Physics 4E — Quiz 4 Solutions — Spring 2020

Problem 1

$$\psi(x) = C(1-x^2) \quad , \quad |x| \le 1$$

(a) Normalization:

$$1 = \int_{-1}^{1} dx\, |\psi(x)|^2 = C^2 \int_{-1}^{1} dx\,(1 - 2x^2 + x^4) = 2C^2\left(1 - \frac{2}{3} + \frac{1}{5}\right) =$$

$$= 2C^2\, \frac{15-10+3}{15} = \frac{16}{15} C^2 = 1 \Rightarrow \boxed{C = \sqrt{\frac{15}{16}}} \boxed{= 0.968}$$

(b) $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$; by symmetry, $\langle x\rangle = 0$

$$\langle x^2\rangle = \int_{-1}^{1} dx\, x^2 |\psi(x)|^2 = C^2 \int_{-1}^{1} dx\,(x^2 - 2x^4 + x^6) = 2C^2\left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right) =$$

$$= \cancel{2} \cdot \frac{\cancel{15}}{\cancel{16}_{\,8}} \cdot \frac{35-42+15}{\cancel{3}\cdot\cancel{5}\cdot 7} = \frac{8}{8\cdot 7} = \frac{1}{7} \Rightarrow \boxed{\Delta x = \frac{1}{\sqrt{7}} = 0.378\,\text{Å}}$$

(c) $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$; by symmetry, $\langle p\rangle = 0$

$$p = \frac{\hbar}{i}\frac{d}{dx}, \quad p^2 = -\hbar^2 \frac{d^2}{dx^2}. \quad \langle p^2\rangle = \int_{-1}^{1} dx\, \psi(x)^* p^2 \psi(x)$$

$$p^2\psi(x) = -\hbar^2 \frac{d^2}{dx^2} C(1-x^2) = +\hbar^2 \cdot 2C \Rightarrow$$

$$\langle p^2\rangle = 2C\hbar^2 \int_{-1}^{1} dx\, C(1-x^2) = 2C^2\hbar^2 \cdot 2\left(1 - \frac{1}{3}\right) = \cancel{4}^{\,2} \cdot \frac{\cancel{15}^{\,5}}{\cancel{16}} \cdot \hbar^2 \cdot \frac{\cancel{2}}{\cancel{3}} = \frac{5}{2}\hbar^2$$

$$\Rightarrow \boxed{\langle p^2\rangle = \frac{5}{2}\hbar^2} \Rightarrow \boxed{\Delta p = \sqrt{\frac{5}{2}}\,\hbar = \frac{1.58\,\hbar}{\text{Å}}} \text{(c)}$$

$$\Delta x \Delta p = \frac{1}{\sqrt{7}}\sqrt{\frac{5}{2}}\,\hbar = \sqrt{\frac{5}{14}}\,\hbar = \boxed{0.598\,\hbar > \hbar/2} \text{(d)}$$

(e) $K = \dfrac{\langle p^2\rangle}{2m_e} = \dfrac{5}{2}\dfrac{\hbar^2}{2m_e \text{Å}^2} = \dfrac{5}{4} \times 7.62\,\text{eV} = \boxed{9.53\,\text{eV}}$ (e)

Problem 2

The expectation value of the energy for a harmonic oscillator is

$$E = \langle K \rangle + \langle U \rangle = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2} m\omega^2 \langle x^2 \rangle$$

By symmetry, $\langle p \rangle = 0$ and $\langle x \rangle = 0$, so

$$\langle p^2 \rangle = (\Delta p)^2 \text{ and } \langle x^2 \rangle = (\Delta x)^2$$

(a) For the ground state, the energy is $E_0 = \frac{\hbar\omega}{2}$. Therefore,

$\langle K \rangle = \langle U \rangle = \frac{\hbar\omega}{4}$. Therefore,

$$\frac{(\Delta p)^2}{2m} = \frac{\hbar\omega}{4} \Rightarrow (\Delta p)^2 = \frac{\hbar m\omega}{2} \; ; \; \frac{1}{2} m\omega^2 (\Delta x)^2 = \frac{\hbar\omega}{4} \Rightarrow (\Delta x)^2 = \frac{\hbar}{2m\omega}$$

$$\Rightarrow (\Delta p)^2 (\Delta x)^2 = \frac{\hbar m\omega}{2} \cdot \frac{\hbar}{2m\omega} = \frac{\hbar^2}{4} \Rightarrow \boxed{\Delta x \Delta p = \frac{\hbar}{2}}$$

(b) For the first excited state, the energy is $E_1 = \frac{3}{2}\hbar\omega$. Therefore,

$\langle K \rangle = \langle U \rangle = \frac{3}{4}\hbar\omega \Rightarrow$

$$\frac{(\Delta p)^2}{2m} = \frac{3\hbar\omega}{4} \Rightarrow (\Delta p)^2 = \frac{3\hbar m\omega}{2} \; ; \; \frac{1}{2} m\omega^2 (\Delta x)^2 = \frac{3\hbar\omega}{4} \Rightarrow (\Delta x)^2 = \frac{3\hbar}{2m\omega}$$

$$\Rightarrow (\Delta x)^2 (\Delta p)^2 = \frac{9\hbar^2}{4} \Rightarrow \boxed{\Delta x \Delta p = \frac{3}{2}\hbar = 1.5\hbar}$$

(c)

$$E = \frac{1}{2} m\omega^2 A^2 = \frac{3}{2}\hbar\omega \Rightarrow \boxed{m\omega = \frac{3\hbar}{A^2}} \Rightarrow \text{from (b)}$$

$$(\Delta x)^2 = \frac{3\hbar}{2m\omega} = \frac{3\hbar \cdot A^2}{2 \cdot 3\hbar} = \frac{A^2}{2} \Rightarrow \boxed{\Delta x = \frac{A}{\sqrt{2}} = 2.12\,\mathring{A}} \text{ since } A = 3\,\mathring{A}$$

(d)

$$E = \frac{3}{2}\hbar\omega = \frac{3}{2}\frac{\hbar \cdot 3\hbar}{m_e A^2} = \frac{9}{2}\frac{\hbar^2}{m_e (3\,\mathring{A})^2} = \frac{1}{2} \times 7.62\,\text{eV}$$

$$\Rightarrow \boxed{E = 3.81\,\text{eV}}$$

Problem 3

(a) Reflection coefficient is

$$R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}$$

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}} = \sqrt{\frac{2 \times 3}{7.62}}\ \mathring{A}^{-1} = 0.887\ \mathring{A}^{-1}$$

$$k_2 = \sqrt{\frac{2m(E-U_1)}{\hbar^2}} = \sqrt{\frac{2 \cdot 1}{7.62}}\ \mathring{A}^{-1} = 0.512\ \mathring{A}^{-1}$$

$$R = \left(\frac{0.887 - 0.512}{0.887 + 0.512}\right)^2 = \boxed{0.072}$$

So for every 10,000 electrons/s $\boxed{\text{719 get reflected /s}}$

(b) The transmission probability is given by

$$T = e^{-2\sqrt{\frac{2m}{\hbar^2}(U_2 - E)}\,(b-a)} = e^{-2\sqrt{\frac{2}{7.62}\cdot(7-3)}\cdot 1.5} = e^{-3.074}$$

$$\Rightarrow \boxed{T = e^{-3.074} = 0.0462}$$

So 10,000 − 719 = 9281 are incident on the second barrier,

9281 × 0.0462 ~~get~~ = 429 get transmitted

So for 10,000 incoming electrons/s, 429 electrons/s are found at $x > b$

(c) The wavefunction has different forms in the different regions.

For example, in region I:

$$\psi(x) = Ae^{ik_1x} + Be^{-ik_1x}$$

$$p\,\psi(x) = \frac{\hbar}{i}\frac{d}{dx}\psi(x) = ik_1Ae^{ik_1x} - ik_1Be^{-ik_1x} \neq \lambda\psi(x)$$

(c) so clearly, $\psi(x)$ is not an eigenfunction of the momentum operator

⇒ has no eigenvalue

(d) $\psi(x)$ is an eigenfunction of the Hamiltonian, with eigenvalue $E = 3\,eV$