→ Magnetic Helicity

- another conserved quantity in ideal MHD is magnetic helicity K

$$K = \int_V d^3x \; \underline{A} \cdot \underline{B}$$

V is taken to be the volume of a 'flux tube'.

- what, yet another invariant!?!

→ K is different ⇒ has topological interpretation

$$K = \int_V d^3x \; \underline{A} \cdot \underline{\nabla} \times \underline{A}$$

→ $\underline{x} \to -\underline{x}$ flips sign of K

→ K is a pseudo-scalar

∴ has orientation or "handedness"...

Proceed via:
- show K conservation
- discuss interpretation of K
- comment on utility ⇒ Taylor Relaxation

N.B.: Important → K is gauge invariant

i.e. if $\underline{A} \to \underline{A} + \underline{\nabla}\chi$

$$K \to K + \int_V d^3x \, \underline{\nabla}\chi \cdot \underline{B}$$

$$= K + \int_V d^3x \, \underline{\nabla} \cdot (\underline{B}\,\chi)$$

$= 0$, to surface term. $\left\{ \underline{B} \cdot \hat{n} = 0 \right.$ on surface of tube

Now, consider a blob of MHD fluid in motion

can show $\dfrac{dK}{dt} = \cdot$

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \eta \underline{J}$$

$$\underline{E} = -\frac{1}{c}\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi$$

$\Rightarrow$

$$\frac{\partial \underline{A}}{\partial t} = \underline{v} \times \underline{\nabla} \times \underline{A} - c\underline{\nabla}\phi - c\eta \underline{J}$$

$$\frac{\partial \underline{B}}{\partial t} = -\underline{v} \cdot \underline{\nabla}\, \underline{B} + \underline{B} \cdot \underline{\nabla}\, \underline{v} - \underline{B}\, \underline{\nabla} \cdot \underline{v} + \eta \nabla^2 \underline{B}$$

$$\frac{dK}{dt} = \frac{d}{dt} \int_V d^3x \, (\underline{A} \cdot \underline{B})$$

$$= \int d^3x \left(\frac{d\underline{A}}{dt} \cdot \underline{B} + \underline{A} \cdot \frac{d\underline{B}}{dt} \right) + \int \underline{A} \cdot \underline{B} \, \frac{d}{dt} d^3x$$

$$\frac{dK}{dt} = \int d^3x \left(\frac{\partial \underline{A}}{\partial t} \cdot \underline{B} + (\underline{V} \cdot \nabla \underline{A}) \cdot \underline{B} + \underline{A} \cdot \frac{\partial \underline{B}}{\partial t} \right.$$

$$\left. + \underline{A} \cdot (\underline{V} \cdot \nabla \underline{B}) + \underline{A} \cdot \underline{B} \; \nabla \cdot \underline{V} \right)$$

where $\frac{d}{dt} d^3x = \nabla \cdot \underline{V}$

i.e. $\frac{d}{dt} dV = \frac{d}{dt} d\underline{\sigma} \cdot d\underline{\ell} + d\underline{\sigma} \cdot \frac{d}{dt} d\underline{\ell}$

$$= - d\underline{\ell} \cdot \nabla \underline{V} \cdot d\underline{\sigma} + (\nabla \cdot \underline{V})(d\underline{\sigma} \cdot d\underline{\ell}) + d\underline{\ell} \cdot \nabla \underline{V} \cdot d\underline{\sigma}$$

$$= \nabla \cdot \underline{V} \; d^3x$$

s.t. and $\underline{B} \cdot \hat{n}$ on surface of tube.

$$\frac{dK}{dt} = \int d^3x \left[\left(\underline{B} \cdot \cancel{\underline{V}} \times \underline{B} - c \cancel{\underline{B}} \cdot \nabla \phi - c\eta \, \underline{J} \cdot \underline{B} \right) \right.$$

$$\left. + \underline{A} \cdot \left(\nabla \times (\underline{V} \times \underline{B}) \right) + \nabla \cdot \left((\underline{A} \cdot \underline{B}) \underline{V} \right) + \underline{A} \cdot \eta \nabla^2 \underline{B} \right]$$

where $\underline{A} \cdot (\underline{V} \cdot \nabla \underline{B}) + \underline{B} \cdot (\underline{V} \cdot \nabla \underline{A}) + \underline{A} \cdot \underline{B} \; \nabla \cdot \underline{V} = \nabla \cdot (\underline{V} \, \underline{A} \cdot \underline{B})$

$$\frac{dK}{dt} = \int d^3x \left[\nabla \cdot \left((\underline{A} \cdot \underline{B}) \underline{V} \right) + \nabla \cdot \left((\underline{V} \times \underline{B}) \times \underline{A} \right) + \cancel{(\underline{V} \times \underline{B})} \cdot (\nabla \times \underline{A}) \right.$$

$$\left. - c\eta \, \underline{J} \cdot \underline{B} - \eta (\underline{A} \cdot \nabla \times \underline{J}) \, c \right]$$

$$\Rightarrow \frac{dK}{dt} = \int d^3x \left\{ \underline{\nabla} \cdot \left[(\underline{A} \cdot \underline{B}) \underline{V} + (\underline{V} \times \underline{B}) \times \underline{A} + c\eta (\underline{A} \times \underline{J}) \right] - c\eta\, \underline{J} \cdot \underline{B} - c\eta\, \underline{J} \cdot \underline{B} \right\}$$

$$= \int d\underline{S} \cdot \left[(\underline{A} \cdot \underline{B}) \underline{V} + (\underline{V} \times \underline{B}) \times \underline{A} + c\eta\, \underline{A} \times \underline{J} \right]$$

$$- 2 \int d^3x \left[c\eta\, \underline{J} \cdot \underline{B} \right]$$

$$= \int d\underline{S} \cdot \left[\cancel{(\underline{A} \cdot \underline{B}) \underline{V}} - \cancel{(\underline{A} \cdot \underline{B}) \underline{V}} + \cancel{(\underline{A} \cdot \underline{V}) \underline{B}} \right] - c\eta \int d\underline{S} \cdot \underline{J} \times \underline{A}$$

$\underline{B} \cdot \hat{n} = 0$, on tube

$$- 2c\eta \int d^3x (\underline{J} \cdot \underline{B})$$

$$= -\int c\eta\, d\underline{S} \cdot \left[\underline{\nabla B} \cdot \underline{A} - \cancel{\underline{A} \cdot \underline{\nabla B}} \right] - 2c\eta \int d^3x\, \underline{J} \cdot \underline{B}$$

$$= -2c\eta \int d^3x (\underline{J} \cdot \underline{B})$$

$\Rightarrow$ have shown:

$$\boxed{\frac{dK}{dt} = -2c\eta \int d^3x (\underline{J} \cdot \underline{B})}$$

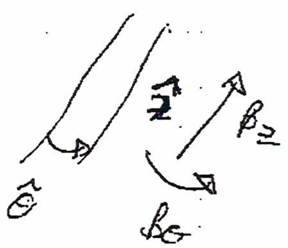
and clearly! $\frac{dK}{dt} \to 0$ as $\eta \to 0$ (non-singular $\underline{J}$)

$\therefore$ Magnetic Helicity is conserved in ideal MHD.

→ Magnetic Helicity conserved, but what does it mean?

- helicity is non-trivial ⇒ more than just helical field lines.

interesting to note: $\iota(r) = \frac{r B_z}{R B_\theta(r)} = \frac{1}{R \mu(r)}$



$\mu(r) = \frac{B_\theta(r)}{r B_z}$ → field line pitch. (length scale over which winding varies)

cylindrical plasma → $\underline{B} = \underline{B}(r)$

Now, $A_\theta = \frac{1}{r} \int_0^r r \, B_z \, dr$

$$A_z = - \int_0^r B_\theta \, dr$$

so $$\underline{A}\cdot\underline{B} = \frac{B_\theta}{r}\int_0^r r B_z\,dr - B_z\int_0^r B_\theta\,dr$$

$$= \mu B_z \int_0^r \frac{B_\theta}{\mu}\,dr - B_z\int_0^r B_\theta\,dr$$

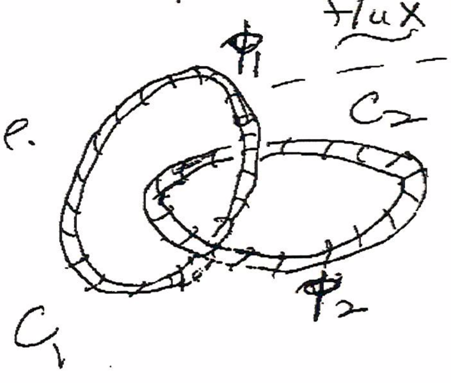
$$\underline{A}\cdot\underline{B} = B_z\left[\mu\int_0^r \frac{B_\theta}{\mu}\,dr - \int_0^r B_\theta\,dr\right]$$

$= 0$ for constant μ

$\therefore$ non-zero helicity requires $\mu = \mu(r)$
i.e. — pitch varies with radius / twist
$\Rightarrow$ $\boxed{\text{magnetic shear}}$

— physically $\rightarrow$ helicity means self-linkage of 2 flux tubes

i.e.



tube 1: flux
$\Phi = \int d\underline{A}\cdot\underline{B} = \phi_1$
x-section area; const

tube 2: $\Phi = \phi_2$

field in loops, only

Now, for volume V_1 of tube 1

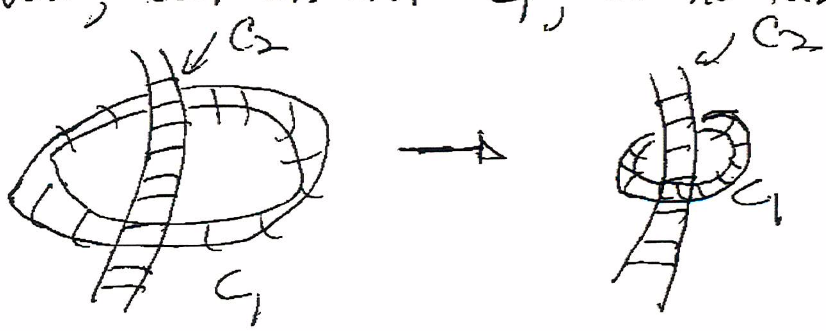
$$K = \int_{V_1} \underline{A} \cdot \underline{B} \, d^3x = \oint_{C_1} d\ell \int_{A_1} dS \, A \cdot \underline{B}$$

C_1: along loop; A_1: X-sect area

$$= \oint_{C_1} \underline{A} \cdot d\underline{\ell} \int_{S_1} \underline{B} \cdot \hat{n} \, dA$$

$$= \phi_1 \oint_{C_1} \underline{A} \cdot d\underline{\ell}$$

Now, can shrink C_1, as no field outside loops



$\rightarrow$ in X section:

$$\text{but} \int_{C_1} A \cdot d\ell = \int_{A \text{ enclosed}} B \cdot dS = \phi_2$$

so . . . $K_1 = \phi_1 \phi_2$ $\Rightarrow$ product of fluxes

similarly $K_2 = \phi_2 \phi_1$

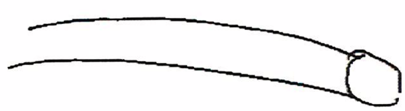
$\therefore$ $K = 2\phi_1 \phi_2$

if n windings $K = K_1 + K_2 = \pm 2n\phi_1 \phi_2$.

$\Rightarrow$ helicity is measure of self-linkage of magnetic configuration.

Why care $\rightarrow$ Taylor Conjecture (1974) (J. B. Taylor)

- in magnetic confinement, of great interest to determine how fields, currents self-organize

- RFP  $\sim$ toroid $\sim$ toroidal current

well fit by $B_z = B_0 J_0(\alpha r)$
$B_\theta = B_0 J_1(\alpha r)$

$\underline{J} \times \underline{B} = 0$
$\Leftrightarrow$ force free

$\Rightarrow$ why so robust?, especially since RFP's so turbulent

- Taylor conjectured conservation of magnetic helicity constrains relaxation to force-free state.

Key Point
- helicity conserved in flux tubes, to η
- toroidal plasma $\Rightarrow$ many small tubes

etc.

- recall Sweet-Parker model: magnetic reconnection / resistive dissipation effective on small scales.

$\Rightarrow$ Taylor Conjecture: At finite η, helicity of small tubes dissipated but global helicity conserved.

i.e.

$$\int \underline{A} \cdot \underline{B} \, d^3x = K_0 \rightarrow \text{conserved.}$$

plasma volume

∴ Taylor conjectured that actual magnetic configuration could be explained by minimum principle:

$$\delta\left[\int_V d^3x \, \frac{B^2}{8\pi} + \lambda \int_V d^3x \, \underline{A}\cdot\underline{B}\right] = 0$$

i.e. minimize magnetic energy subject to constraint of conserved global helicity.

Comments:

→ it works! – indeed amazingly well – for RFPs, spheromaks, etc. Departures only recently being discovered

→ inspired idea of helicity injection as way to maintain configurations

→ it is a conjecture → no proof.

Hypothesis: Selective Decay
- energy cascades → small scale
- helicity cascades → large scale (less dissipation)

– Relevance to driven system?
i.e. in real RFP, transformer on.

→ dynamics? – <u>how</u> does relaxation occur

→ more in discussion of kinks, tearing..