\mathrm{k}\Omega$. The dimensionless resistance of a quantum wire of length $L + \delta L$ is then given by

$$\mathcal{R}(L+\delta L) = \mathcal{R}(L) + \mathcal{R}(\delta L) + 2\,\mathcal{R}(L)\,\mathcal{R}(\delta L) + 2\cos\alpha \sqrt{\mathcal{R}(L)\,\big[1+\mathcal{R}(L)\big]\,\mathcal{R}(\delta L)\,\big[1+\mathcal{R}(\delta L)\big]}\ ,$$

where $\alpha$ is a *random phase* uniformly distributed over the interval $[0, 2\pi)$. Here,

$$\mathcal{R}(\delta L) = \frac{\delta L}{2\ell}\ ,$$

is the dimensionless resistance of a small segment of wire, of length $\delta L \lesssim \ell$, where $\ell$ is the 'elastic mean free path'.

(a) Show that the distribution function $P(\mathcal{R}, L)$ for resistances of a quantum wire obeys the equation

$$\frac{\partial P}{\partial L} = \frac{1}{2\ell} \frac{\partial}{\partial \mathcal{R}} \left\{ \mathcal{R}\,(1+\mathcal{R})\,\frac{\partial P}{\partial \mathcal{R}} \right\} .$$

(b) Show that this equation may be solved in the limits $\mathcal{R} \ll 1$ and $\mathcal{R} \gg 1$, with

$$P(\mathcal{R}, z) = \frac{1}{z}\, e^{-\mathcal{R}/z}$$

for $\mathcal{R} \ll 1$, and

$$P(\mathcal{R}, z) = (4\pi z)^{-1/2}\, \frac{1}{\mathcal{R}}\, e^{-(\ln \mathcal{R} - z)^2/4z}$$

for $\mathcal{R} \gg 1$, where $z = L/2\ell$ is the dimensionless length of the wire. Compute $\langle \mathcal{R} \rangle$ in the former case, and $\langle \ln \mathcal{R} \rangle$ in the latter case.

Solution:

(a) From the composition rule for series quantum resistances, we derive the phase averages

$$\begin{aligned}
\langle \delta\mathcal{R} \rangle &= \Big(1 + 2\,\mathcal{R}(L)\Big)\frac{\delta L}{2\ell} \\
\langle (\delta\mathcal{R})^2 \rangle &= \Big(1 + 2\,\mathcal{R}(L)\Big)^2 \left(\frac{\delta L}{2\ell}\right)^2 + 2\,\mathcal{R}(L)\,\Big(1 + \mathcal{R}(L)\Big)\,\frac{\delta L}{2\ell}\left(1 + \frac{\delta L}{2\ell}\right) \\
&= 2\,\mathcal{R}(L)\,\Big(1 + \mathcal{R}(L)\Big)\,\frac{\delta L}{2\ell} + \mathcal{O}\big((\delta L)^2\big) ,
\end{aligned}$$

whence we obtain the drift and diffusion terms

$$F_1(\mathcal{R}) = \frac{2\mathcal{R} + 1}{2\ell} \qquad , \qquad F_2(\mathcal{R}) = \frac{2\mathcal{R}(1+\mathcal{R})}{2\ell} .$$

Note that $2F_1(\mathcal{R}) = dF_2/d\mathcal{R}$, which allows us to write the Fokker-Planck equation as

$$\frac{\partial P}{\partial L} = \frac{\partial}{\partial \mathcal{R}} \left\{ \frac{\mathcal{R}\,(1+\mathcal{R})}{2\ell}\,\frac{\partial P}{\partial \mathcal{R}} \right\} .$$

(b) Defining the dimensionless length $z = L/2\ell$, we have

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial \mathcal{R}} \left\{ \mathcal{R}\,(1+\mathcal{R})\,\frac{\partial P}{\partial \mathcal{R}} \right\} .$$

In the limit $\mathcal{R} \ll 1$, this reduces to

$$\frac{\partial P}{\partial z} = \mathcal{R}\,\frac{\partial^2 P}{\partial \mathcal{R}^2} + \frac{\partial P}{\partial \mathcal{R}} ,$$

which is satisfied by $P(\mathcal{R}, z) = z^{-1}\, \exp(-\mathcal{R}/z)$. For this distribution one has $\langle \mathcal{R} \rangle = z$.

In the opposite limit, $\mathcal{R} \gg 1$, we have

$$\frac{\partial P}{\partial z} = \frac{\partial}{\partial \mathcal{R}}\left(\mathcal{R}^2 \frac{\partial}{\partial \mathcal{R}}\right) = \frac{\partial^2 P}{\partial \nu^2} + \frac{\partial P}{\partial \nu} \, ,$$

where $\nu \equiv \ln \mathcal{R}$. This is solved by the log-normal distribution,

$$P(\mathcal{R}, z) = (4\pi z)^{-1/2} \, e^{-(\nu+z)^2/4z} \, .$$

Note that

$$P(\mathcal{R}, z) \, d\mathcal{R} = \widetilde{P}(\nu, z) \, d\nu = \frac{1}{\sqrt{4\pi z}} \, e^{-(\nu-z)^2/4z} \, d\nu \quad ,$$

One then obtains $\langle \nu \rangle = \langle \ln \mathcal{R} \rangle = z$. Furthermore,

$$\langle \mathcal{R}^n \rangle = \langle e^{n\nu} \rangle = \frac{1}{\sqrt{4\pi z}} \int\limits_{-\infty}^{\infty} d\nu \, e^{-(\nu-z)^2/4z} \, e^{n\nu} = e^{k(k+1)z}$$

Note then that $\langle \mathcal{R} \rangle = \exp(2z)$, so the mean resistance grows *exponentially* with length. However, note also that $\langle \mathcal{R}^2 \rangle = \exp(6z)$, so

$$\langle (\Delta \mathcal{R})^2 \rangle = \langle \mathcal{R}^2 \rangle - \langle \mathcal{R} \rangle^2 = e^{6z} - e^{4z} \quad ,$$

and so the standard deviation grows as $\sqrt{\langle \mathcal{R}^2 \rangle} \sim \exp(3z)$ which grows faster than $\langle \mathcal{R} \rangle$. In other words, the resistance $\mathcal{R}$ itself is not a *self-averaging* quantity, meaning the ratio of its standard deviation to its mean doesn't vanish in the thermodynamic limit – indeed it diverges. However, $\nu = \ln \mathcal{R}$ *is* a self-averaging quantity, with $\langle \nu \rangle = z$ and $\sqrt{\langle \nu^2 \rangle} = \sqrt{2z}$ .