HW7 Additional Problems

#1 a). Recall the even solutions derived in class and in the discussion section notes on "Quantum Wells":

$$\psi_k(x) \propto \begin{cases} \cos kx & |x| \leq \frac{L}{2} \\ e^{-\delta x} & x \geq \frac{L}{2} \\ e^{\delta x} & x \leq -\frac{L}{2} \end{cases} \qquad \text{w/} \quad k = \sqrt{\frac{2mE}{\hbar^2}} \qquad \delta^{-1} = \sqrt{\frac{2m(U-E)}{\hbar^2}}$$

With the quantization condition on k (or E)

$$\tan\frac{kL}{2} = \frac{1}{\delta k} = \frac{1}{kL}\sqrt{u - (kL)^2} \qquad u = \frac{U}{\left(\frac{\hbar^2}{2mL^2}\right)}$$

This can be solved numerically by first defining the dimensionless variable $X = kL$ and solving

$$\tan\frac{x}{2} = \frac{\sqrt{u - x^2}}{x}$$

(See mathematica notebook in discussion section notes for how to do this)

$U = 100\text{ eV} \quad L = 0.2\text{ nm} \qquad \frac{\hbar^2}{2m} = 3.8\times10^{-2}\text{ eV nm}^2$

Hence $u = \dfrac{4\text{ eV nm}^2}{3.8\cdot10^{-2}\text{ eV nm}^2} = 1.05\times10^2$

Solving this for $x = 2.62 \Rightarrow \sqrt{\dfrac{E_0}{\left(\frac{\hbar^2}{2mL^2}\right)}} = 2.62$

which gives $E_0 = (2.62)^2 \dfrac{3.8\times10^{-2}\text{ eV nm}^2}{0.04\text{ nm}^2}$

$$\boxed{E_0 = 6.52\text{ eV}}$$ Which agrees well w/ ex. 6.8

b.) The limit $E \ll U$ is the same as $u \gg x^2$
Therefore the solution for the even and odd case reduce to

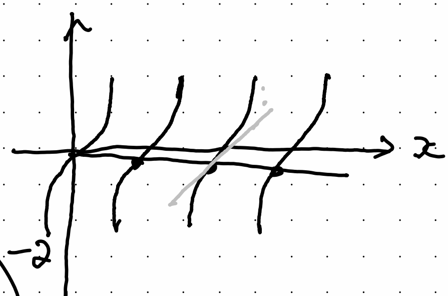
$$\tan\frac{x}{2} = \frac{\sqrt{u-x^2}}{x} \sim \frac{\sqrt{u}}{x} + O\left(\frac{x}{\sqrt{u}}\right) \qquad \text{even}$$

$$\tan\frac{x}{2} = \frac{-x}{\sqrt{u-x^2}} \sim \frac{-x}{\sqrt{u}} + O\left(\left(\frac{x}{\sqrt{u}}\right)^3\right) \qquad \text{odd}$$

Consider the odd solutions in this limit, as they are much easier to deal w/ to describe the scaling of the levels.

$$\frac{1}{2}(x - 2\pi n) \sim \frac{-x}{\sqrt{u}}$$

$$\left(1 + \frac{2}{\sqrt{u}}\right)x = 2\pi n$$



$$\Rightarrow E_n^{odd} \sim \frac{\hbar^2\pi^2 n^2}{2mL^2}\left(\frac{1}{\sqrt{u}} + \frac{1}{2}\right)^{-2}$$

$$\frac{\delta}{L} = \frac{1}{\sqrt{(U-E)/\left(\frac{\hbar^2}{2mL^2}\right)}} \sim \frac{1}{\sqrt{u}} \quad \text{in the limit } E \ll U$$

$$E_n^{odd} \sim \frac{2^2\hbar^2\pi^2 n^2}{2m(L+2\delta)^2}$$

Since these are how the odd solutions scale, the general levels will scale as

$$E_n \sim \frac{\hbar^2\pi^2 n^2}{2m(L+2\delta)^2}$$

Plugging in $n = 2m+1$ we recover the scaling of the odd solutions derived above,

$\boxed{\#2}$ i).

We may determine B from continuity conditions at $x=0$:

$$1+B = 1.5 \implies B = 0.5$$

ii). $R = |B|^2 = 0.25 \quad T = 1-R = 0.75$

Thus for every 1000 electrons 750 are transmitted.

iii). From continuity in ψ' we get

$$ik_1 - ik_1 B = 1.5\, ik_2$$

We have

$$k_1 = \sqrt{\frac{2m}{\hbar^2} E} \qquad k_2 = \sqrt{\frac{2m}{\hbar^2}(E-V)}$$

barrier height 8eV

$$\implies k_2/k_1 = \frac{1}{3} = \sqrt{1 - \frac{V}{E}}$$

incident energy

$$\implies \frac{V}{E} = \frac{8}{9} \implies E = 9 \text{ eV}$$

Thus the kinetic energy of the incident and reflected waves is

9 eV

whereas the kinetic energy of the transmitted is

1 eV

$\boxed{\#3}$

Using the formula $T(E) \sim \exp\left(\frac{-2}{\hbar}\sqrt{2m}\int\sqrt{U(x)-E}\,dx\right)\ldots$

- In the left case $T_L(E) = \exp\left(\frac{-2}{\hbar}\sqrt{2m}\sqrt{\frac{2U_0}{3}}a\right)$

  " middle " $T_M(E) = \exp\left(\frac{-2}{\hbar}\sqrt{2m}\sqrt{\frac{U_0}{3}}a\right)$

Hence $T_M(E) = (T_L(E))^{\frac{1}{\sqrt{2}}} = \left(\frac{10^2}{10^4}\right)^{\frac{1}{\sqrt{2}}} = 0.039$

so for particles w/ energy incident $2U_0/3$

390 of the 10000 tunnel through.

- $T_R(E) = (T_M(E))^{b/a} = 0.01$

  $\Rightarrow b/a = \log(10^{-2})/\log((10^{-2})^{\frac{1}{\sqrt{2}}}) = \sqrt{2}$

  $\Rightarrow b = \sqrt{2}\,a$

- If instead the particles have mass M then their transmission amplitude is

$$\left(10^{-2/\sqrt{2}}\right)^{\sqrt{\frac{M}{m}}} = 10^{-2} \Rightarrow M = 2m$$