→ Return of the β-effect

Recall: and the Rhines Scale emerges.

– 2D

$$\partial_t \nabla^2\phi + \nabla\phi\times\hat{z}\cdot\nabla\nabla^2\phi$$
$$+ \mu\nabla^2\phi - \nu\nabla^2\nabla^2\phi = \tilde{f}$$

– β-plane

$$\partial_t \nabla^2\phi + \nabla\phi\times\hat{z}\cdot\nabla\nabla^2\phi + \text{etc}$$
$$= -\beta V_x$$

$$\Downarrow$$
$$(\nabla\Omega)$$

⇒ Rossby waves.

β-effect: introduces Rossby wave, zonal flow

i.e. linear, inviscid wave solution

Rossby wave:

$$\omega = -\beta k_x / k^2$$

and: zonal flow

$$k_x = 0, \quad \omega = 0.$$
$$k_y \neq 0,$$

(2)

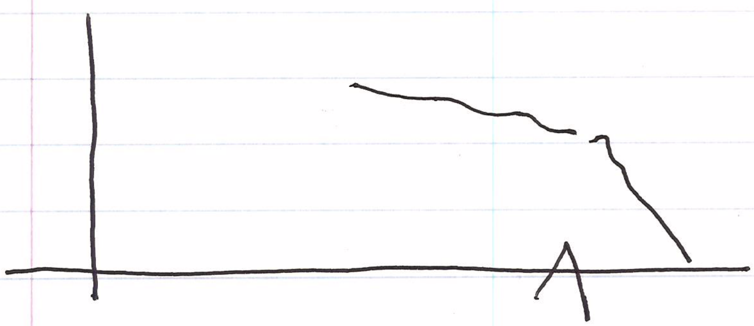
$\left\{\begin{array}{l}\text{Scale Flow} \\ \text{Interaction} \\ \quad \text{Waves, Turbulence}\end{array}\right. \longleftrightarrow$ Rhines Scale

$$\partial_t \nabla^2\phi + \underline{v}\cdot\underline{\nabla}\nabla^2\phi - \nu\nabla^2\nabla^2\phi = -\beta\tilde{v}_y + \tilde{f} \quad \tilde{f} \to \text{NL}$$

$\Rightarrow$

$$\partial_t \phi_{\underline{k}} + \underbrace{i\omega_k}_{\text{wave frequency}} \phi_{\underline{k}} + \sum_{\underline{k}'} \underbrace{C_{\underline{k},\underline{k}'}}_{cc} \phi_{-\underline{k}'}\phi_{\underline{k}+\underline{k}'}$$

$$+ \underbrace{\nu_k}_{\text{dissp.}} \phi_k = \underbrace{f_{\underline{k}}}_{\text{forcing}}$$



When is wave important?

$\Rightarrow$ Compare wave frequency vs NL.

A more general way to say this:

$$\langle \phi(t')\phi(t)\rangle_{\underline{k}} = |\phi_{\underline{k}}|^2 \underbrace{e^{-i\omega_k(t-t')}}_{\text{wave frequency}} \; \underbrace{e^{-(t-t')/\tau_{c\underline{k}}}}_{\text{eddy scrambling rate}}$$

Crossing: $\omega_{\underline{k}}$ vs $1/\tau_{c\underline{k}}$

Crudely:

$$1/\tau_{c\underline{k}} \sim k\tilde{v}$$

$$\omega_{\underline{k}} \sim -\frac{k_x \beta}{k^2}$$

$$\Rightarrow \quad k\tilde{v} \sim \frac{k_x \beta}{k^2}$$

$$\therefore \quad k^2 \sim \beta/\tilde{v}$$

$$l \sim (\tilde{v}/\beta)^{1/2} \longrightarrow \text{Rhines scale}$$

$$l_R = (\tilde{v}/\beta)^{1/2}$$

β-plane Spectral Eqn.

Wave turbulence

$$\partial_t \phi_k + i\omega_k \phi_k + A_k \phi_k + \sum_{k'} C_{k,k'} \phi_{k+k'} \phi_{k'}$$

wave

c.c

derive

$$= 0$$

⇒ wave + eddy character ⇔ each k

$$\Rightarrow \langle \phi_k(t)\phi(t') \rangle_k = |\phi_k|^2 e^{-i\omega_k (t-t')} e^{-(t-t')/\tau_{kc}}$$

$t > t'$

wave new

decay

ω vs $1/\tau$

Can derive eqn. for $|\phi_k|^2$, → Ex. (Final)

ala ~~...~~ eqn. for Burgers. $|N_k|^2$

- $\nu_k < 0$, $k \to 0$.

Zakharov

but θ changes!

{ negative viscosity, inverse cascade

[dispersive waves

- $$\theta_{k,p,q} = \frac{(1/\tau_k + 1/\tau_p + 1/\tau_q)}{(\omega_k - \omega_p - \omega_q)^2 + (1/\tau_k + 1/\tau_p + 1/\tau_q)^2}$$

→ $\delta(\omega_k - \omega_p - \omega_q)$ → W.T.T. (weak)

wave resonance, a la' Fermi G.R.

→ $1/\tau$ → S.T. (strong) → cla. hydro.

so $\quad l \ll l_R \Rightarrow \omega_k \tau_{c_k} < 1$

$\Rightarrow$ eddy character dominates

$l \gg l_R \Rightarrow \omega_k \tau_{c_k} > 1$

$\Rightarrow$ wave character dominates

N.B. $\quad R_c \to \infty \Rightarrow \beta \to 0$

all scales eddy character dominates

→ Observe:

$$1/T_k \sim k\tilde{v}_k \rightarrow \left\{ \begin{array}{l} \text{inverse cascade} \\ \text{inertial range} \end{array} \right.$$

$$\sim v(\ell)/\ell$$

$$\sim \epsilon^{1/3}\ell^{-2/3} \sim k^{2/3}\epsilon^{1/3}$$

$$\omega_k \sim -\frac{\beta k_x}{k^2}$$

⇒ wave character will 'overtake' eddy character at low k!

That k → Rhines scale. → why? → changes transfer physics.

— crudely,

$$k\tilde{v} \sim k\beta/k^2$$

$$k_R \sim (\beta/\tilde{v})^{1/2} \rightarrow \boxed{\ell_R \sim (\tilde{v}/\beta)^{1/2}}$$

scale at which 'wave character' takes over → Rhines scale

→ slightly less crude

$$\beta k_x / k^2 \simeq \frac{\epsilon^{1/3}}{\ell^{2/3}}$$

{ waves fed by inverse cascade range

$$\beta \ell \simeq \epsilon^{1/3} / \ell^{2/3}$$

$$\boxed{\ell_R \sim \epsilon^{1/5} / \beta^{3/5}}$$

vs

$$\ell^{5/3} \simeq \left(\epsilon^{1/3} / \beta\right)$$

→ Vallis Maltrud

→ really anisotropic

ℓ_R vs rhist ℓ_R

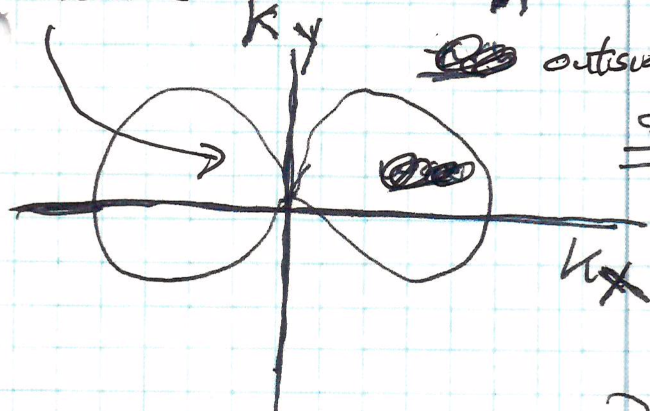
natural emerges as scale

$$\epsilon^{1/3} k^{2/3} = \frac{\beta k_x}{k^2}$$

{ wave-turbulence boundary is anisotropic

inside - wave

outside - eddy



wave → turb bndry

→ what of waves??

$$\delta(\omega_p + \omega_q - \omega_k)\ ?$$

cf. Kartashova

$$\omega = -\beta k_x / k^2$$

⇒ strong dispersion → difficult resonance condition to satisfy.

→ (Lanczos - Hasselmann Matching) — ~~anisotropy~~ test — symmetry

→ enter, then 2 wave + 1 ZF
resonance. ($k_x = 0$)

⇒ triads scattering energy are
2 wave + 1 ZF.

⇒ energy naturally coupled to ZF.

⇒ Rhines scale emerges as
ZF scale.

Enter the Zonal Flow

Fluids in Flatland

→ Rhines Mechanism

→ Rossby Turbulence Inverse Cascade

→ PV Mixing, Taylor Identity

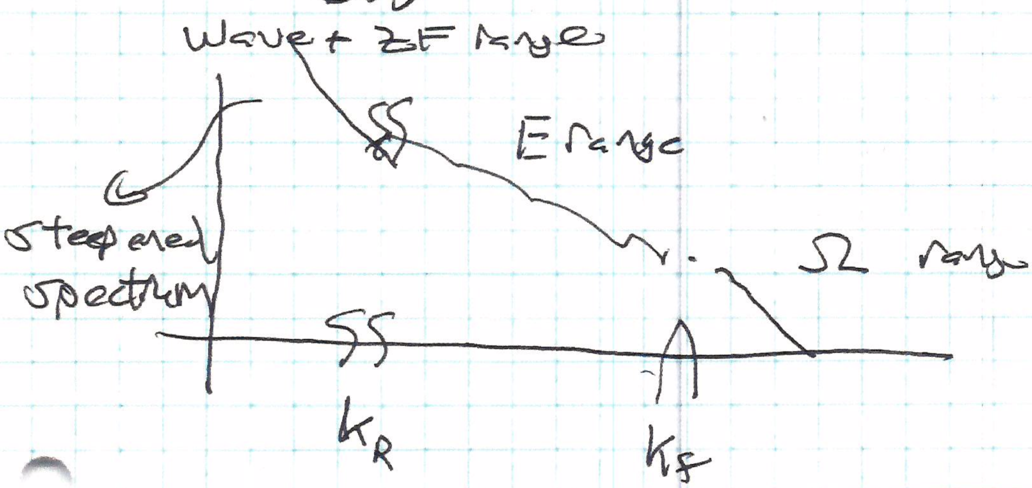
→ Pseudomomentum and Non-Acceleration Thms.

→ QG, ~~Eliassen-Palm~~

→ Jets/Phenomena.

→ Recall:

↳ $l_R \sim (\tilde{V}/\beta)^{1/2}$ → Rhines scale



For $k < k_R$;

$$\underbrace{\omega_k + \omega_{k'}}_{\text{pass energy}} + \Omega = 0$$

$$k + k' + z\hat{y} = 0$$

anisotropy critical → need β effect.

ZF.

Transfer process :

eddy straining → - wave + ZF → wave

- wave + wave → ZF

scattering

∴

- expect ZF to appear on Rhines scale i.e. maximal coupling

- however, while Rhines mechanism explains appearance of ZF in β plane forced at small scale, does not address ZF structure, etc i.e. - direction - strength (damping, μ)

- often said that ZF "produced from inverse cascade", but