"Engineering Flows: Models and Mixing Length Theory"

- Law of Wall and Prandtl Mixing Length Theory
- Another Look at Wakes
- Heat Transfer: Laminar and Turbulent

All three make heavy use of

- scale invariance

- analogy of turbulent mixing
  with ~~[illegible]~~ mixing/diffusion
  in gases.

i.e. $D \sim v_{Th}\, l_{mfp}$ — Mixing Length Theory

$v_{Th} \rightarrow \tilde{V}$ i.e. V_A

$l_{mfp} \rightarrow x$ i.e. distance from wall

x — mixing length

N.B.
- Useful, in concert with phenomenology
- no real physical resemblance of diffusive mixing to turbulent mixing (recall Richardson).

→ [Momentum] Flux Driven Turbulence

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Turbulent Pipe Flow

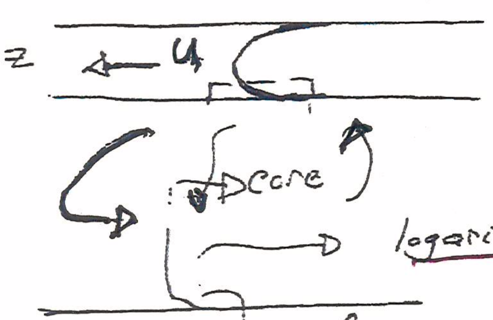
(cf. Landau, Lifshitz "Fluid Mechanics")

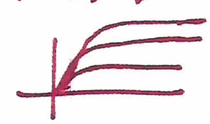
Till now → homogeneous flow in a periodic box
→ cascade in scale space (Kolmogorov) 1941

Now → inhomogeneous flow in a pipe
→ momentum transport in a turbulent boundary layer $\boxed{\text{(Prandtl)}}$ 1931, 32

Consider turbulent pipe flow: $Re \gg 1$ $\gtrsim 10^6$

eg in toilet





→ universal

logarithmic profile (inertial sublayer)
(virtually all high Re experiments → "profile consistency")

linear profile
(viscous sublayer)

Common features of pipe flow:

- linear → logarithmic $u(x)$ profile near boundary
- logarithmic profile persists over a broad range of Re

$$\left(Re = 2ua/\nu \right)$$

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∴ logarithmic profile "universal" (Prandtl "Law of the Wall")

– resistance ~~discontinuously~~ increases with increasing Re,

$$\lambda = \frac{2a\,\Delta p/\ell}{\frac{1}{2}\rho U^2}$$

$\Delta p/\ell \to$ pressure drop/length

$(\to \tau_{KE})$

$\frac{1}{2}\rho U^2 \to$ mean flow energy

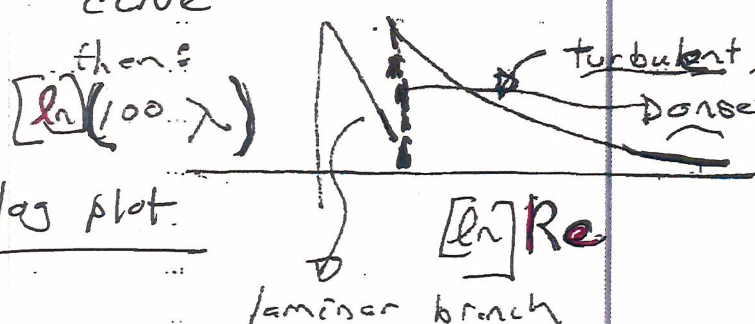
better $1/\lambda$ on KE vs ΔP

incremental increase KE dropping with Re

not incr. stored energy much with $\uparrow \Delta P$

Resistance curve

then: log plot

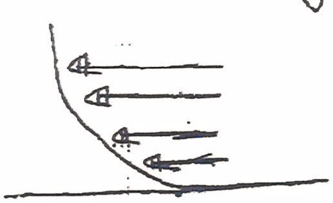


some subtlety in how define Re

– turbulent resistance curve universal.

no indep scale dep.

What is going on? → physics of resistance?



no slip boundary condition

$$U = U(x) \xrightarrow[x\to 0]{} 0$$

gradient

$$\therefore\ U = U(x) \Rightarrow \left\{ \text{momentum flux to wall.} \right.$$

$$\partial_t \underline{v} + \underline{v}\cdot\nabla\underline{v} - \nu\nabla^2\underline{v} = -\frac{\nabla P}{\rho}$$

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→ Momentum flux to wall ⇒ stress on the wall

→ wall stress must balance pressure drop for steady flow

τ, ρ, some U_*

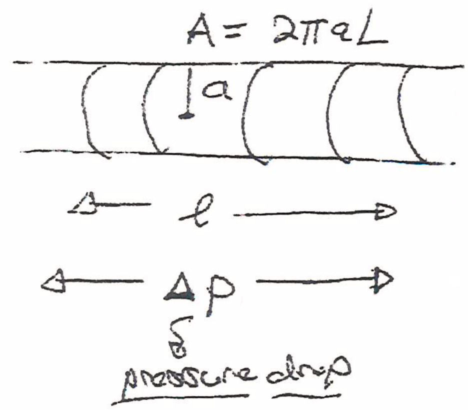
so wall stress : ρU_*^2

$U_* \equiv$ friction velocity

i.e. $v\cdot\nabla v - \nu\nabla^2 v = -\nabla p/\rho \sim \frac{\Delta p}{\rho}$

$\Rightarrow v_*^2 2\pi R \sim \Delta p \pi R^2$

$$\rho U_*^2 2\pi a\, \ell = \Delta p\, \pi a^2$$ defines U_*



Force on wall $\simeq \rho U_*^2 A_{wall}$

(Pressure Drop) A_{flow} = Force on Fluid

stationarity ⇒ $\rho U_*^2 (2\pi a \ell) = (\Delta p)\, \pi a^2$

$$\boxed{U_* = \left[(\Delta p/2\rho)\left(\frac{a}{\ell}\right)\right]^{1/2}}$$

Friction Velocity

$u_* \equiv$ friction velocity
$\equiv$ "typical" velocity of turbulence in turbulent pipe — think of as energy containing range.

Deriving the inertial sublayer profile:

(i) dimensional reasoning

in pipe flow inertial sublayer, have

3 dimensional parameters ρ, τ, x

ρ: density

τ: wall stress, u_*

x: distance from wall

Key Point: Assumption of scale invariance

on scale $l_{vs} = \frac{\nu}{u_*} < x < a$

only length in theory (resl. ν).

→ universality of logarithmic profile motivates scale invariance assumption

Recall

now, seek velocity gradient du/dx,

$\frac{du}{dx}$: u_*, x, ρ

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so simplest form for dU/dx is:

$$\left[\frac{dU}{dX} = \frac{U_*}{X} \right.$$

$$\Rightarrow \boxed{U = \frac{U_*}{K} \ln(X/X_0) = \frac{U_*}{K} \ln X + \text{const.}}$$

[c.f. Barenblatt]

→ logarithmic profile (consequence of scale invariance in pipe flow)

→ $K \approx .4$ universal constant → von-Karman constant (empirical)

$X_0 \leftrightarrow$ width of viscous sublayer $\sim \nu / U_*$

ii.) Physical Reasoning

stationary flow ⇒

momentum flux to wall = pressure drop

Mixing Length Theory initiated by Boussinesq.

2ψQ

$\therefore \quad \langle \tilde{V}_x \tilde{V}_z \rangle = u_*^2$

Reynolds stress

$\rho \langle \tilde{V}_x \tilde{V}_z \rangle = \Gamma_p = \Pi_{x,z}$

$\hookrightarrow$ momentum flux

$\Pi_{x,z} / \rho = u_*^2$

Now, to calculate $\langle \tilde{V}_x \tilde{V}_z \rangle$:

→ take velocity fluctuation as generated by mixing of $U(x)$, so

→ $\tilde{V}_z \approx -\underbrace{l}_{\text{"mixing length"}} \frac{\partial U}{\partial x}$

$\tilde{V}_z$ results from mixing of mean profile U

analogous to Chapman-Enskog expansion, i.e.

$l \leftrightarrow l_{mfp}$

$\tilde{V}_x \leftrightarrow V_{Th}$

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~~logarithmic~~ ~~[illegible]~~ ~~further~~

no scale.

here, scale invariance $\Leftrightarrow$ $\ell \sim x$

mixing length set by distance from wall

so

$$\langle \tilde{v}_x \tilde{v}_z \rangle \cong -\langle v_x \ell \rangle \frac{\partial U}{\partial x}$$

$$\cong -U_* x \frac{\partial U}{\partial x}$$

$U_* \rightarrow$ drive dependence

$\nu_T = U_* x$ $\Rightarrow$ "eddy viscosity" "turbulent viscosity" > Key concept

Fick's

$\Rightarrow$ rate of turbulent ~~[illegible]~~ diffusion of momentum ~~[illegible]~~

then momentum balance $\Rightarrow$ ~~[illegible]~~

$$U_* x \frac{\partial U}{\partial x} = U_*^2$$

$$\Rightarrow \boxed{U = \frac{U_*}{\kappa} \ln(x/x_0)}$$ $\rightarrow$ Logarithmic Profile

$\rightarrow$ Law of the Wall

FAQ

Some comments:

→ as in K41, clear phenomenology critical to guiding the approximations ⟷ scale invariance

⇒ "Mixing length theory always works .... provided you know the mixing length..."
—P.D.

→ why a single value of velocity, i.e. u_*?

Consistent with mixing length hypothesis, velocity fluctuations generated by mixing of mean flow gradient, i.e.

⇒

$$\tilde{V} \sim l \frac{\partial u}{\partial x} \sim x \frac{\partial u}{\partial x}$$

$$\sim \cancel{x} \frac{u_*}{\cancel{x}}$$

absence of preferred scale

internal consistency

Consistent. ∴ Assumption consistent with:
- logarithmic profile
- scale invariance.

What happens at wall ?

⇒ viscous sublayer / cut-off of inertial layer ?

∴ when : $\nu_T < \nu$ {molecular viscosity dominates mixing

$$\Rightarrow \quad u_* x \lesssim \nu$$

$$x \lesssim \nu / u_* \equiv x_0 \quad \rightarrow \quad x_\nu = \frac{\nu}{u_*}$$

viscous sublayer scale.

dissip'n scale

In viscous sublayer, flow linear :

$$\nu \frac{\partial u}{\partial x} = u_*^2$$

$$\therefore \quad u = \frac{u_*^2}{\nu} x$$

⇒ note effect of turbulence is to :

- flatten profile – {higher transport at fixed wall stress
- reduce central velocity
- limit Q (quality factor)

– matching, for const:

$$x_0 = \nu/u_* \quad \text{so}$$

$$u = \frac{u_*}{\kappa}\ln\left(\frac{u_* x}{\nu}\right)$$

Note: Flow in viscous sublayer is <u>turbulent</u>, but <u>mixing</u> there affected by dissipation range scales ⇒ linear profile

Now – turbulent dissipation ?

Consider NSE:

$$\frac{\partial \hat{V}}{\partial t} + \hat{V}\cdot\nabla\hat{V} + \langle V_z\rangle \frac{\partial}{\partial z}\hat{V} + \hat{V}_x \frac{\partial}{\partial x}\langle V_z\rangle$$

$$= -\nabla\hat{p} + \nu\nabla^2\hat{V}$$

$\hat{V}\cdot$ and avg ⇒

$$\frac{\partial}{\partial t}\langle \hat{V}^2\rangle + \langle \hat{V}\cdot\hat{V}\cdot\nabla\hat{V}\rangle + \langle V_z\rangle\overset{\text{odd}}{\cancel{\left\langle \hat{V}\cdot\frac{\partial}{\partial z}\hat{V}\right\rangle}}$$

$$+ \langle \hat{V}_x\hat{V}_z\rangle\frac{\partial}{\partial x}\langle V_z\rangle = -\underset{\text{i.b.p.}}{\cancel{\langle \hat{V}\cdot\nabla\hat{p}\rangle}} - \nu\langle(\nabla\hat{V})^2\rangle$$

For net energy budget:

$$\partial_t \mathcal{E} = -\langle \tilde{V}_x \tilde{V}_z \rangle \frac{\partial \langle V_z \rangle}{\partial x} - \nu \langle (\nabla \tilde{V})^2 \rangle$$

$$\left(+ T \atop \left[\frac{\partial}{\partial x} \langle \tilde{V}^3 \rangle \right] \right)$$

radiat. ~~energy~~ spr.

input to fluctuations by relaxation of mean shear flow (Reynolds work)

dissipation of fluctuation energy by viscosity

$\therefore$ can define:

$$\epsilon = \langle \tilde{V}_x \tilde{V}_z \rangle \frac{\partial U}{\partial x}$$

turbulent dissipation rate

input to turbulence from Reynolds work mean flow.

and using mixing length theory:

$$\langle \tilde{V}_x \tilde{V}_z \rangle = U_* X \frac{\partial U}{\partial x}$$

$$\Rightarrow \quad \epsilon = (U_* X) \left(\frac{\partial U}{\partial x} \right)^2 = \nu_T \left(\frac{\partial U}{\partial x} \right)^2$$

rate of "heating" by turbulent relaxation input → mean flow mixing

obviously: $\nu \langle (\nabla \tilde{v})^2 \rangle = \nu_T \left(\frac{\partial U}{\partial x} \right)^2$

small scale dissipation

and,

$\epsilon = (u_* x) \left(\frac{u_*}{x} \right)^2$ (ignoring R)

$= \frac{u_*^3}{x}$

no proof, c.f. 4/5 Law.

$\epsilon \leftrightarrow$ wall distance.

→ sets dissipation rate, as fctn x

i.e. $\epsilon = \frac{v_0^3}{\ell}$ $\quad v_0 \leftrightarrow u_*$, $\ell \leftrightarrow x$

→ ϵ finite as $\nu \to 0$ (i.e. viscous sublayer gradient diverges then)

Additional References:

— S. B. Pope, "Turbulent Flows"

— H. Tennekes and J. Lumley, "A First Course in Turbulence"

and of course,

Landau & Lifshitz.

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→ Now, interesting to tabulate comparison between Pipe Flow and K41 Problem

Pipe Flow (Prandtl)	K41 (Kolmogorov)
scales: a, x, ν/u_*	l_0, l_n, l_d
invariance: $X \to$ real space	$l \to$ scale space
inertial sublayer viscous sublayer	inertial range dissipation range
lance: $u_*^2 = \nu_T \frac{\partial u}{\partial x}$	$\epsilon = \frac{v(l)^2}{\tau(l)}$
namics: eddy viscosity $\nu_T = u_* X$	turn-over rate $1/\tau(l) = \frac{v(l)}{l}$
ult: $u = \frac{u_*}{\kappa} \ln(x)$	$v(l) = \epsilon^{1/3} l^{1/3}$
universal profile	universal spectral scaling
sipation: $\nu = \nu_T$ $x_0 = \nu/u_*$	$v(l)/l = \nu/l^2$ $l_d = \nu^{3/4}/\epsilon^{1/4}$.

⇒ Practical Issues

Resistance Law ⇒ Pipe Flows

have: $\frac{\nu}{u_*} < x \leq a$ ← radius

can push to $x \sim a$, with logarithmic accuracy

$$U \cong \frac{u_*}{k} \ln\left(\frac{u_* a}{\nu}\right)$$

but

$$v_* = u_* = \left(\frac{a}{\ell}\frac{\Delta P}{2\rho}\right)^{1/2}$$

⇒ can re-write:

$$U = \left(\frac{a\,\Delta P}{2\rho\,\ell k^2}\right)^{1/2} \ln\left(a\left(\frac{a\,\Delta P}{2\rho\ell}\right)^{1/2}\Big/\nu\right)$$

Convenient to define:

$$\lambda = \frac{2a\,\overset{\text{drive}}{\Delta P}/\ell}{1/2\,\rho U^2} \rightarrow \text{friction factor / resistance coefficient}$$

⇒ taking $Re = 2aU/\nu$

can re-write friction law as:

$$1/\sqrt{\lambda} = .88 \ln\left(Re\sqrt{\lambda}\right) - .85$$

phenom.

$Re \doteq 2aU/\nu$

$$\lambda = \frac{2a\, \Delta p/\ell}{1/2\, \rho U^2}$$

– good fit to pipe flow data.

Additional Cases

Turbulent Wakes, Thermal Boundary Layers

Here:

→ turbulent wakes, complete wake story begun earlier
  → bckgnd
  → scaling
  → eddy mixing

→ Thermal BL / Heat Transfer

→ bckgnd, set up, types
  Pr

→ heat transfer problem
  – heat transfer coeff
  – Nu
  – laminar, turbulent

→ intro to temp fluctn turbulence – passive scalar.

References : Boundary layers, wakes, heat transfer

→ Landau & Lifshitz : excellent, 'physicist style' treatment of these 'engineering' subjects

→ V. Krasnov : Good summary, many examples

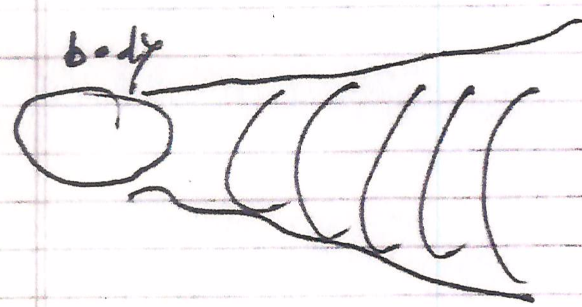
→ H. Tennekes, J. Lumley : Basic discussion, Good first course.

→ S. B. Pope : classic Engineering text. Detailed zoology.

B.) Wakes – Simple physics   cf: { Prandtl – Tietjens, Falkovich, Landau }

Wake is:

- region of departure from potential flow behind object moving thru water and experiencing drag



- wake is inexorably coupled to drag
- Message of wakes:

$\boxed{\text{A little drag forces a global adjustment of flow structure}}$

- drag – [thinking in frame where object at rest, drag results from loss of flow momentum to object.

Which spreads faster downstream
Laminar or Turbulent wake?

4a

[redacted]

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(ii) Turbulent Wakes $Re \sim UR/\nu \gg 1$

$$\underline{U}\cdot\nabla\underline{v} + \underline{v}\cdot\nabla\underline{v} - \cancel{\nu\nabla^2\underline{v}} = -\frac{\nabla p}{\rho}$$

ignore

$$\Rightarrow \quad \frac{U}{X} v_x \sim \frac{\tilde{v}_y}{W} v_x$$

$$\Downarrow$$

[wake spreads by advection, not diffusion]

$\tilde{v}_y$ ~ turbulent velocity $\boxed{W \sim \frac{\tilde{v}_y}{U} X}$

Take wake turbulence isotropic,

so $\boxed{\tilde{v}_x \sim \tilde{v}_y}$ {Fair? Test}

$$W \sim x\tilde{v}_x / U$$

but from drag:

$$\tilde{v}_x \sim F_d / \rho U W^2$$

$\Rightarrow$

$$W \sim X \frac{F_d}{\rho u^2 W^2} \approx X \left(F_d / \rho u^2 W^2 \right)$$

$$W^3 \sim F_d X / \rho u^2$$

$$\Rightarrow \boxed{\begin{aligned} W &\sim \left(F_d / \rho u^2\right)^{1/3} X^{1/3} \\ &\sim \left(C_D R^2\right)^{1/3} X^{1/3} \end{aligned}}$$

then, comparing widths:

laminar: $w/R \sim (X/R)^{1/2}\, Re^{-1/2}$

$Re \sim UR/\nu$

turbulent: $w/R \sim (X/R)^{1/3}\, C_D^{1/3}$

[interestingly laminar wake expands with downstream length more rapidly!

Why ?

→ turbulence can relax ∇V behind object (due separation) rapidly, and faster than ν. Thus surrounding flow penetrates the dead water region more rapidly.

Also observe: Wake Re drops with x.

⇒

$$Re \sim \frac{W V_x}{\nu} \sim \frac{w V_x}{\nu} \sim \frac{\cancel{W}}{\nu}\,\frac{F_d}{\rho u W^{\cancel{2}}}$$

y direction (sps)   wake flow Re

$$Re \sim F_d / \rho u W \nu$$

$$\sim \frac{u^{\cancel{2}} R^2 \cancel{\rho_0} C_D}{\nu \cancel{\rho_0} \cancel{u} (C_D R^2)^{1/3} x^{1/3}}$$

$C_D \sim 1$

$$\sim \left(\frac{uR}{\nu}\right)\left(R/x\right)^{1/3}$$

$$\boxed{Re(x) \sim Re_o \left(R/x\right)^{1/3}}$$

and $Re(x) \to 1$ at

$$x_L \sim R\,(Re_o)^3$$

distance behind boat where turbulent wake transitions to laminar.

i.e. akin l_d: transition from turbulent mixing to viscous mixing

N.B. In wake, vortical/rotational region can expand into irrotational region, but never reverse!

i.e. would really violate H-Thm ...

Wakes - Supplement   Sketch

→ Revisit turbulent wake, using turbulent viscosity, i.e.

$$W \sim (\nu x / u)^{1/2} \qquad (\nu \to D_T)$$

$$\to (D_T x / u)^{1/2}$$

i.e. width of turbulent wake set by turbulent diffn, following Blasius Law.

but $D_T \sim W \tilde{V}$ ⇒ turbulent viscosity at mixing length level.

$$\sim W \left(F_d / \rho u W^2 \right)$$

$$\sim F_d / \rho u W \sim \text{const} / W$$

⇒

$$W \sim \left(F_d x / \rho u^2 W \right)^{1/2}$$

$$W^{3/2} \sim \left(F_d / \rho u^2 \right)^{1/2} x^{1/2} \sim (C_D R^2)^{1/2} x^{1/2}$$

$$\sim C_D^{1/2} R \, x^{1/2}$$

$$W \sim (C_D)^{1/3} R^{2/3} x^{1/3}$$

$$\Rightarrow \boxed{W/R \sim C_D^{1/3} (X/R)^{1/3}}$$

agrees ✓.

Now, $D_T \sim \tilde{\nu} W$

$$\sim \frac{(\tilde{\nu} W^2)}{W}$$

$$\sim \frac{\rho U \tilde{\nu} W^2}{\rho U W}$$

$$\sim \frac{Q}{W} \sim Q/R\,(X/R)^{1/3}$$

∴ - Point is that turbulent viscosity, mixing drops downstream, relative to constant viscous mixing. ✱

- follows from $\tilde{\nu} W \sim \overbrace{Q}^{\to\,\text{const.}}/W$

- explains why turbulent wake spreads more slowly than laminar wake.

Thermal Boundary Layer + Heat Transfer

Consider stationary (in mean sense) flow + heat conduction

$$\cancel{\partial_t T} + \underline{v}\cdot\underline{\nabla} T = \chi \nabla^2 T$$

→ thermal diffusivity

$$\chi = K/\rho c_p$$

↓ heat conductivity

$$\rho\, \underline{v}\cdot\underline{\nabla}\underline{v} = -\underline{\nabla} P + \rho\nu \nabla^2 \underline{v} + \underline{g}\,\rho$$

for now exclude buoyancy

so : dimensionless #

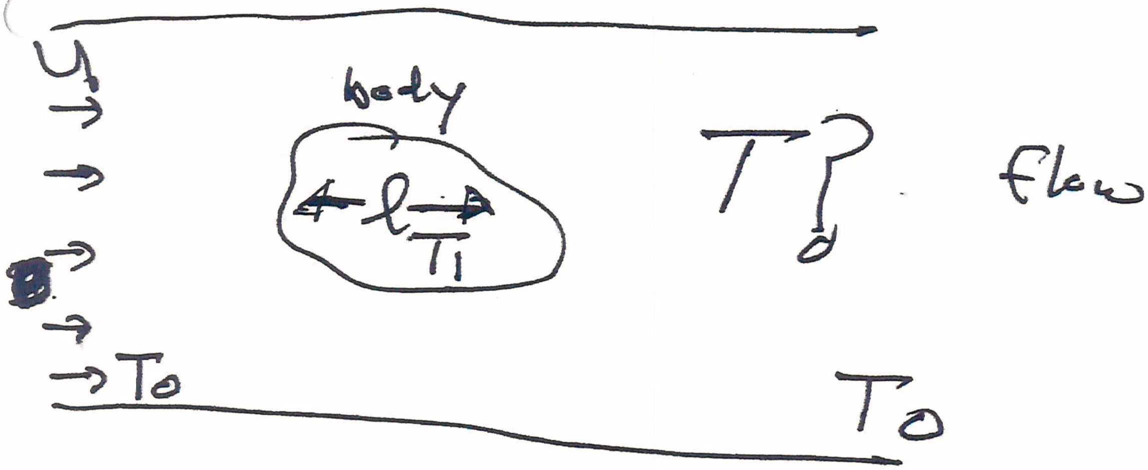
→ Re, as usual

→ $Pr = \nu/\chi$

n.b. → if buoyancy, $Ra = \dfrac{g\alpha(\delta T)L^3}{\nu\chi}$

↓ Rayleigh #.

Now, generic problem:



- body scale l, at temp T_1
- incoming flow $\underline{U}$, at T_0

→ $\boxed{\text{what is temp field?}}$

<u>i.e.</u> can flow cool body??

$$\frac{T - T_0}{T_1 - T_0} = f\left(\frac{r}{l}, Re, Pr\right)$$

↳ downstr.

$$\frac{V}{U} = f\left(r/l, Re\right)$$

↳ inc.

is scaling of result,

Furthur ways of keeping score:

→ if concerned with cooling body

$$h = \alpha = q / (T_1 - T_2)$$

q → surface heat flux of body

h → heat transfer coefficient / effectiveness

T_1 → body T

T_2 → flow T

$q \sim -k_T \nabla T$

as $q = -k \nabla T \big|_{\text{surface}}$ ⇒ [h is strongly tied to boundary layer dynamics] ✓

→ dim-less ratio:

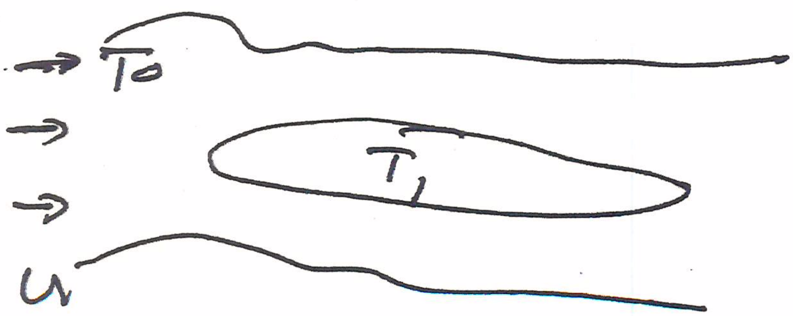
$$N = \frac{h\ell}{k} \sim \frac{D_{\text{eddy, Thermal}}}{\chi} \sim \frac{\chi_{\text{eddy}}}{\chi}.$$

N → Nusselt #

$$\stackrel{\text{so}}{=} \quad N = f(Re, P) \text{ for B-L heat transfer.}$$

N.B.: Note trade-offs in cooling problem
i.e. resistance of pipe, heat transfer.

So ①



How does Nu scale in laminar BL?

How effective is laminar flow in cooling?

$$q = -k \left.\frac{\partial T}{\partial n}\right|_{\text{bndry}}$$

$$\sim k \frac{(T_1 - T_0)}{\delta}$$ → surface heat flux

δ → bndry layer width

but, we know for laminar BL,

$$\delta \sim \ell/(Re)^{1/2}$$ i.e. Blasius.

So for Pr~1.

$$Nu \sim \frac{h\ell}{k} \sim \left(\frac{q}{T_i - T_o}\right)\ell/k$$

$$\sim \frac{k(T_i - T_o)}{\delta}\,\frac{\ell}{k\,(T_i - T_o)}$$

$$\sim \sqrt{Re}$$

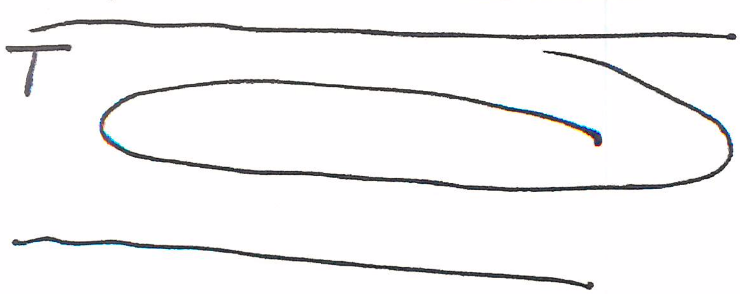
so $\boxed{N \cong \sqrt{Re}\; f(P_r)}$ → Nusselt number

$\boxed{h \sim \frac{k}{\ell}\sqrt{Re}}$ → heat transfer coeff.

~ (note size scaling)

~ note C_p importance!

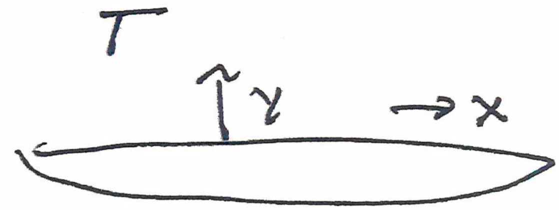
② Turbulent B.L.



sufficient to calculate temp field in flow.

$$q = -K_T \frac{dT}{dy}$$

thermal eddy vis.



$$K_T = \rho c_p \underbrace{V_* y}_{\nu_T}$$

$V_* \to$ friction velocity for BL

so

$$dT/dy = \frac{q}{\rho c_p V_*} / y$$

turb. boundary layer for Temp field.

$$\boxed{T = \frac{q}{\rho c_p V_*} \ln(y/y_0) + f(P)}$$

$y_0 = \nu / V_*$

additional diddle const may enter.

$(P_r \sim 1)$

$$N = \frac{K_T}{k} \sim \frac{V_* a}{k}$$

→ And, flow is turbulent, with temp fluctuations.

Production: $-\frac{Q}{T_0}\frac{\nabla T}{T_0} \to \frac{d}{dt}\frac{\tilde{T}^2}{T_0^2}$

$$\sim \left(\frac{\tilde{T}}{T_0}\right)^2 \frac{V}{L}$$

so $\equiv \alpha$

$$\alpha \equiv \frac{v(\ell)}{\ell}\,\tilde{t}(\ell)^2$$

$$\sim \frac{\epsilon^{1/3}}{\ell^{2/3}}\,\tilde{t}(\ell)^2 \Rightarrow \tilde{t}(\ell) \sim \ell^{1/3}\frac{\alpha^{1/2}}{\epsilon^{1/6}}$$

$$t(\ell)^2 \sim \left(\alpha/\epsilon^{1/3}\right)\ell^{2/3} \to \underline{k^{-5/3}}$$

i.r. scaling for $\tilde{T}/T$ fluctns.

$Pr >$
$\nu > \chi$
χ sees smooth flow, small scales. $\frac{\nu}{\chi}$

but? $Pr \ll 1$ / $\gg 1$ → how reconcile dissipn ranges?

One field may see other smooth! TBC.