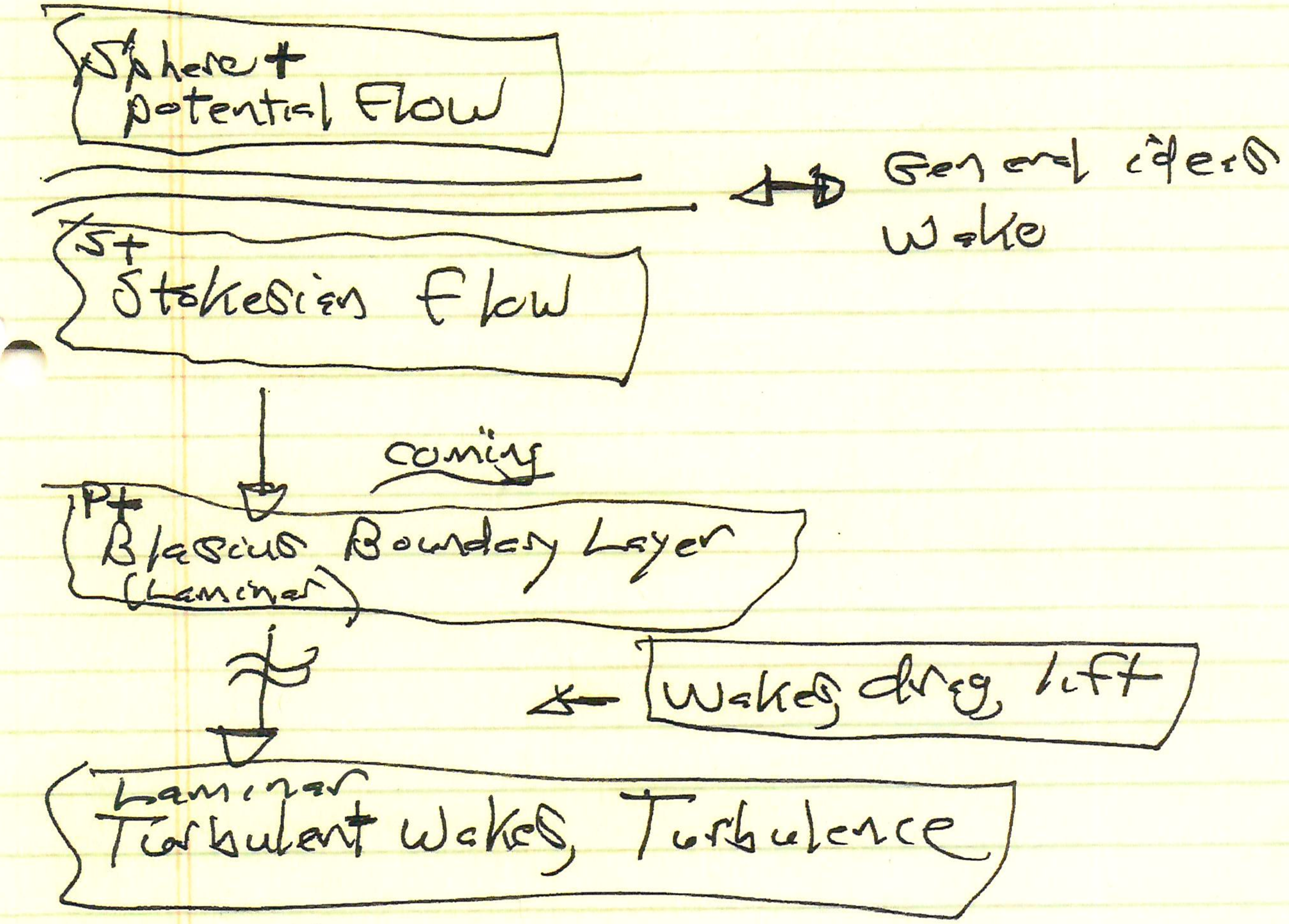
Interfaces — Rayleigh–Taylor 1:
— Surface Waves
— Kelvin-Helmholtz (wake

Physics 216/116

Lecture IVa — Instabilities I

So far: — basic eqns.
— Potential Flow
— Low Re flow



Sphere + potential flow

General ideas wake

S+ Stokesian flow

coming

P+ Blasius Boundary Layer (Laminar)

Wakes, drag, lift

Laminar Turbulent wakes, Turbulence

all: {energy source for flow is body motion (U)}

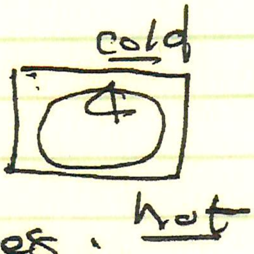
so: Instability ⇒ { continuation → KH
diversion ⇒ RT, RB

[stored free energy ⇒ fluid motions ⇒ chaos, turbulence dissipation.]

⇒ **Relaxation** – critical stage usually is linear instability

① – stored free energy {equilbrm + small pert ⇒ growth}

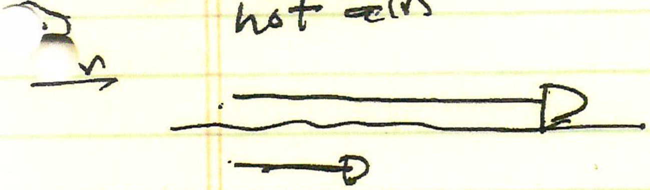
i.e.



Rayleigh-Benard convection

⇒ ∇T ⟺ thermal buoyancy energy

⇒ rises hot air



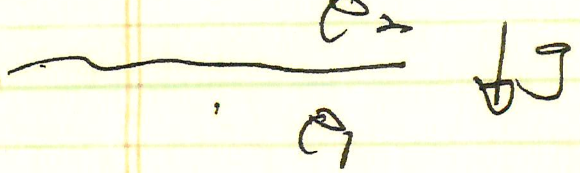
Kelvin-Helmholtz,
shear flow

⇒ ∇V ⟺ kinetic energy
flow shear

relevant to breakdown of wake (after separation)

⇒ onset of turbulent wake

③ or



Rayleigh-Taylor

⇒ $\nabla \rho + g$ (buoyancy)
but, heat not central.
⇒ gravitational potential energy.

Real Story/Question:

Relaxation → Linear Instability (rate) → Nonlinear Evolution saturation → Final state { (incl. dissipation) often turbulent.

Hydro stability is "huge" subject

c.f. Chandrasekhar [Theory of Hydrodynamic and Hydromagnetic Stability]

Here: — First step.

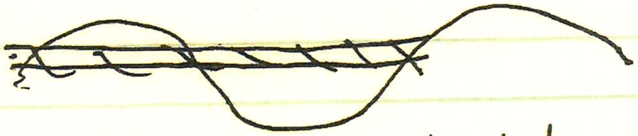
- 2 classes → ① interfacial instabilities → RT, KH
  ② Convection → RB

+ homework

1.) Interfacial Instabilities

if L st $1/L = \left(\frac{1}{\rho}\frac{\partial \rho}{\partial z}\right), \frac{1}{V}\frac{\partial V}{\partial z}$

$k_x L \ll 1$



⇒ treat gradient as held in an interface/layer.

surface

⇒ strategy: – 2 homogeneous media

⊕

– matching conditions

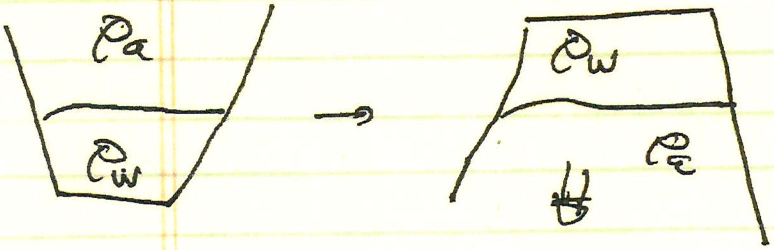
⟺ significant overlap with theory of {surface phenomena, droplets, etc. |

⇒ biophysics:

n.b. in 'Life at low Reynolds number', surface tension relevant.

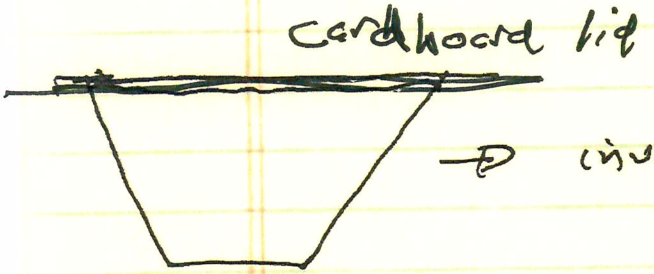
→ Primo Example 1: Rayleigh–Taylor

(c.f. posted papers, especially Taylor 1950).



why? ⇒

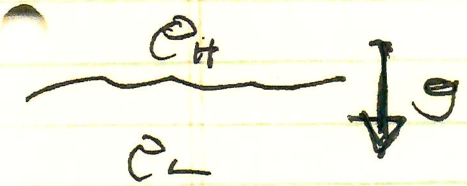
Ripples on surface grow ⇒ R.T. instability



⇒ invert

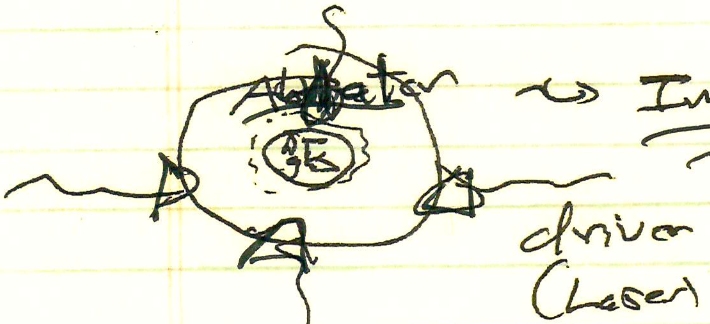
nothing happens!

⇒ cardboard effectively takes $\gamma_{\text{surf.}} \to \infty$.



→ contrived ?!

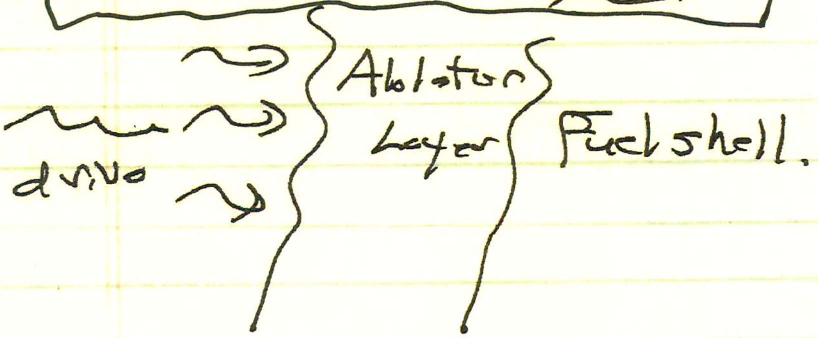
but **ICF** (controlled and otherwise)



→ Implosion

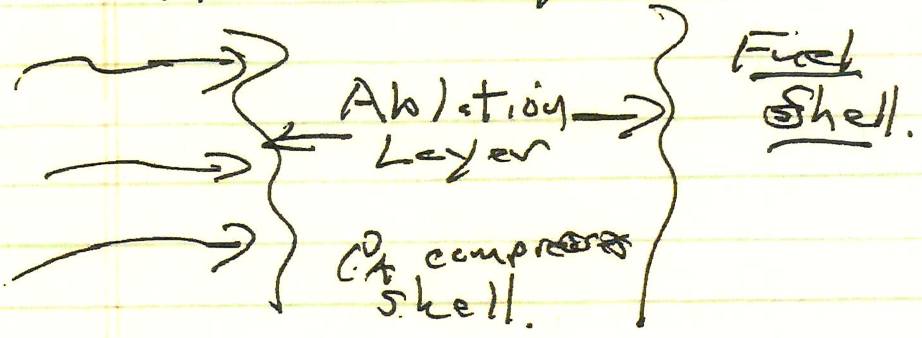
⇒ ablation-driven rocket:

driver complexes

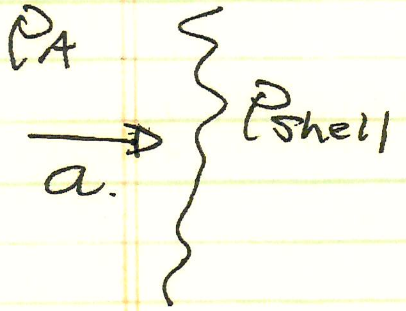


treat compression dynamics as rocket equation

⇒ drive causes ablator layer to heat and expand, thus compressing inner fuel layer



Consider situation:



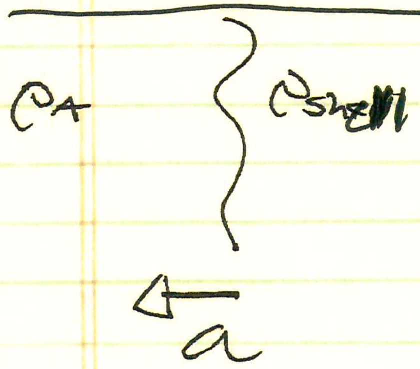
$\rho_{shell} > \rho_a$

"Light/low density fluid (gas) pushing on (i.e. accelerating onto) heavy fluid."

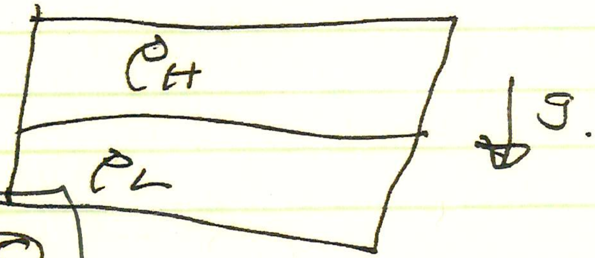
c.f. Taylor's paper.

i.e.

in frame of ablator:

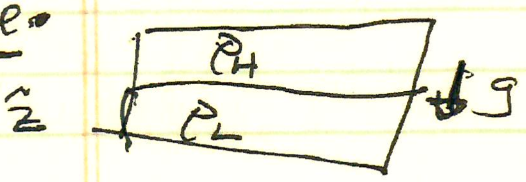


⇒ equivalent to



Both cases: $\boxed{\nabla P \, \nabla \rho < 0}$

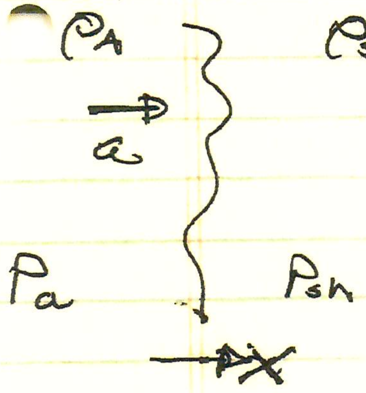
i.e.



$\nabla \rho > 0$

$\nabla P < 0$.

$\nabla P = -\rho g$



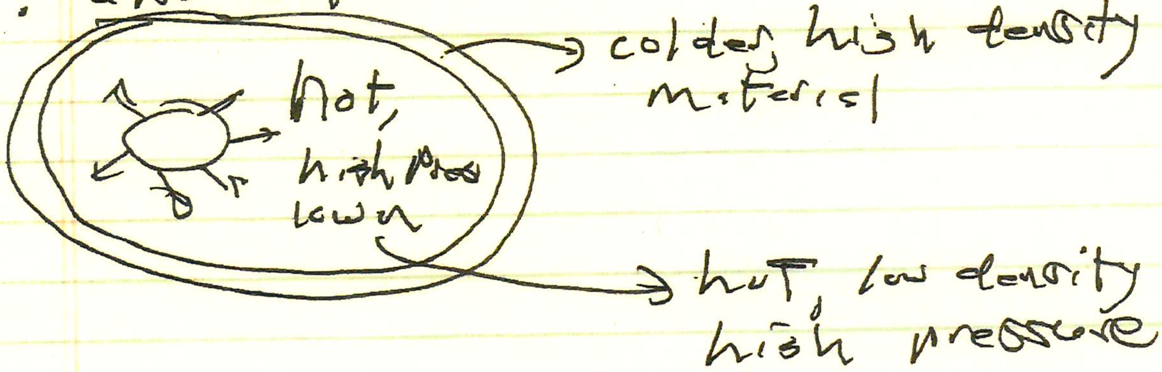
$\nabla_x \rho > 0$

$\nabla_x P < 0$.

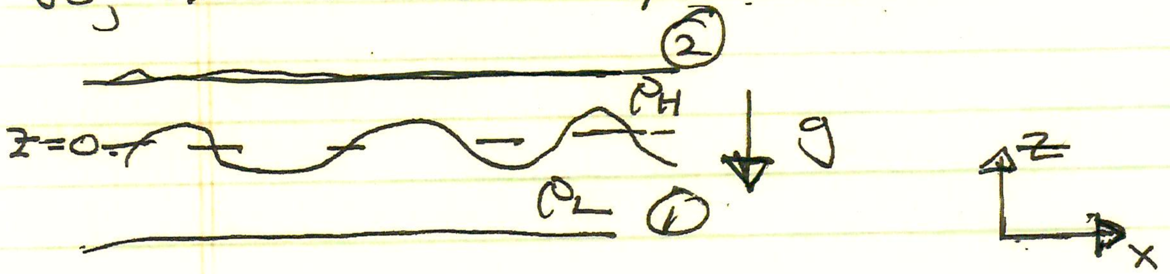
$P_a > P_{shell}$

i.e. both $\nabla P \nabla \rho < 0$

n.b.: also supernovas



so, hereafter: simple case



— $\underline{\nabla} \cdot \underline{V} = 0$ i.e. ($\gamma < k c_s$ etc.)

— ideal fluid (add visc. on HW)

Equilibrium

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v} = -\frac{\underline{\nabla} p}{\rho} + \rho\,\underline{g}$$

$\underline{\rho \text{ const.}}$

$\underline{v} \rightarrow 0$

$$\nabla^2 p = 0$$

→ vert, $\partial_z^2 p = 0$

$$p = p_0 + p' z$$

but $\underline{dp/dz = -\rho g}$

$$p_2' = -\rho_2 g$$

$$p_1' = -\rho_1 g$$

$\Delta p > 0$

going ↓

~ interface ($k_x L_z$ [redacted] $\ll 1$), vorticity localized in interface. So treat fluid as irrotational. No initial $\underline{\omega}$, so ω stays zero.

$\underline{\omega} = 0$, $\underline{v} = \underline{\nabla}\phi$

$\underline{\nabla}\cdot\underline{v} = 0$ $\nabla^2\phi = 0$

e^{-kz} ②

interface — — — $z=0$

e^{+kz} ① $\phi = \sum_k e^{ikx}\phi_k(z)$

$(\partial^2/\partial z^2 - k^2)\phi_k(z) = 0$

$$\phi_k = \begin{cases} e^{-kz} & z>0 \\ e^{kz} & z<0 \end{cases} \qquad (k>0)$$

at interface $(z=0)$ = matching conditions

①
→ pressure balance across interface [redacted] $P(0_+) = P(0_-)$

(else interface in motion on acoustic time scales)

$V_z(0_+) = V_z(0_-)$ {continuity of Velocity

$$\left.\frac{\partial\phi}{\partial z}\right|_{\substack{2\\ z\to 0}} = \left.\frac{\partial\phi}{\partial z}\right|_{\substack{1\\ z\to 0}}$$

i.e.

$$\int_{0_-}^{0_+}\left[\frac{\partial^2\phi}{\partial z^2} - k^2\phi\right] = 0$$

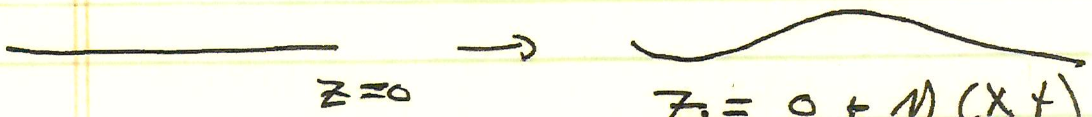
i.e

Note: V_z b.c. immediately forces

$$-k\phi_2 = k\phi_1 \Rightarrow \phi_2 = -\phi_1$$

What of dynamics?

— interface ripples



$z_i = 0 + \eta(x,t)$

- displacement of interface
- η specifies interface position

Note: $\phi = \phi(x, z_i, t)$ — $\left\{\begin{array}{l}\text{at interface}\\ \text{position}\end{array}\right.$

$= \phi(x, 0+\eta, t)$

linear theory

$\cong \phi(x, 0, t)$

i.e. linear theory $\left\{\begin{array}{l}\phi(x, z_i, t) \to \phi(x, 0, t)\\ k\eta <<< 1\end{array}\right.$

Now must account for force of gravity with displaced interface in Bernoulli equation:

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v}\right) = -\underline{\nabla} p - \rho \underline{g}$$

$$\rho\left(\frac{\partial}{\partial t}\underline{v} + \underline{\nabla}\left(\frac{v^2}{2}\right) - \overset{0}{\cancel{\underline{v}\times\underline{\omega}}}\right) = -\underline{\nabla} p - \rho g \vec{z}$$

$(g>0)$

$\underline{v} = \underline{\nabla}\phi$, $v_z = \partial_z \phi$

$\int_0^\eta dz\, v_{z_1} = \phi_1$, $\int_\eta^0 dz\, v_{z_2} = \phi_2$

so

$$\boxed{\frac{\partial\phi}{\partial t} + \frac{(\underline{\nabla}\phi)^2}{2} = -\frac{p}{\rho} - g\eta}$$

[absent gravity ; $p = -\rho\, \partial\phi/\partial t$

so [illegible]

and at interface.

$$\left.p = -\rho g \eta - \rho \frac{\partial \phi}{\partial t} \right]$$ linearize

and finally have equation/dynamic boundary condition for displacement:

$$\boxed{\frac{d\eta}{dt} = \partial_z \phi \Big|_0 = v_z}$$

$\Rightarrow$

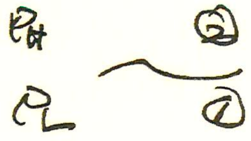
$$\frac{\partial \eta}{\partial t} + \underset{\underset{\nabla\phi}{\uparrow}}{\underline{v}} \cdot \underline{\nabla}\, \eta = \frac{\partial \phi}{\partial z}\Big|_0$$

interface velocity must match fluid velocity at interface

For stability : linearize

$$\frac{\partial \tilde{\phi}}{\partial t} = -\frac{\tilde{p}}{\rho} - g\eta$$

$$\frac{\partial \tilde{\eta}}{\partial t} = \frac{\partial \tilde{\phi}}{\partial z}\Big|_0 .$$



So noting $\rho_2 \neq \rho_1$

$$\tilde{p}^{(1)}\Big| = \tilde{p}^{(2)}\Big|$$

$$\tilde{\phi}^{(1)}\Big|_0^0 = -\phi^{(2)}\Big|_0^0$$

$$\tilde{p}_2 = \tilde{p}_1$$

$$\rho_2 \frac{\partial \tilde{\phi}^{(2)}}{\partial t} + g\rho_2 \tilde{\eta} = \rho_1 \frac{\partial \tilde{\phi}^{(1)}}{\partial t} + g\rho_1 \tilde{\eta}$$

$$g(\rho_2-\rho_1)\tilde{\eta} = \rho_1 \frac{\partial \tilde{\phi}^{(1)}}{\partial t} - \rho_2 \frac{\partial \tilde{\phi}^{(2)}}{\partial t}$$

$$= (\rho_1+\rho_2)\frac{\partial \tilde{\phi}^{(1)}}{\partial t}$$

$$\boxed{\begin{aligned} \frac{\partial \tilde{\phi}^{(1)}}{\partial t} &= g\left[\frac{(\rho_2-\rho_1)}{(\rho_2+\rho_1)}\right]\tilde{\eta} \\ \frac{\partial \tilde{\eta}}{\partial t} &= \frac{\partial \tilde{\phi}^{(1)}}{\partial z}. \end{aligned}}$$

$$\Rightarrow \boxed{\frac{\partial^2 \tilde{\phi}^{(1)}}{\partial t^2} = g\left[\frac{(\rho_2-\rho_1)}{(\rho_1+\rho_2)}\right]\frac{\partial \tilde{\phi}^{(1)}}{\partial z}.}$$

using $\phi \sim e^{-i\omega t} e^{kz} e^{ikx}$

$$\Rightarrow -\omega^2 = \left[g(\rho_2 - \rho_1)/(\rho_1 + \rho_2)\right] k.$$

$\rho_2 = \rho_H$
$\rho_2 = \rho_L$

$$\boxed{\gamma^2 = gAk \quad , \quad A = \left(\frac{\rho_H - \rho_L}{\rho_H + \rho_L}\right)}$$

– free energy

– inertia

↓
Atwood #

i.) $\rho_H = H_2O$ $\quad \lambda \sim 1$ cm.
$\rho_L =$ air $\quad \tau_g \sim .1$ sec.

(fast!)

ii.) $\rho_2 =$ air $\quad \rho_{air}/\rho_{H_2O} \to 0$
$\rho_1 =$ water

$$\omega = \sqrt{kg} \quad \rightarrow$$ dispersion relation for surface gravity wave.

(stable wave counterpart of R-T)

iii.) $\gamma \simeq (gAk)^{1/2}$

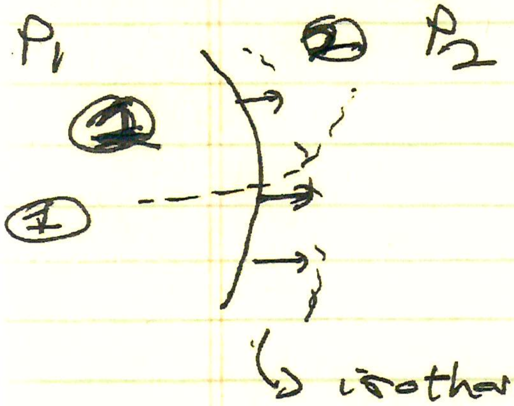
→ shorter wavelengths grow faster?!

⇒ small scale effects?

cut-off, regulator?

- viscosity (HW)
- surface tension ⊛
- finite layer width ($k_y L_z \gtrsim 1$)

Surface Tension → { force due increase in surface area interface



① expands

inflate balloon

↳ isothermal displacement

$$dF = -P_1\,dV - P_2(-dV) + \sigma\,dA$$

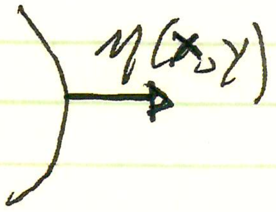
dF ↓ change in free energy

P_1 ↓ ① expands into ②

$\sigma\,dA$ ↓ change in surface area of interface

PV work.

$$dV = dA\, \underset{\substack{\uparrow \\ \text{displacement}}}{d\eta}$$



so

$$dA = \int dx\,dy \left(1+\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2\right)^{1/2} - \int dx\,dy$$

small displacement (slope):

$$\simeq \int dx\,dy \left(1+\frac{1}{2}\left(\frac{\partial\eta}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial\eta}{\partial y}\right)^2\right) - \int dx\,dy$$

$$= \int dx \int dy\ \frac{1}{2}\left[\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2\right]$$

IBP

$$= \int dx \int dy\ \underset{\substack{\uparrow \\ \text{curvature of} \\ \text{surface displacement}}}{\left(-\nabla^2\eta\right)}\, d\eta$$

so

$$dF = \int\left[(\rho_2-\rho_1)\,dA_0 - \sigma\nabla^2\eta\, dA_0\right]\delta\eta$$

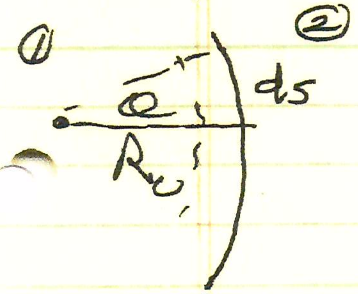
$$= \int\left[(\rho_2-\rho_1) - \sigma\nabla_\perp^2\eta\right] dA_0\, d\eta$$

so criterion for equilibrium:

$$\boxed{P_2 - P_1 = T \nabla^2 \eta}$$

More generally: $dF = (P_2 - P_1)\, dA_0\, d\eta + T\, dA$

Now consider arbitrary (i.e. not "weakly curved" interface):



$$dS = (R_0 + d\eta)\, d\theta$$

$$\simeq dl_0 \left(1 + d\eta / R_1\right)$$

radius curv

(Gauss Thm)

In general, surface parametrized by 2 radii of curvature, R_1, R_2:

$$dA = \int dl_1\, dl_2 \left(1 + \frac{d\eta}{R_1}\right)\left(1 + \frac{d\eta}{R_2}\right) - \int dl_1\, dl_2$$

$$= \int dl_1\, dl_2 \left(1/R_1 + 1/R_2\right) d\eta$$

so $dF = \int \left[(P_2 - P_1) + T\left(1/R_1 + 1/R_2\right)\right] dA_0\, d\eta.$

so, for equilibrium with interface (general)

$$\boxed{\sigma\left(1/R_1 + 1/R_2\right) = P_1 - P_2}$$

[Laplace's Law

- Given 2-phase equilibrium (separate domains), can use Laplace Law to estimate droplet size for immiscible liquids

i.e. $P_1 > P_2 \Rightarrow R \sim \sigma/(P_1 - P_2)$ |

Now, back to R-T, S-W:

$\rho_H >> \rho_L$

$\rho_H + \rho_L \to \rho_H$

$$P \to P - \rho\gamma_T \nabla_h^2 \eta$$

$\gamma_T \equiv \sigma/\rho.$ → σ for each interface i.e. water-air, etc.

$\Rightarrow$

$$\gamma_{R-T} = \left(kgA^{1} - \underset{\uparrow\ \text{cut-off}}{\gamma_T} k^3\right)^{1/2}$$

$$k_{max}\Big|_{\text{cnot}} \sim \left(g/\gamma_T\right)^{1/2}.$$

→ limits range of unstable modes.

For stable case:

$$\omega^2 = gk + \frac{\sigma}{\rho} k k^2$$

} gravity–capillary

↓ gravity wave (long)

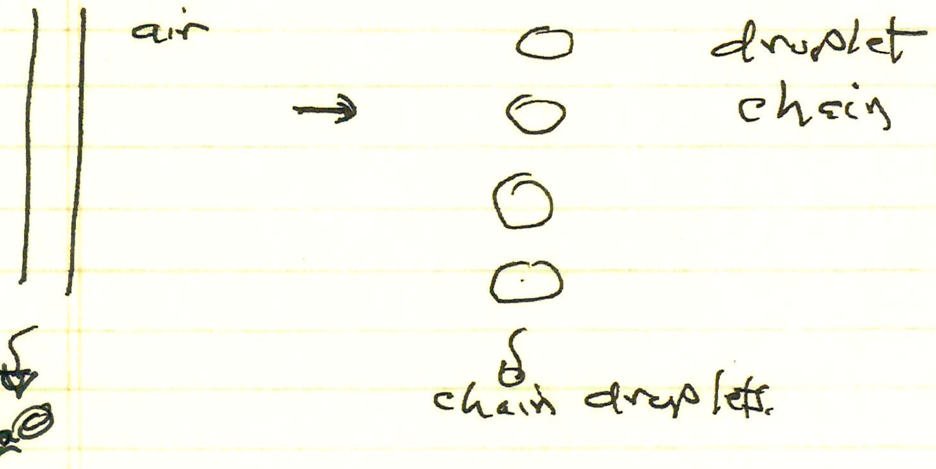
↓ capillary wave (short)

$$l_{cap} \sim (\sigma/\rho g)^{1/2}$$

{ in ocean cross-over at few cm.
Capillarity important at ≤ 5cm.

N.B.:

Capillarity (S.T.) can induce instability – classic is line of fluid break-up to string of pearls



akin sausage instability in MHD