Patterns in Convection I, cont'd

— recall, derived $\left\{ \begin{array}{l} \text{Swift–Hohenberg} \\ \text{Newell–Whitehead} \end{array} \right.$ envelope eqns. In gradients:

→ linear marginality curve ⇒ curvature ⟺ envelope approach

→ saturation $Ra = Ra_{crit} + \epsilon$; slightly above marginality; inversion + phase symmetry

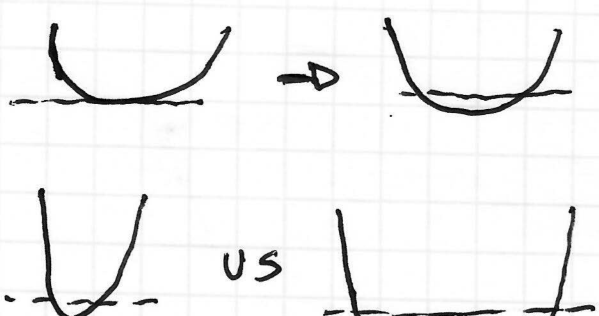
→ symmetry of base state, i.e.

$$\underline{q} = \underbrace{q_0 \hat{x}} + \underline{k}$$

$q_0 \to$ single cell size

Key pt. $\tau_0 \partial_t W = Ra_{crit} + \delta Ra$

$$- \xi_0^2 (q - q_0)^2$$



"parabolic" marginal curve admits restricted range of modes in dynamics

→ physics: modulation of base state. Pattern observed deviates from linear prediction.

$\Rightarrow$ Newell – Whitehead Eqn.:

$$\tau_0 \partial_t A = rA + \xi_0^2 \left(\partial_x + \frac{1}{2ik_c} \partial_y^2 \right)^2 A - g_{eff} |A|^2 A$$

for amplitude (A complex) $\rightarrow$ prototype of "Amplitude Eqn."

furthur

$$A = a e^{i\phi}$$

$a \rightarrow$ amplitude, $\phi \rightarrow$ phase

$\Rightarrow$ for $\partial_y = 0$;

$$\tau_0 \partial_t a = \left(r - \xi_0^2 (\partial_x \phi)^2\right) a + \xi_0^2 \partial_x^2 a - g_{eff} a^3$$

$$\partial_t \phi = \frac{\xi_0^2}{\tau_0} \left(\partial_x^2 \phi + 2 \frac{\partial_x a}{a} \partial_x \phi \right)$$

For exact, stationary solutions:

$$\partial_x a = 0$$

$$\phi = \delta k\, x + \phi_0 \quad \rightarrow \text{\underline{phase winding}}$$

so

$$0 = \left(r - \xi_0^2 (\delta k)^2\right) a - g a^3$$

$$a = \left[\left(r - \xi_0^2 \delta k^2\right)/g_{eff}\right]^{1/2}$$

needs, $\boxed{\delta k < \sqrt{r}/\xi_0}$

→ long wavelength, as $r \sim O(\epsilon)$

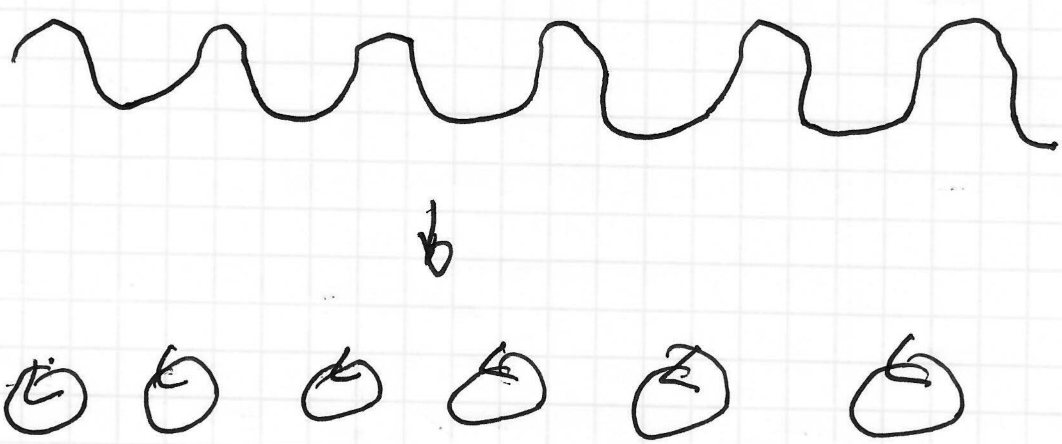
→ termed "phase winding" solution

so

$$W = \frac{1}{2}\left(a e^{i\phi_0} e^{ik_0 x} e^{i\delta k x} + c.c.\right)$$

→ weak modulation of amplitude

Now → what type of secondary instabilities might occur?



2. x

cells

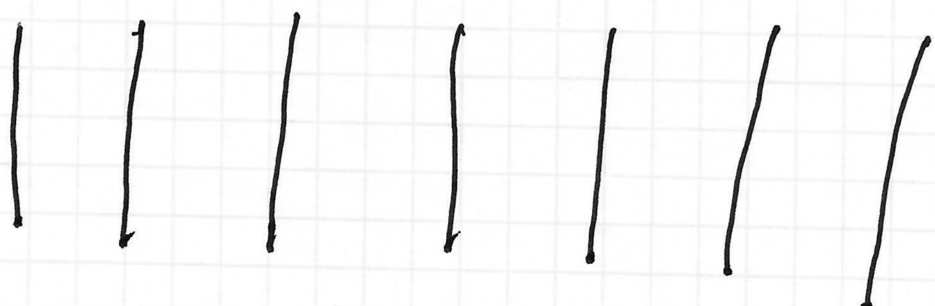
Consider symmetry:

→ translation (no boundary in analysis)

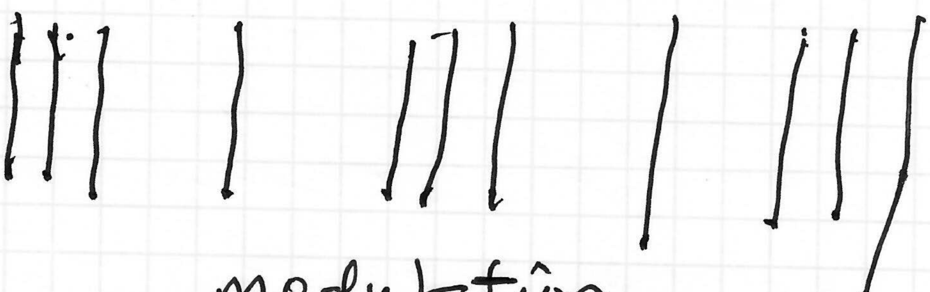
→ rotation (instability mechanism (primary) invariant to rotation in x, y)

① → Breaking translation

stripe → rolls



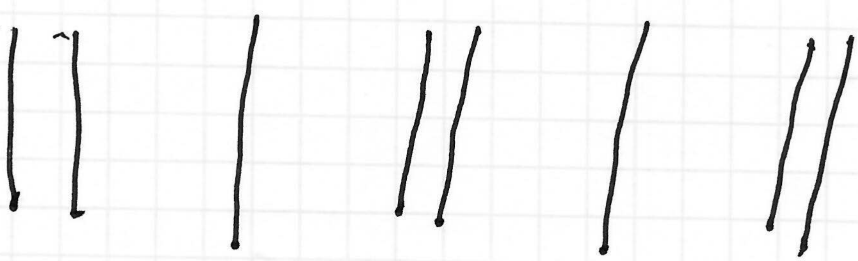
critical



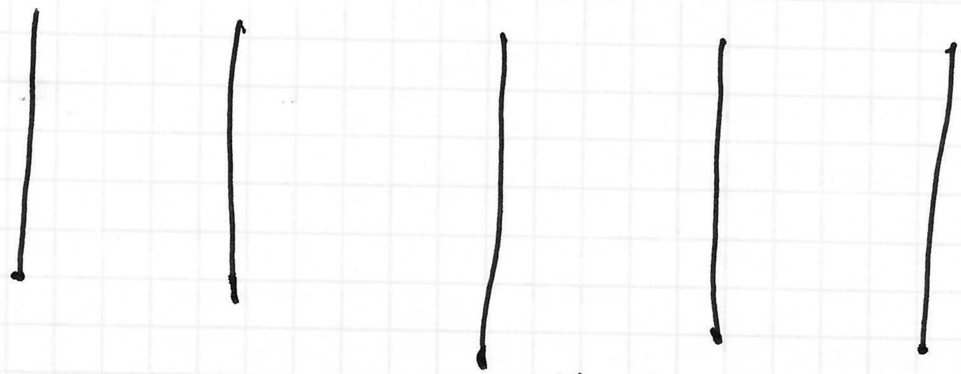
modulation

like sign vortices attract. → bunch.

cluster ↔ negative diffn.



collapse to roll pairs (i.e. mergers)



final pattern.

Process is one of modulation $\Rightarrow$ coalescence $\Rightarrow$ condensation

Points:

— modulation brings rolls closer, triggers $\Rightarrow$

— vortex merger (attraction)

— configuration lowers energy (i.e. $(\nabla A)^2 \downarrow$)

$\Rightarrow$ instability:

Phenomena referred to as Eckhaus Instability $\rightarrow$ i.e. E.I.

Calculation:

Have N–W eqn:

$$\tau_0 \partial_t A = rA + \xi_0^2 \left(\partial_x + \frac{1}{2ik_0}\partial_y^2\right)^2 A - g_{eff}|A|^2 A$$

$$A(x,y,t) = \tilde{A}(x,y,t)\, e^{i\delta k x}$$

(recall $|\delta k| < \sqrt{r}/\xi_0$)

so $(k_c \to 1)$, $g_0 \to 1$ (general)

$$\frac{d\tilde{A}}{dt} = (r - \delta k^2)\tilde{A} + 2i\delta k(\partial_x - i\partial_y^2)\tilde{A} + (\partial_x - i\partial_y^2)^2\tilde{A} - |\tilde{A}|^2\tilde{A}.$$

uniform: $\tilde{A}_0 = (r - \delta k^2)^{1/2}$

Now,

$$\tilde{A} = \tilde{A} + a \qquad \text{(n.b. notation)}$$

{ perturbation about uniform phase winding state

$$a = u + iv$$

and linearization $\Rightarrow$

$$\begin{cases} \partial_t u = \left[-2(r - \delta k^2) + \partial_x^2 + \delta k\, \partial_y^2 - \partial_y^4\right] u - \left(2\delta k - \partial_y^2\right) \partial_x v \\ \partial_t v = \left(2\delta k - \partial_y^2\right) \partial_x u + \left(\partial_x^2 + \delta k\, \partial_y^2 - \partial_y^4\right) v \end{cases}$$

so, cranking straightforwardly:

$$u = U e^{\sigma t} \cos(q_x x) \cos(q_y y)$$

$$v = V e^{\sigma t} \sin(q_x x) \cos(q_y y)$$

$\Rightarrow$ dispersion relation:

$$0 = \sigma^2 + \left[2(r - \delta k^2) + q_x^2 + q_y^2 \delta k + q_y^4\right] \sigma + \left(2(r - \delta k^2) + q_x^2 + q_y^2 \delta k + q_y^4\right)\left(q_x^2 + q_y^2 \delta k + q_y^4\right) - q_x^2 \left(2\delta k + q_y^2\right)^2$$

For Eckhaus, $\varepsilon_y = 0$.

$$\sigma^2 + 2\left((r - \delta k^2) + \varepsilon_x^2\right)\sigma + \varepsilon_x^2\left(2(r - 3\delta k^2) + \varepsilon_x^2\right) = 0$$

$$(\sigma - s_1)(\sigma - s_2) = 0$$

$$\sigma^2 - (s_1 + s_2)\sigma + s_1 s_2 = 0$$

Roots real $\Rightarrow$ stability if

$s_1 + s_2 < 0$ $\qquad$ $s_1 < 0, \; s_2 < 0$

$$\Rightarrow s_1 s_2 > 0$$

so, instability if:

$$\left(2(r - 3\delta k^2) + \varepsilon_x^2\right) < 0$$

$$\boxed{\varepsilon_x^2 < 2(3\delta k^3 - r)}$$

requires: $\boxed{|\delta k_1| > \sqrt{r/3}}$

Instability requires:

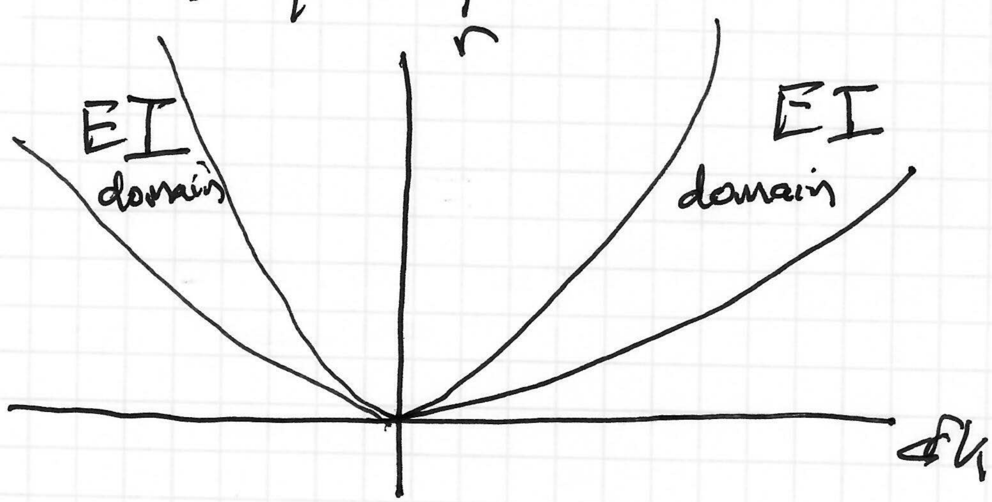
$|\delta k| < \sqrt{r}/\xi_0$ → phase winding

$|\delta k| > \sqrt{r/3}$ → instability

$q_y = 0$ ⇒ Eckhaus

$\xi_0 = 1$ $\sqrt{r/3} < |\delta k| < \sqrt{r}$

→ hydrodynamic mode → broken translational invariance

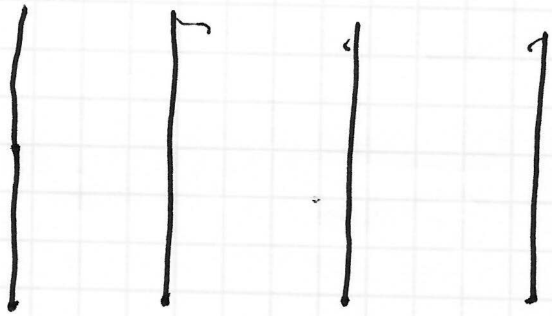


N B:
can represent as negative diffn.

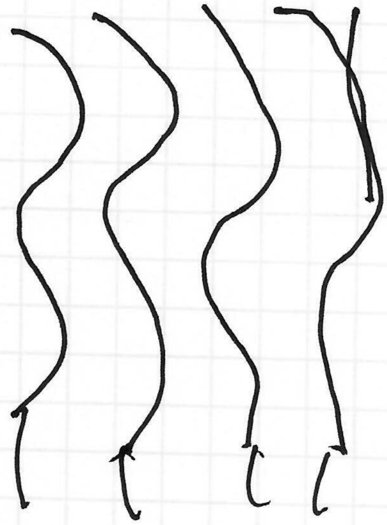
② Rotational Invariance → see negative diffn phase.

→ Another symmetry is rotational invariance in x, y plane for primary instability.

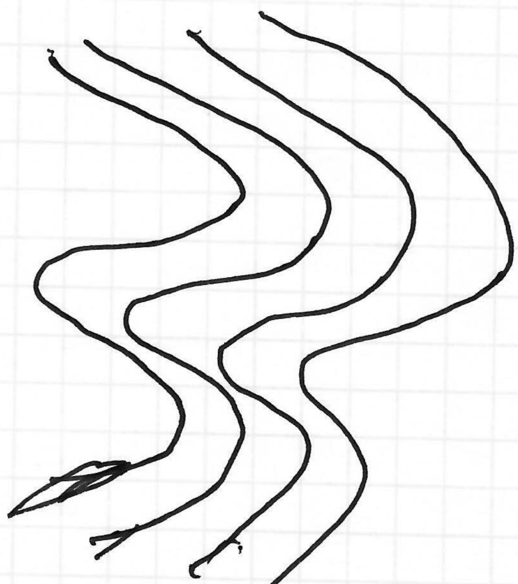
⇒ Zig-Zag



uniform



bending
(null rotational invariance)



saturation when
$(\partial A/\partial y^2)^2$ kicks in. → Flex.

In analysis, take $\epsilon_x = 0$, ϵ_y finite; then:

$$s_- = -2(r - \alpha k^2) - \epsilon_y^2 \alpha k - \epsilon_y^4 < 0.$$

But

$$s_+ = -\epsilon_y^3 (\epsilon_y^2 + \delta k)$$

$$> 0$$

for $\delta k < 0$ i.e.

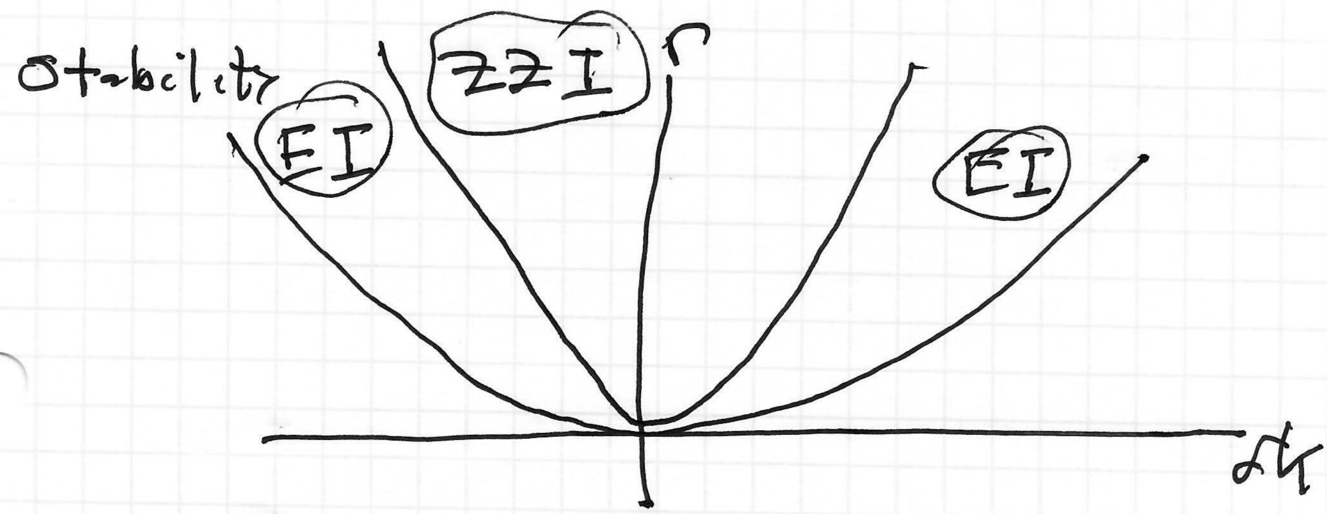
$$\delta k + \epsilon_y^2 < 0 \Rightarrow \boxed{\delta k < -\epsilon_y^2}$$

$\Rightarrow$ **Zig-Zag instability**

$\rightsquigarrow$ bending.

Eckhaus and Zig-Zag are two typical pattern forming instabilities.

$\Rightarrow$ Tips of huge ice-bergs.



→ Other Pattern Problems / Related:

* ⓐ → Waves ⇒ NLS

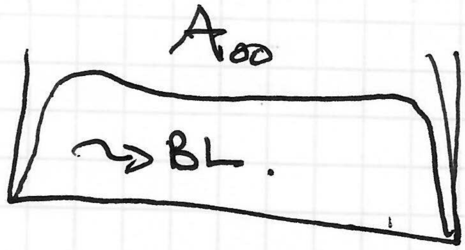
special case of CGL

ⓑ → Geometry ⇒ boundary conditions on amplitude eqns

ⓒ → Grain boundaries / dislocations ⇒ pattern discontinuities

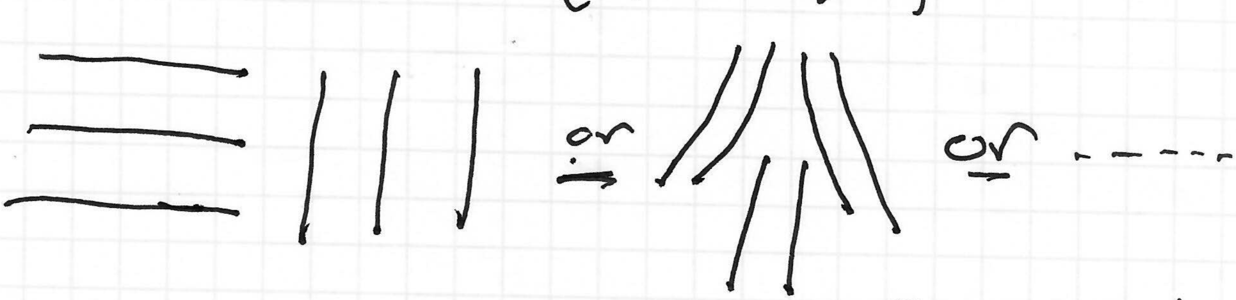
* ⓓ → Phase Diffusion ⇒ clustering (negative diffn.)

ⓑ



obviously quantized (h.o. eqn.)

ⓒ



(director field, etc.).

ⓐ Waves ⇒ NLS as envelope eqn.

c.f. Classic plasma problem of Langmuir Turbulence reduces to NLS in subsonic case.

But NLS is far more generic.

Also, observe:

dispersion

CGL: $\partial_t A = r A + (1+i\alpha)\partial_x^2 A - (1+i\gamma)|A|^2 A$

NL frequency shift

Now, conservative limit: $r \to 0$

$|\alpha, \gamma| \gg 1$

$\gamma \overline{V} A$
$V \sim |A|^2$
quadratic feedback on mean.

$$\boxed{i\,\partial_t A = \alpha\,\partial_x^2 A + \gamma |A|^2 A}$$

i.e. NLS $\quad i\hbar \dfrac{\partial \psi}{\partial t} = H\psi$

V_0

$$= -\frac{\hbar^2}{2m}\nabla^2\psi + [|\psi|^2]\psi$$

Point: NLS is generic

$\Leftrightarrow$ mean field interaction:

For linear, dispersive wave train:

$$\Phi = \int dk\, F(k) \exp[ikx - i\omega(k)t]$$

$\omega = \omega(k) \rightarrow$ dispersion relation

wave train

$k_0 \rightarrow k_0 + \delta k$

small

modulation

$k_0 \rightarrow$ carrier

$$\omega = \omega(k_0 + k - k_0)$$

$$\doteq \omega_0 + (k-k_0)\,\omega_0' + \frac{(k-k_0)^2}{2}\,\omega_0''$$

$$\Phi = \int dk\, F(k) \exp\left[ikx - i\left(\omega_0 + (k-k_0)\,\omega_0' + \frac{(k-k_0)^2}{2}\,\omega_0''\right)t\right]$$

envelope

$\downarrow$

$$\equiv \varphi\, e^{i(k_0 x - \omega_0 t)}$$

$$\varphi = \int dK\, F(k_0 + K) \exp\left[iKx - i\left(K\omega_0' + \frac{K^2}{2}\omega_0''\right)t\right]$$

obviously $\Rightarrow$ φ accounts for modulation

$\Longleftrightarrow$ <u>is</u> <u>envelope</u>

Now, φ clearly satisfies:

$$i\left(\partial_t\varphi + \omega_0'\partial_x\varphi\right) + \frac{1}{2}\omega_0''\partial_x^2\varphi = 0$$

so <u>modulation</u> has dispersion relation

$$\omega = k\omega_0' + \frac{q^2}{2}\omega_0''$$

i.e. $\varphi = q_0 e^{i(kx-\omega t)}$.

Also observe: could also entertain <u>nonlinear</u> <u>frequency</u> shift (phase invariant)

amplitude

$$\omega = k\omega_0' + \frac{k^2}{2}\omega_0'' - \varepsilon a^2$$

$|\varphi|^2$

(generic effect), then φ <u>satisfies</u>:

$$i\left(\partial_t\varphi + \omega_0'\partial_x\varphi\right) + \frac{\omega_0''}{2}\partial_x^2\varphi + \varepsilon|\varphi|^2\varphi = 0.$$

where: $\varphi = a \exp(ikx - i\omega t)$

ω = as above.

Now, frame co-moving with V_{gr}

$\Rightarrow$

$$\boxed{i \partial_t \varphi + \frac{\omega_0''}{2} \partial_x^2 \varphi + \gamma |\varphi|^2 \varphi = 0}$$

NLS.

explains why h.o. in dk.

Point:

$\rightarrow$ NLS is generic to:

- weakly nonlinear dispersive wave train

- nonlinear frequency shift $\sim a^2 \sim |\varphi|^2$

$\rightarrow$ key point is:

- sgn ω_0''
- sgn γ.

→ NLS has imaginary coeffs.

⇒ need treat via

$\psi = A e^{i\phi}$

so usual crank (linearly):

$$\boxed{\Omega^2 = \omega'' q A_0^2 k^2 + \frac{\omega''}{4} k^4}$$

so instability if: $\boxed{\omega'' q < 0}$ $\omega'' > 0$

⇒ Benjamin - Feir instability

⇒ Modulation grows

→ Linear relative of self-focusing.

→ 1D: NLS ⇒ soliton solution

→ 3D: collapse → singularity (exact).