

Coarsening - Continuum Models

2.) Reuall coarsening / basic aspects, phenomenology

Consider:

- spin system
- Ising Model like (local spin)

$$H = - \sum_{\langle i, j \rangle} J_{ij} S_i S_j$$

{ lower energy for alignment

- Temperature T
- crit temp for order T_c

$T > T_c \rightarrow$ disordered phase

$T < T_c \rightarrow$ ordered

"Quenching" - system instantaneously cooled to $T < T_c$

Why: - $1/\tau_{cool} = \frac{1}{T} \frac{dT}{dt} \rightarrow$ cooling rate externally controlled parameter

- $\tau_R =$ system response rate

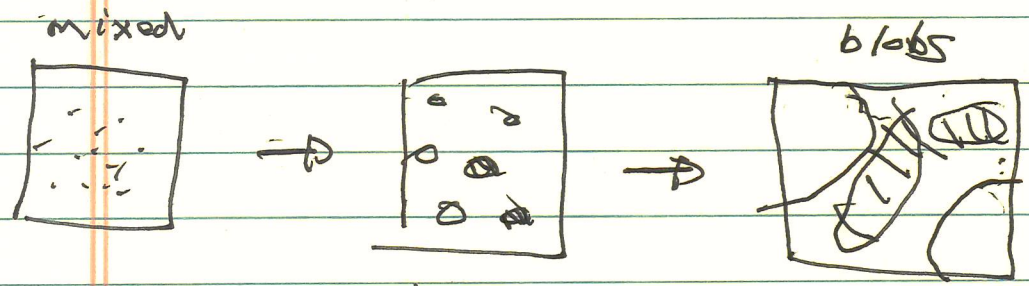
i.e. time rate for system to evolve

To isolate intrinsic from external physics,

$$1/\tau_{cool} \gg 1/\tau_R$$

\rightarrow quenched

- system coarsens \rightarrow forms domains of like spec (lower energy)



see Kravitsky, pg 237

Coarsening: 'unmixing' \rightarrow Phase Separation \rightarrow blobby state

N.B. := Coarsening involves up-gradient transport

- "Negative diffusion" phenomenon

small fluctuations \rightarrow aggregate blobs.

- single, evolving length scale $L(t)$, i.e. $x \rightarrow x/L(t)$ yields self-similarity

$\rightarrow L(t) \sim t^Z$ scaling

- Z is universal (indep details)

\Rightarrow phase transition

- Models of Coarsening:

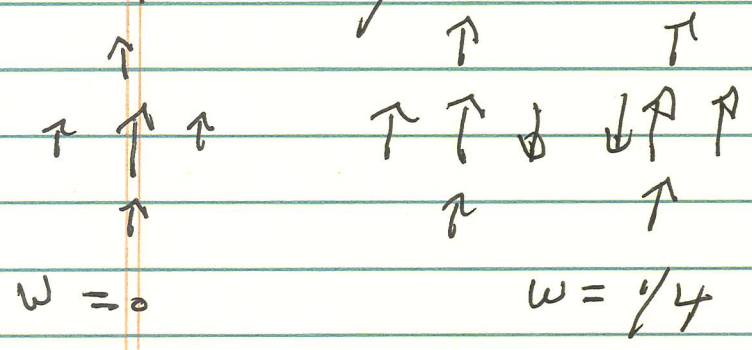
→ Microscopic (log resolution)

~ Voter Model:

$$W_x(s) = \frac{1}{2} \left[1 - \frac{s(x)}{2} \sum_{y \in \langle x \rangle} s(y) \right]$$

spin flip probability

aligned → less prob
opposite → more prob.



etc.

and can iterate, aka CA.

~ Ising / Glauber

etc.

→ Microscopic ⇒ Coarse grained
i.e. large scale

Critical issue: Are dynamics conservative or not?

Non-Conservative \rightarrow TKG-2

Conservative: $\frac{d}{dt} O.P. + \underline{D \cdot J} = 0$

O.P. = $\frac{1}{pd} \sum \tau = m(x, t)$ as before

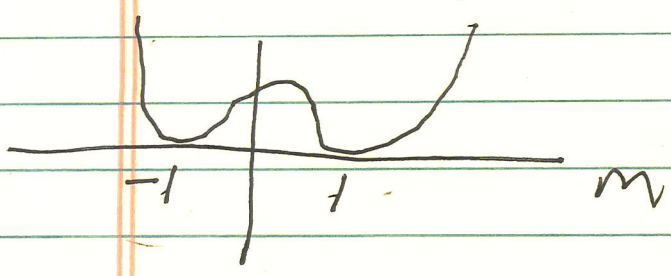
i.e. sum of springs in d-dimensional block

$L > l > a$

Free Energy: 2 - minima

$F[m] = \int dx \left[\frac{(dm)^2}{2} + V(m) \right] = \int dx F$

$V(m) = \frac{1}{2} (1 - m^2)^2$



Functional Derivative

$$F = \int dx \left[\frac{(\dot{m})^2}{2} + V(m(x)) \right]$$

$$\delta F = \int dx \left[2\dot{m} \frac{\delta \dot{m}}{2} + \frac{\partial V}{\partial m} \delta m \right]$$

$$= \int dx \left[-\ddot{m} \right] + \int dx \frac{\partial V}{\partial m} \delta m + \cancel{\delta F}$$

$$= \int dx \left[-\ddot{m} + V'(m) \right] \delta m$$

$$\frac{\delta F}{\delta m} = -\ddot{m} + V'(m)$$

N.B. No noise here.

- For evolution,

$$\frac{\partial m}{\partial t} = -\frac{\delta F}{\delta m}$$

i.e. system evolves according to thermodynamic force

⇒

$$\frac{\partial m}{\partial t} = D^2 m + 2m(1-m^2)$$

TDGL

TDGL (1D - Fisher)

- Dissipative,

$$\frac{dF}{dt} = \int dx \frac{\delta F}{\delta m} \frac{\partial m}{\partial t}$$

$$= - \int \left(\frac{\delta F}{\delta m} \right)^2 dx \leq 0$$

→ 1D $\frac{\partial n}{\partial t} = \gamma n - \alpha n^2 + D \frac{\partial^2 n}{\partial x^2}$

Logistics + Diffusion.

- But, Mass separation is conservative

i.e.

$$\frac{\partial}{\partial t} \int dx m = 0, \text{ up to boundary}$$

$$\frac{\partial m}{\partial t} + \nabla \cdot \underline{J} = 0$$

$$\underline{J} = - \nabla \frac{\delta F}{\delta m}$$

gradient driven flow

\underline{J} produced by gradient in

thermodynamic force ($\underline{J} \sim -\nabla \mu$)
 $\left(\mu = \frac{\delta F}{\delta m} \right)$

then

$$\frac{\partial m}{\partial t} - \nabla^2 \frac{\delta F}{\delta m} = 0$$

$$\frac{\partial m}{\partial t} = + \nabla^2 \frac{\delta F}{\delta m}$$

$$\frac{\delta F}{\delta m} = -\nabla^2 m + V(m)$$

$$\begin{aligned} \frac{\partial m}{\partial t} &= -\partial^2 \left(\partial^2 m - \dot{V}(m) \right) \\ &= -\partial^2 \left(\partial^2 m - (1-m^2)m \right) \\ &= -(\partial^2)^2 m - \partial^2 m + \partial^2 (m m^2) \end{aligned}$$

\uparrow + hyper diffy \uparrow negative diffy

$$\frac{\partial m}{\partial t} = -\partial^2 \left(m + \partial^2 m - m m^2 \right)$$

$$\frac{\partial m}{\partial t} = -\partial^2 \left(+\partial^2 m + m(1-m^2) \right)$$

Cahn-Hilliard Equation

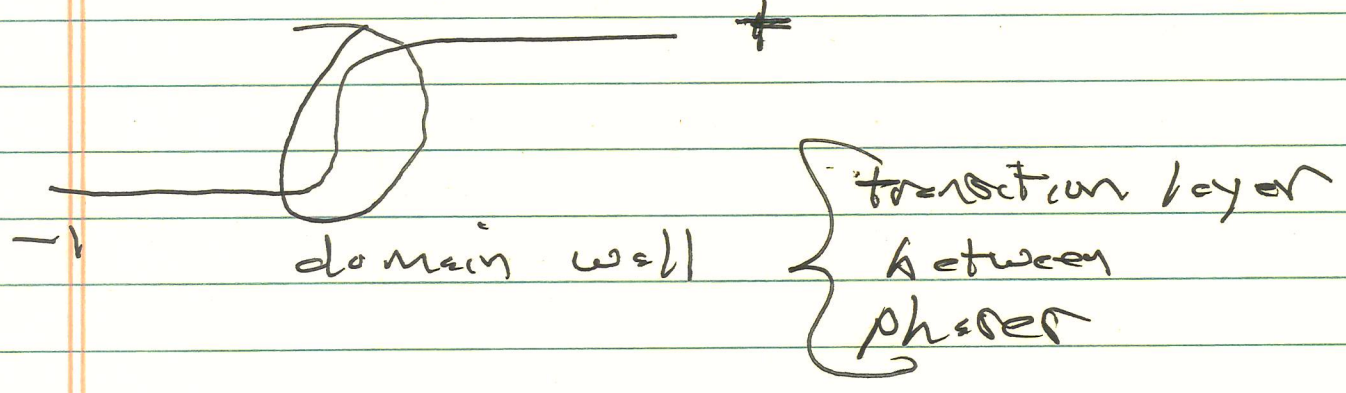
→ Conservative

→ Studies of Solutions for

}	Non-conservative	Coarsening
	Conservative	(T → 0)

c.e. What happens?

~ Focus on domains domain walls



~ straight domain wall

$$\frac{\partial m}{\partial t} = - \frac{\delta F}{\delta m} \quad V(m) = \frac{(k - m^2)^2}{2}$$

$$= D^2 m - \nabla^2 V(m)$$

1D, static

(of course

$$\frac{d^2 m}{dx^2} = V'(m)$$

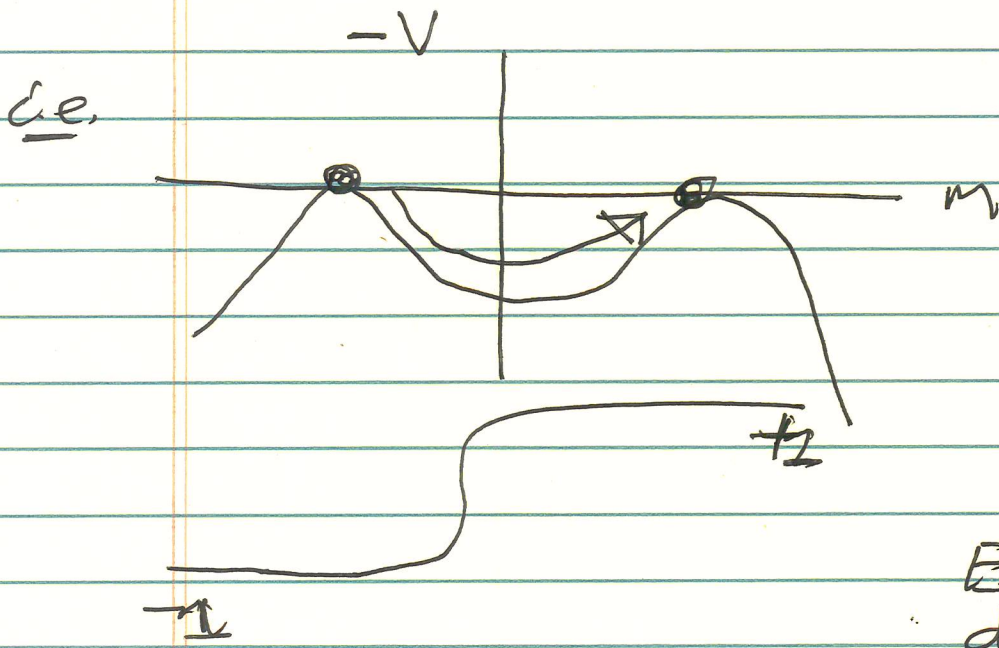
$m = \pm 1$ solutions work, too).

Now: $x \rightarrow$ time t

mechanical analogy

domain. $V \rightarrow$ potential $-V$
mech.

$$\ddot{x} = \mp \frac{d}{dx} \left(\frac{\partial m}{\partial x} \right)^2 - V(x)$$



$$E = 0 \text{ b.c.}$$

$$\frac{dm}{dx} = (2V)^{1/2}$$

$$\underline{\text{So}} \quad x = \int_{m(x)}^{m(x)} \frac{dx}{\sqrt{2V(x)}}$$

$$\Rightarrow \boxed{m(x) = \tanh x}$$

kink solution
 \rightarrow domain wall

in φ^4 : kink

$$\mathcal{L} = \frac{(\partial_t \varphi)^2}{2} - \frac{(\partial_x \varphi)^2}{2} - V(\varphi)$$

$$\underline{\text{So}} \quad \partial_t \varphi = 0$$

$$-\partial_x^2 \varphi + V'(\varphi) = 0 \quad \checkmark$$

$$f = \frac{1}{2} \left(\frac{dm}{dx} \right)^2 + V(m) = 2V(m) = \frac{1}{\cosh^4 x}$$

and

Total free energy / length \equiv surface tension ∇

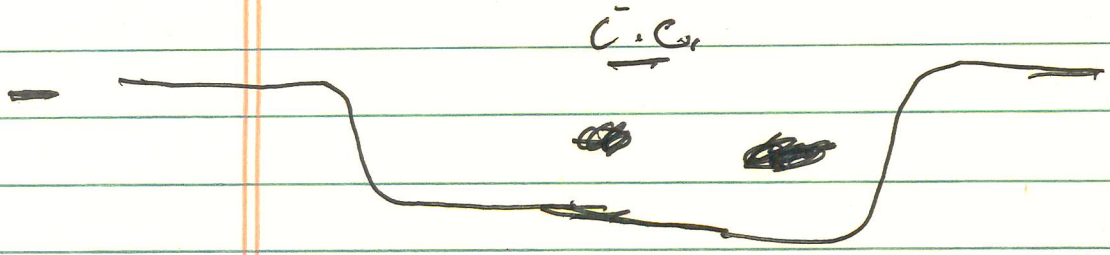
$$\begin{aligned} \nabla &= \int_{-\infty}^{+\infty} 2V(m) dx = \int_{-\infty}^{+\infty} 2V(m) \frac{dx}{dm} dm \\ &= \int_{-1}^{+1} \sqrt{2V(m)} dm = 4/3 \end{aligned}$$

0

action

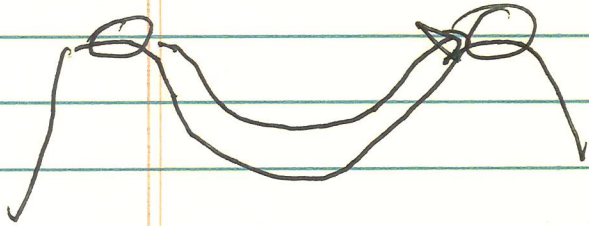
$$S = \int_{-\infty}^{+\infty} dx \left[\frac{1}{2} \left(\frac{dm}{dx} \right)^2 + V(x) \right]$$

Some insights \rightarrow exp/acting mechanical analogy



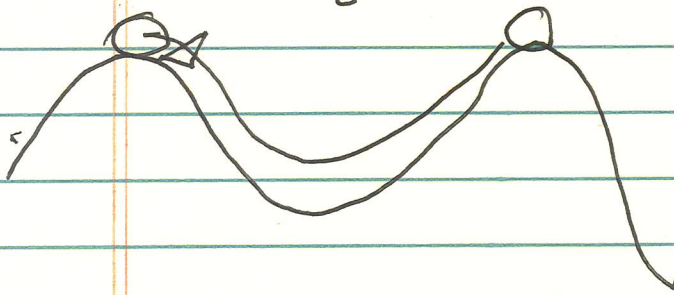
well - well pair \rightarrow stable !!

No!

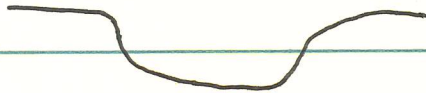


and return?

→ no



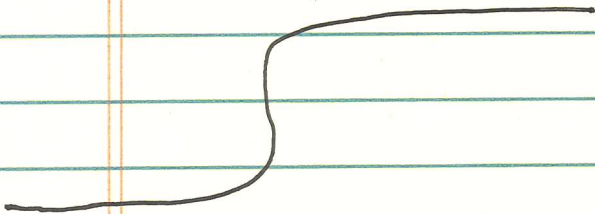
so



is unstable.

→ can characterize types of wells

→ Stability of Domain Wall?



$$M = M_0(x) + \tilde{m}(x,t)$$

→ P.T. ?

$$\frac{d}{dt} \frac{\partial m}{\partial t} = - \frac{\partial F}{\partial m}$$

$$= \frac{\partial^2 m}{\partial x^2} - V'(m)$$

$$m = m_0 + \tilde{m}$$

$$\frac{\partial m_0}{\partial t} + \frac{\partial \tilde{m}}{\partial t} = \frac{\partial^2 \tilde{m}}{\partial x^2} + \frac{\partial^2 m_0}{\partial x^2}$$

$$- V'(m_0 + \tilde{m})$$

$$= \frac{\partial^2 \tilde{m}}{\partial x^2} + \frac{\partial^2 m_0}{\partial x^2}$$

$$- V'(m_0) - \tilde{m} V''(m_0)$$

$$\tilde{m} = \sum_n A_n e^{-E_n t} \psi_n(x)$$

$$E_n \psi_n = - \frac{d^2}{dx^2} \psi_n + V''(m_0) \psi_n$$

→ Fluctn Eqn. for stability,

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$$E_n \Psi_n = \left[-\frac{d^2}{dx^2} + (6 \tanh^2 x - 2) \right] \Psi_n$$

2 levels:

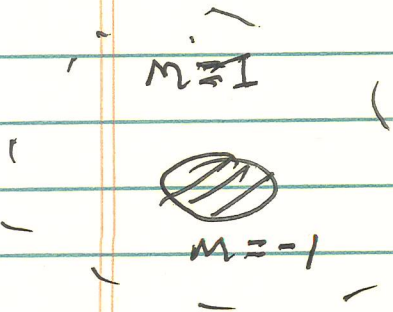
$$E_0 = 0, \quad \Psi_0 = 1/\cosh^2 x$$

(translation mode) $= (\tanh x)' = \frac{d \tanh x}{dx}$

$$E_1 = 3, \quad \Psi_1 = \sinh x / \cosh^2 x$$

$$E \geq 4 \rightarrow \text{continuum} \quad 6 \tanh^2 x - 2 \rightarrow \underline{\underline{4}}$$

Higher Dimensions



droplet of $m = -1$
phase in sea
of $m = +1$ phase?

~ In terms of being minority overwhelmed by opposites

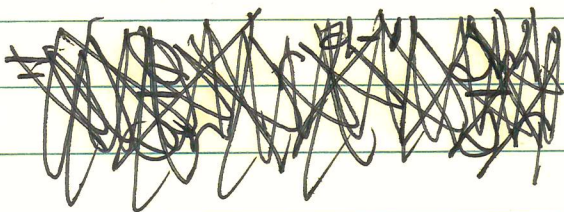
⇒ small droplet shrinks!

Now

$$\frac{\partial m}{\partial t} = \nabla^2 m - V'(m)$$

d-dimensional sphere:

$$\frac{\partial m}{\partial t} = \frac{\partial^2 m}{\partial r^2} + \frac{(d-1)}{r} \frac{\partial m}{\partial r} - V'(m)$$



Now

$R \equiv$ radius droplet

$R \gg \lambda$ width
of interface



$$m(r, t) = F(r - R(t))$$

$$\Rightarrow F'' + \frac{(d-1)}{r} F' + \frac{dR}{dt} F' - V'(F) = 0$$

$$F' \left[F'' + \frac{(d-1)}{r} F' + \frac{dR}{dt} F' - V'(F') \right] = 0$$

$$\int_0^{\infty} dr = \int_{-R}^{\infty} dx \quad R \gg \lambda$$

$$\rightarrow \int_{-\infty}^{\infty}$$

18

$$\int_{-\infty}^{\infty} \left[\frac{(d-1)}{r} F'^2 + \frac{dR}{dt} F'^2 \right] dx$$

$$- \int V(F') F' dx = 0$$

2

$$\int_{-\infty}^{\infty} \left[\frac{(d-1)}{r} F'^2 + \frac{dR}{dt} F'^2 \right] dx$$

$$- \int V(F) dx = 0$$

oo $F=0-N$ for interface

$$\frac{d-1}{R} + \frac{dR}{dt} = 0$$

$F' \neq 0$ only for $r \sim R$, so $r \sim R$

$$R^2 = R_0^2 - 2(d-1)t$$

\downarrow
critical radius

interface
is
key

so droplet disappears for:

$$t = R_0^2 / 2[d-1]$$

larger live
drops longer
etc.

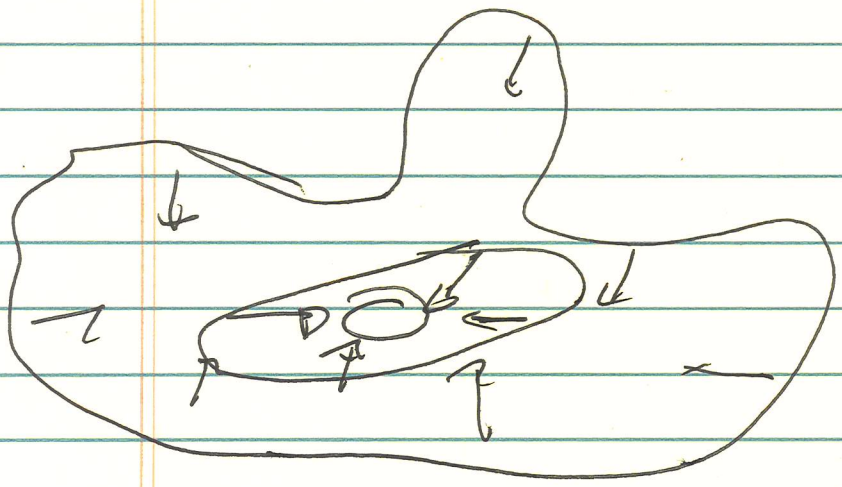
\leadsto Especially useful 2D,

$R \sim$ local radius curvature
local radius curvature

so

$$V \sim -1/R$$

velocity
normal to
surface



Global 2D interface ~~evolving~~ ^{evolving} by ~~curvature-driven~~ ^{curvature-driven} flow necessarily becomes more circular.

If d -dimensions:

- $d-1$ radii curvature R_j

→ mean curvature

$$v = -(d-1)H$$

$$H = \frac{1}{d-1} \sum \left(\frac{1}{R_j} \right)$$

Allen - Cahn Eqn.

Note :

- dynamic local \rightarrow local curvature drive.

not apply conservative

- dynamic indep $V(m)$

\rightarrow all from $\nabla^2 \tau$ (thin) interface

- obviously fails in 1D.

- classic case of curvature driven flow.

Two equivalent approaches to Fermi Liquid theory

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(Dated: 4 December 2019)

A degenerate, interacting Fermi gas is presented in Landau's quasiparticle picture. The simplest case of Fermi liquid theory for Helium-3 is reviewed. Next, the more general problem is treated by the Green's function formalism. The connection between the two presentations of the theory is identified.

Introduction.—The understanding of transport in metals began with a seminal work by Drude¹, who developed a bare-bones model to obtain an estimate for the conductivity. Only a few years later, crucial advances were made in the new quantum theory, which allowed Sommerfeld² to improve the model. In essence, he modeled the electrons in a metal as a non-interacting Fermi gas. However, physical quantities derived from this model were remarkably consistent with experimental observation. In a real metal, the Coulomb interactions in a degenerate electron gas are by no means weak when compared to the kinetic energies. Understandably, the accuracy of Sommerfeld's model puzzled theorists for decades.

The crucial insight that led to the resolution of this conundrum came from the study of superfluid helium. In 1956, Landau³ proposed that the main effect of the interactions in a degenerate Fermi gas can be understood by treating the system of strongly interacting electrons as a system of noninteracting "quasiparticles", resulting in what is now called Fermi liquid theory (FLT). In the following decade, the methods of field theory were used to justify Landau's proposition. Today, FLT is known as a widely successful model used to model the properties of a broad range of fermionic systems.

Preliminaries.—The purpose of this section will be to introduce the quantities of interest in FLT and to provide the simplest examples of its application. The material in this section was drawn from the original papers^{3,4}, several well-known textbooks on the subject⁵⁻⁸, as well as Prof. Congjun Wu's excellent notes⁹.

The assumption of there being a one-to-one correspondence between the electron and the quasiparticle is very powerful, and allows one to retain the concept of the Fermi sphere. This assumption is only valid if there are no "phase transitions" or instabilities as the interaction strength of the electron system is increased. The first consequence and most obvious consequence of this mapping is that the number of particles remains the same. The total spin and the momentum carried by the system also remains unchanged¹⁰. To derive several simple relations, consider the response of the system to a fluctuation. The change in energy can be written as a function of the quasiparticle density n and its energy $\varepsilon(\mathbf{p})$:

$$\delta E = \int \varepsilon(\mathbf{p}) \delta n d^3 p \quad (1)$$

In writing the energy as a function of momentum only, it has been assumed that there is no external magnetic

field or spin-orbit coupling effects. The correction to the energy is given by

$$\delta \varepsilon(\mathbf{p}) = \frac{1}{V} \int f_{\sigma\sigma'}(\mathbf{p}, \mathbf{p}') \delta n(\mathbf{p}') d^3 p'. \quad (2)$$

Here, σ, σ' label the spins. Throughout this discussion, spin will be suppressed for the sake of brevity. The function $f_{\sigma\sigma'}(\mathbf{p}, \mathbf{p}')$ contains all of the interactions and plays a central role in the theory. It is dependent on the momenta of the interacting particles and their spins. In a system invariant under independent spin and spatial rotations, the interaction function f can be written as a sum of a symmetric and antisymmetric part:

$$f_{\sigma\sigma'}(\mathbf{p}, \mathbf{p}') = f^s(\mathbf{p}, \mathbf{p}') + \sigma\sigma' f^a(\mathbf{p}, \mathbf{p}'). \quad (3)$$

These are the only combinations allowed without an external field or spin orbit coupling, and can be seen as the sum of a direct (Hartree) and an exchange (Fock) term. The latter term contains a contraction of the familiar Pauli matrices, labeled by σ, σ' . The more general case can also be considered; generalizations to include a magnetic field or spin-orbit interactions can be found in the literature^{11,12}.

The existence of a Fermi sphere for quasiparticles allows for a further simplification. Since the only relevant interactions will be taking place on the Fermi surface, one can neglect any collisions involving significant momentum transfer, so the only relevant quantity will be the angle between collisions. This allows one to expand the two contributions to the interaction function in terms of Legendre polynomials:

$$f^s(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') = \sum_{l=0}^{\infty} f_l^s P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}'), \quad F_l^s = N(0) f_l^s \quad (4a)$$

$$f^a(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') = \sum_{l=0}^{\infty} f_l^a P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}'), \quad F_l^a = N(0) f_l^a \quad (4b)$$

The value $N(0)$ is the density of quasiparticle states at the Fermi surface, and the coefficients of the expansion $F_l^{s,a}$ are related to the response of the system to various external perturbations. For instance, the F_0^a coefficient modifies the spin susceptibility of the quasiparticles, and F_0^s renormalizes the compressibility:

$$\chi = \frac{N(0)\chi_0}{N(0) + \chi_0 F_0^a} \quad (5a)$$

$$\beta = \frac{1}{n^2} \frac{N(0)}{1 + F_0^s} \quad (5b)$$

the expectation value of any single-particle operator f :

$$F(t) = \int \psi^\dagger(x) f(x) \psi(x) d^3r. \quad (11)$$

By using Eqn. (10), it can be seen that

$$F(t) = -i \int f(x) G(x, x) d^3r, \quad (12a)$$

$$G(x, x) = \lim_{\mathbf{r}' \rightarrow \mathbf{r}} \lim_{t' \rightarrow t+0} G(x - x'). \quad (12b)$$

This form is reminiscent of the classical correlation function, and indeed, the formalism is nearly the same. Formally, the Green's function is the operator that inverts the Hamiltonian. For a free system in the grand canonical ensemble with chemical potential μ , it is given by

$$G^{(0)}(\mathbf{p}, \omega) = \frac{1}{\omega - (\varepsilon(\mathbf{p}) - \mu) + i\eta \text{sign}(|\mathbf{p}| - p_F)} \quad (13)$$

The η is an infinitesimal that specifies the proper manner in which to avoid the poles in the complex plane, and p_F is the radius of the Fermi sphere. Two closely related quantities, G^R, G^A fix the sign of η to ± 1 , respectively. Of special interest is G^R , which is the function that corresponds to the physical observables. In this form, the poles of G^0 are the energy states of the free Fermi gas; since they are real, the states have an infinite lifetime. It will also be useful to notice that the poles have unit residue.

Naturally, an interacting system must have a more complicated expression for its Green's function. However, the real part and imaginary part of the poles still represent the eigenmodes and their attenuation, respectively. The residue at those poles, denoted Z , is commonly referred to as the renormalization factor, and it can be shown to be a real value between 0 and 1⁸.

As with any perturbation theory, the higher order terms in the theory can be obtained in terms of the quantities obtained from the perturbed state. In this context, the higher order Green's functions are composed of certain combinations of the bare quantity in Eq. (13). However, the algebraic expressions for the higher order terms quickly become unwieldy. The technique of Feynman diagrams allows one to compactly represent these quantities. The solid lines represent the bare Green's functions, and the dashed lines signify the (two-body) interaction. The corrections to the two-body Green's function arising from the Hartree-Fock approximation are shown in Fig 1. The relevant algebraic expressions for these corrections pictured in Fig. 1 (a), (b) are presented below, in that order.

$$G^{(1)}(\mathbf{p}, \omega) = - (G^0(\mathbf{p}, \omega))^2 \int U(0) G^0(\mathbf{p}', \nu) \frac{d^4 p'}{(2\pi)^4}, \quad (14a)$$

$$G^{(1)}(\mathbf{p}, \omega) = (G^0(\mathbf{p}, \omega))^2 \int U(\mathbf{q}) G^0(\mathbf{p} - \mathbf{q}, \omega - \nu) \frac{d^4 q}{(2\pi)^4}. \quad (14b)$$

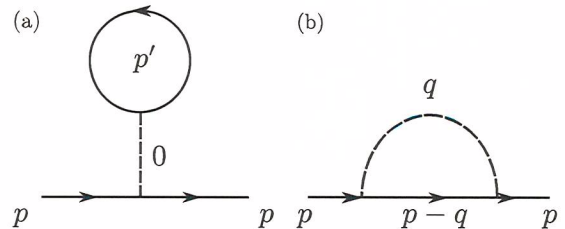


FIG. 1. Examples of Feynman diagrams. (a) The correction to the two-particle Green's function from the direct interaction. The corresponding expression is given by Eqn. (14a). (b) The correction to the two-particle Green's function from the exchange interaction. The corresponding expression is given by Eqn. (14b).

The rules for a general diagram can be found in any good textbook on quantum field theory, such as the references mentioned in the beginning of this section. Finally, Eqns.(14a) and (14b) can be rewritten in the more suggestive form,

$$G(\mathbf{p}, \omega) = G^0(\mathbf{p}, \omega) + G^0(\mathbf{p}, \omega) \Sigma(\mathbf{p}, \omega) G(\mathbf{p}, \omega). \quad (15)$$

The quantity $\Sigma(\mathbf{p}, \omega)$ is defined as the self-energy. Physically, it is the energy due to the inter-particle interactions. This term also appears if we rewrite Eqn.(13) for the full Green's function:

$$G(\mathbf{p}, \omega) = \frac{1}{\omega + \Sigma(\mathbf{p}, \omega) - (\varepsilon(\mathbf{p}) - \mu)}. \quad (16)$$

This self-energy can be extracted experimentally, and so the Green's function for a system can be found, allowing one to obtain the observables.

So far, all averages were taken to be over the ground state of the given system, which is only correct at zero temperature. At finite temperatures, the excited states can no longer be neglected. The generalization of the theory to finite temperatures is not obvious. The technique, pioneered by T. Matsubara, begins by transforming to an imaginary time $t \rightarrow -i\tau$. The imaginary time is constrained to an interval $-1/T < \tau < 1/T$. A few lines of algebra shows that this procedure yields Eqn. (10) with the difference $t \rightarrow -i\tau$, and the average $\langle \dots \rangle$ becomes a thermal trace over all eigenstates. The (anti)symmetry properties of bosons (fermions) under exchange constrains the Green's function to be (anti)periodic under the shift $\tau \rightarrow \tau + 1/T$. The finite-temperature version of Eqn. (13) is given by

$$G^0(\mathbf{p}, \omega_n) = \frac{1}{i\omega_n - (\varepsilon(\mathbf{p}) - \mu)}, \quad \omega_n = n\pi T. \quad (17)$$

n is an even(odd) integer in the case of bosons(fermions). The diagrams and the expressions for the corrections remain largely the same. The major difference is any integrals over frequency, such as in Eqn. (14a), are replaced by a sum over all Matsubara frequencies.

it is not presented here. In the end, both methods fail to agree with experiment, in that the factor preceding T^2 departs significantly from $4\pi^2$ for a significant class of materials. The authors propose that the discrepancy arises due to electron-phonon interactions and support their claims with a phenomenological model. In the end, the equivalency of the two approaches to the problem is manifest, albeit with one being more general than the other.

Limitations and conclusion.—It was assumed that turning on the interactions would allow us to define quasiparticles and retain the concept of the Fermi-sphere. This assumption that the interaction do not change the problem qualitatively is not true in general. A well-known example where the FLT cannot be applied is in situations with attractive potential between the constituents. For instance, such a situation can take place due to electron-phonon interaction. As a result, the Fermi surface is destroyed due to Cooper pair formation, and the gradual increase of the interaction strength is no longer reasonable. In any case, a crucial assumption made in FLT is that the potential be repulsive.

Another failure of FLT was illustrated in a well-known model due to Luttinger²². The Luttinger liquid, which is a one-dimensional (1D) system of interacting fermions is also a system that cannot be obtained from the FLT. In fact, the system is exactly solvable, and it possesses no sharp particle-like excitations in its spectrum.

Both cases can be tracked down to the definition of a quasiparticle. A quasiparticle is a long-lived excitation, with an attenuation coefficient small compared to its lifetime. Mathematically, this information is contained in the renormalization factor Z introduced in a previous section. The case $Z = 0$ corresponds to a situation where the ground state cannot be obtained by the perturbation theory, and the notion of a fermionic quasi-particle is no longer appropriate to describe the physics of the real system.

In the recent years, a new research direction addresses the physics of so-called non-Fermi liquid (NFLs). The 1D model introduced above is a toy model for a NFL. It is difficult to put all of these different NFLs under

one umbrella, but one characteristic of this behavior is their resistivity, which is typically linear in temperature T . In contrast, FLT predicts a resistivity proportional to the scattering time, which is proportional to T^2 . A relatively recent review summarizes the recent experimental and theoretical efforts to understand these materials²³, but a satisfactory understanding of the physics in these materials has not yet been achieved. A similar enigma has presented itself in the study of superconductivity, which has also proved to be a notoriously difficult problem to solve. It will be exciting to see what progress will be made on both of these topics in the nearest future.

¹P. Drude, *Annalen der Physik* **306**, 566 (1900).

²A. Sommerfeld, *Zeitschrift für Physik* **47**, 1 (1928).

³L. D. Landau, *JETP* **30**, 920 (1956), (in Russian).

⁴L. D. Landau, *JETP* **32**, 59 (1957), (In Russian).

⁵P. W. Anderson, *Basic Notions Of Condensed Matter Physics* (Taylor & Francis Ltd., 2018).

⁶E. M. L. L.D. Landau, *Statistical Physics*, Vol. 9 (Elsevier LTD, Oxford, 1995).

⁷C. P. Gordon Baym, *Landau Fermi-liquid theory*, wiley ed. (Wiley-VCH, 1991).

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