Physics 210B

L.II_2_b _ Fokker-Planck II.

→ So far:

- derived equation for pdf of process specified by Langevin Eqn.

i.e. $$\frac{\partial v}{\partial t} + \frac{\gamma}{M} v = \frac{\tilde{f}}{M}$$

$$\partial_t f = -\frac{\partial}{\partial \underline{v}} \cdot \left\{ \underbrace{\frac{\langle \Delta \underline{v} \rangle}{\Delta t} f}_{\substack{\text{drift/drag} \\ \text{(deterministic)}}} - \frac{\partial}{\partial \underline{v}} \cdot \underbrace{\frac{\langle \Delta \underline{v}\, \Delta \underline{v} \rangle}{2\Delta t} f}_{\substack{\text{diffusion} \\ \text{(random)}}} \right\}$$

- restrictions ⇒ $\langle \Delta \underline{v}\, \Delta \underline{v} \rangle < \infty$

- simple example.

Now:

- motion of particle in specified potential

- Sedimentation
- Multiplicative Noise – Logistics (simple)

Now → Spatial Distribution / Pdf.

Consider $\left\{\begin{array}{l}\text{Brownian Motion}\\ \text{random walk}\end{array}\right\}$ in potential $U(\underline{x})$, with noise

Seek $F(\underline{x}, t)$

→ Seek $f(\underline{x}, t) = \int d\underline{v}\, P(\underline{x}, \underline{v}, t)$

$$\equiv n(\underline{x}, t)$$

Now, $\dfrac{d\underline{x}}{dt} = \underline{v}$

particle orbits

$$\frac{d\underline{v}}{dt} = \underbrace{-\frac{\beta}{m}\underline{v}}_{\text{Stokes drag}} \underbrace{-\frac{1}{m}\underline{\nabla} U}_{\text{Deterministic Force}} + \overbrace{\frac{\tilde{f}}{m}}^{\text{Random Force}}$$

Now, if (as appropriate) interested in long timescales, assume Brownian particle reaches 'terminal velocity'

$$t \gg (\beta/m)^{-1}$$

$$\frac{d\underline{x}}{dt} = \underline{v}$$

$$\frac{d\underline{v}}{dt} = -\frac{\beta}{m}\underline{v} + \underline{g}_{ext} + \frac{\underline{\tilde{f}}}{m}$$

$$\underline{v} = \frac{d\underline{x}}{dt}$$

$$\boxed{\frac{d\underline{x}}{dt} = -\frac{\underline{\nabla} U}{\beta} + \frac{\underline{\tilde{f}}}{\beta}}$$

So now write F-P equation for $n(\underline{x}, t)$,

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial \underline{x}} \cdot \left\{ \underbrace{\frac{\langle \Delta \underline{x} \rangle}{\Delta t} n}_{\text{drift/drag.}} - \underbrace{\frac{\partial}{\partial \underline{x}} \cdot \frac{\langle \Delta \underline{x} \Delta \underline{x} \rangle}{2\Delta t} n}_{\text{spatial diffusion}} \right\}$$

So $\dfrac{\langle \underline{\Delta x} \rangle}{\Delta t} = \underline{V} = -\dfrac{\underline{\nabla} U}{\beta}$

$\left(\langle \tilde{f} \rangle = 0 \right)$

$$\langle \underline{\Delta x}\, \underline{\Delta x} \rangle = \int_0^t dt' \int_0^t dt'' \, \frac{\langle \tilde{f}(t')\, \tilde{f}(t'') \rangle}{\beta^2}$$

$$\cong \int_0^t dt_+ \int_{-\infty}^{\infty} dt_- \, \langle \tilde{f}^2 (t_-) \rangle / \beta^2$$

$$\langle \tilde{f}(t')\, \tilde{f}(t'') \rangle = 2 |\tilde{f}|^2 \tilde{\tau}_{ac}\, \delta(t' - t'')$$

and recall from FDT:

$$\beta T = \frac{|\tilde{f}|^2 \tilde{\tau}_{ac}}{2}$$

$$\rightarrow \langle \underline{\Delta x}\, \underline{\Delta x} \rangle = 2 \frac{T}{\beta} t \left[\underline{\underline{I}} \right]$$

for forcing uncorrelated in differing directions

$$= 2 D t \left[\underline{\underline{I}} \right]$$

$\boxed{D = T/\gamma}$ $\rightarrow$ spatial diffusivity i.e. $D = \int_0^\infty du \, \langle \underline{v}(u)\, \underline{v}(0) \rangle$

then

$$\boxed{\frac{\partial n}{\partial t} = -\frac{\partial}{\partial \underline{x}} \cdot \left\{ -\frac{\underline{\nabla} U(\underline{x})\, n}{\beta} - \frac{\partial}{\partial \underline{x}} \cdot \left(\frac{T}{\beta} \underline{\underline{I}}\, n \right) \right\}}$$

- Schmoluchowski Equation (F-P Eqn.)

- Deterministic Drift + Diffusion

- stationary state (1D)

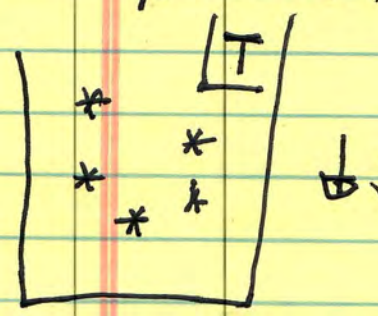
$$\frac{\partial n}{\partial t} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{T}{\beta} n \right) = -\frac{\nabla U n}{\beta}$$

$\Rightarrow$

$$n = n_0 \exp\left[-U(x)/T \right]$$

→ A Specific Application — Sedimentation



Brownian particles in fluid at T in gravitational field.

- Spatial Distribution? $\overset{t\to\infty}{\sim} \exp[-mgz/T]$
  How evolve?

- Particles $\begin{cases} \text{- drift, due gravity} \\ \text{- random walk} \end{cases}$

then, for particles ($m_p \gg \rho_{H_2O} V_{part.}$)

$$m_p \frac{d\underline{v}}{dt} = -\beta \underline{v} - m_p g \hat{z} + \tilde{f}$$

1D, $t \gg (\beta/m)^{-1}$

$$\frac{dz}{dt} = -\frac{g}{\beta}\hat{z} + \frac{\tilde{f}}{m_p}$$

So, $$\frac{\partial n(z,t)}{\partial t} = -\frac{\partial}{\partial z}\left\{ \frac{\Delta z}{\Delta t} n - \frac{\partial}{\partial z}\left(\frac{\langle \Delta z \Delta z \rangle n}{2\Delta t}\right)\right\}$$

From before:

$$\frac{\langle \Delta z \, \Delta z \rangle}{2\Delta t} = \frac{T}{\beta}$$

and

$$\frac{\langle \Delta z \rangle}{\Delta t} = -\frac{g m_p}{\beta} \hat{z} \equiv \underline{V} = -\frac{g m_p}{\beta}$$

Note: $\underset{\text{drift velocity}}{\underline{V}} = +\frac{1}{\beta}(-mg\hat{z}) = \frac{1}{\beta} \underset{\text{force}}{\underline{F}}$

$$\underline{V} = \mu \underline{F}$$

μ ↓ mobility → determine mobility from drift

$\mu = 1/\text{resistance}$

$\boxed{\mu = 1/\beta}$ here

→ generic to friction-produced terminal free-fall.

Now, $D = T/\beta$

$= \mu T$ → Einstein Relation

N.B. Mobility easy to calculate from Diffusivity

⇒ generic to Brownian motion type Langevin Equation and FDT

For steady state:

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial z}\left\{ -\frac{mg}{\beta}\, n - \frac{T}{\beta}\frac{\partial n}{\partial z} \right\} \to 0$$

⇒

$$n = n_0 \exp\left[-\frac{mgz}{T}\right] \checkmark$$

Particles end up in layer $\Delta z \sim T/mg$ at bottom.

Can obtain relaxation to equilibrium profile. See Chandrasekhar

⇒ 

Full evolution
→ diffusion spreading
+
sedimentation.

$$\langle \delta x_\perp^2 \rangle = 2Dt$$

Can write:

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial z^2} + c\frac{\partial n}{\partial z}$$

$D = T/\beta$

$c = \frac{mg}{\beta}$

$n \to \delta(z - z_0) \quad t \to 0$

$D\frac{\partial n}{\partial z} + cn = \Gamma = 0, \quad z = 0$

$t > 0$

$\Rightarrow$

$$n(t, z, z_0) = \frac{1}{2(\pi D t)^{1/2}}\left\{\exp\left[-(z-z_0)^2/4Dt\right] + \exp\left[-(z+z_0)^2/4Dt\right]\right\} + \frac{c}{D\sqrt{\pi}}e^{-cz/D}\int_{\frac{z+z_0-ct}{2(Dt)^{1/2}}}^{\infty} e^{-x^2}\,dx$$

$t \to \infty$

$$\to \frac{c}{D\sqrt{\pi}}e^{-cz/D}$$

Aside: In principle, could obtain F-P eqn. for $f(x, v, t)$

$$\frac{dx}{dt} = \underline{v}$$

$$\frac{d\underline{v}}{dt} = -\frac{\beta}{m}\underline{v} - \frac{1}{m}\underline{\nabla}U + \frac{\tilde{\underline{f}}}{m}$$

i.e.

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f = -\frac{\partial}{\partial \underline{v}}\cdot\left\{\left(-\frac{\beta}{m}\underline{v} - \frac{1}{m}\underline{\nabla}U\right)f - \frac{\partial}{\partial \underline{v}}\cdot \underline{\underline{D}}_v f\right\}$$

U indep. v. $\Rightarrow$

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f - \frac{\underline{\nabla}U}{m}\cdot\frac{\partial f}{\partial \underline{v}} = \left.\frac{df}{dt}\right|_{H.\ Orb.}$$

$$= -\frac{\partial}{\partial \underline{v}}\cdot\left\{-\frac{\beta}{m}\underline{v} f - \frac{\partial}{\partial \underline{v}}\cdot \underline{\underline{D}}_v f\right\}$$

and extract Schmoluchowski in $t \gg (\beta/m)^{-1}$ limit

$$D_v = \frac{\beta}{M} v_{th}^2 \sim \frac{|\tilde{f}_0|^2}{M^2} \tau_{ac}$$

– Multiplicative Noise (Simple)

Consider Logistic Eqn → Populations

$$\frac{dN}{dt} = \underbrace{N}_{\substack{\text{Malthusian} \\ \text{growth} \\ \text{(exponential)}}}(k - N)$$

↳ saturation by competition $\sim N^2$

$N \equiv$ # of population

$$x_{n+1} = \alpha x_n (1 - x_n)$$

Logistic Map

$N = 0$, $N = k$ are fixed pts

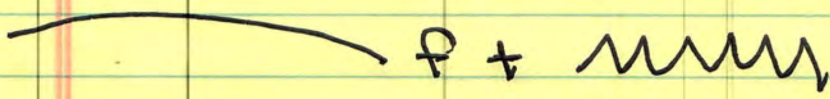
Now, could consider variability in k, and treat as stochastic variable

$$\frac{dN}{dt} = N\left(k_0 + \underbrace{\tilde{\gamma}(t)}_{\substack{\text{variability in} \\ \text{resources}}} - N\right) + \underbrace{\tilde{\alpha}(t)}_{\substack{\text{external input} \\ \text{variability}}}$$

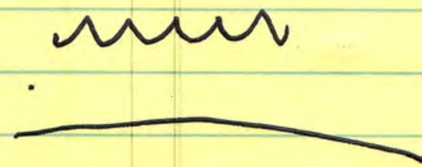
⇒ multiplicative noise
→ rate

⇒ additive noise.

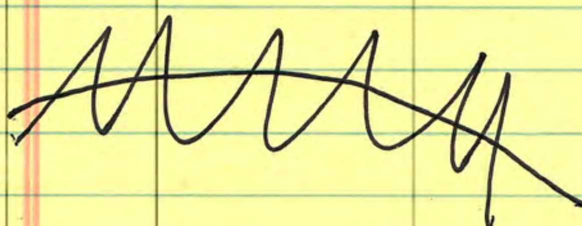
i.e. additive:



noise on top deterministic base =



multiplicative:



multiplies by fast, random quantity

How treat ?

$f(N,t) \rightarrow$ population pdf

$\downarrow$

$\rightarrow$ Fokker-Planck Equation for $f(N)$

$\rightarrow$ here $\langle \tilde{\gamma}(t)\, \tilde{\gamma}(t') \rangle = |\tilde{\gamma}_0|^2 \tau_{ac}\, \delta(t-t')$

Delta correlated for simplicity.

N.B. This is a "textbook model".

$\rightarrow$ additive, as usual

$$\langle \tilde{\alpha} \tilde{\gamma} \rangle \doteq 0$$

Then : $$\frac{dN}{dt} = N\left(k_0 + \tilde{\gamma}(t) - N\right) + \tilde{\alpha}$$

&

$$\frac{\partial}{\partial t} f(N,t) = -\frac{\partial}{\partial N}\left[\left(k_0 N - N^2\right) f(N,t) - \frac{\partial}{\partial N}\left(D\, f(N)\right)\right]$$

For D;

$$\langle \Delta N\, \Delta N \rangle = \int^t dt' \int^t dt'' \, \langle \tilde{\gamma}(t') \tilde{\gamma}(t'') \rangle N^2 \ !$$

$$+ \int^t dt' \int^t dt'' \, \langle \alpha(t') \alpha(t'') \rangle$$

$$= |\gamma_0|^2 \tau_{ac} N^2 t + |\alpha_0|^2 \tilde{\tau}_{ac}$$

Nonlinearity in D

→ one trademark feature of multiplicative noise

→ Note: $N \to \infty \Rightarrow D \to 0$

Rate variation ⇒ pdf spread in proportion to population.

→ Additive correction significant at low N.

Now, ignoring additive correction,

$$\boxed{\partial_t f(N) = -\frac{\partial}{\partial N}\left\{ (k_0 N - N^2) f(N,t) - \frac{\partial}{\partial N}\left(\frac{|\tilde{\gamma}_0|^2}{2} \tau_{ac} N^2 f(N,t) \right) \right\}}$$

is Fokker-Planck Equation

and stationarity:

$$N(k_0 - N) f(N) = \frac{\partial}{\partial N}\left(\frac{|\tilde{\gamma}_0|^2}{2} \tau_{ac} N^2 f(N) \right)$$

so Norm

$$\boxed{f(N) = C \cdot n^{[2(k_0/\sigma^2)-2]} \, e^{-2N/\sigma^2}}$$

$\sigma^2 = |\tilde{\gamma}_0|^2 \tilde{\tau}_{ac}$ Power exponential tail

Need $k_0^2 > (\sigma^2/2)^2 \Leftrightarrow f > 1/n$
$n \to 0$

i.e. $\boxed{k_0 > \frac{|\tilde{\gamma}_0|^2 \tilde{\tau}_{ac}}{2}}$ to avoid log. singularity

Physics of $k_0 > \frac{|\tilde{\gamma}_0|^2 \tilde{\tau}_{ac}}{2}$?

Convenient to linearize around fixed point:

$$\frac{dN}{dt} = (k + \tilde{\gamma} - N)N$$

Validity ?

$$N = k_0 + \tilde{n}$$

$$\frac{d\tilde{n}}{dt} = (k_0 + \tilde{n})(\cancel{k_0} + \tilde{\gamma} - \cancel{k_0} - \tilde{n})$$

$$\cong k_0 \tilde{\gamma} - k_0 n + O(n^2)$$

so

$$\partial_t f(n) = -\frac{\partial}{\partial n}\left[-k_0 n f(n) - \frac{\partial}{\partial n}\left(\frac{k_0^2 \gamma_0^2 \tau_{ac}}{2} f(n)\right)\right]$$

linearize abt fixed pt.

$$= -\frac{\partial}{\partial n}\left[-k_0 n f(n) - \frac{\partial}{\partial n}\left(\frac{k_0^2 \Gamma^2}{2} f(n)\right)\right]$$

⇒ zero flux / stationarity:

$$\boxed{f(n) = \underbrace{c}_{\text{const}} \exp\left[-n^2/k_0 \Gamma^2\right]}$$

Valid for: $\langle(\tilde{n}/N_0)^2\rangle = \langle(\tilde{n}/k_0)^2\rangle < 1$

Now $\langle \tilde{n}^2 \rangle = \frac{\Gamma^2 k_0}{2}$

so $\langle(\tilde{n}/k_0)^2\rangle < 1 \Rightarrow \boxed{\frac{\Gamma^2}{2k_0} < 1}$

→ again ; $\underbrace{\sigma^2 < 2k_0}$

i.e. fluctuations small compared logistic growth.

N.B. :

- can determine time evolution
- can get moments
- spatio - temporal dynamics.