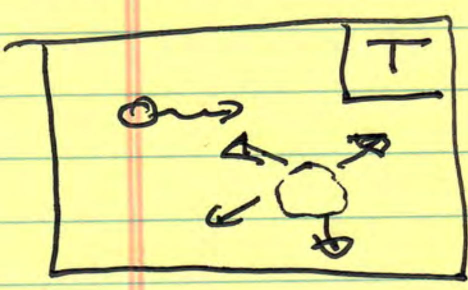
Physics 210 B

II - 2 - a → Fokker-Planck I

- Back to B.M.; $\left\{\begin{array}{l}\text{Statistical} \\ \text{Dynamics} \\ \text{Problem}\end{array}\right.$



i.) how does Pdf - Probability Distribution Function - evolve for Brownian Particle

i.e. $f(\underline{v}, \underline{x}, t)$

→ more interesting/useful than $\langle x^2 \rangle$, $\langle V^2 \rangle$ etc.

and

ii.) If initial cloud of particles, how does it evolve in time.

i.e. $n(t=0) = n_0 \delta(\underline{r} - \underline{r}_0)$

$n(\underline{r}, t)$?

- Of course, will involve Diffusion

$\langle \delta x^2 \rangle \simeq 2 D_x t$

⇒ Fokker-Planck Equation

- calculate $f(x,v,t)$; $P(x,t)$, not moments $\langle x^n \rangle$
- applies to dynamics with $\sim$ **no** memory / short memory
- Evolution as result of sequential, uncorrelated small steps.

⇒ closely related to Central Limit Theorem — background

Aside

N.B.: Central Limit Theorem

- Consider a sum of n independent random variables $\Delta x_1, \Delta x_2 \ldots \Delta x_n$ $(n \gg 1)$.
- $x_n = \Delta x_1 + \Delta x_2 + \ldots \Delta x_n$
  [sum of independent increments]

- For ΔX_i $\qquad \langle \Delta X_i \rangle = 0$

$$\langle \Delta X_j^2 \rangle = \sigma_i^2$$

N.B.: $\Rightarrow$ Variance (only) - second moment - exists (i.e. finite)

$$\langle \Delta X_j^2 \rangle < \infty, \quad \text{all } j$$

~~xxxx~~ $\Rightarrow$ no outliers among $\langle \Delta X_j^2 \rangle$, all alike

Then $\quad S_n^2 = \sigma_1^2 + \sigma_2^2 + \ldots \sigma_n^2$

$$\cong n\sigma^2$$

and $\quad \underset{n\to\infty}{\mathrm{Pdf}}(y) \to \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$

$\rightarrow \quad \mathrm{Pdf}(X_n) \to \frac{1}{\sqrt{2\pi S_n}} \exp\left[-X_n^2/2S_n^2\right]$

$(n \gg 1)$

$$\to \frac{1}{\sqrt{2\pi N\sigma^2}} \exp\left[-X_n^2/N\sigma^2\right]$$

i.e. Pdf of sum of squares is Gaussian, with variance $N\sigma^2$

Key Points:

→ variance of step $\langle \Delta x_j^2 \rangle < \infty$ finite

→ convergence $\langle \Delta x_j^2 \rangle < \infty \not\Rightarrow$ say, $\langle \Delta x_j^4 \rangle < \infty \Rightarrow$ higher moments can induce a "heavy tails" on pdf.

Will discuss further.

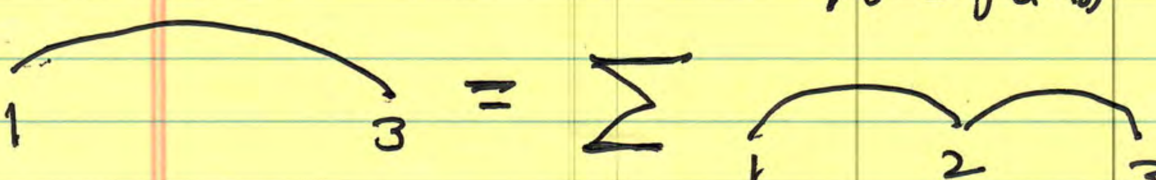
→ Continuing with Fokker-Planck Theory

- consider a system with short/no ~~no~~ memory

⇒ every step in τ independent of prior history (Markov Process)

so $x_1, t_1 \longrightarrow x_3, t_3$
with intermediate step
at x_2, t_2

$$\underbrace{\text{Prob.}(x_3, t_3\, ;\, x_1, t_1)}_{\text{Prob at } x_3\, t_3 \text{ starting from } x_1, t_1} = \underbrace{\int dx_2}_{\text{integrate over intermediate}} \underbrace{P(x_3, t_3 \mid x_2, t_2)}_{2 \to 3 \text{ jump}} * \underbrace{P(x_2, t_2 \mid x_1, t_1)}_{1 \to 2 \text{ jump}}$$



i.e. 

each step independent
all possible intermediates

$$\boxed{P[(x_3, t_3)\, ;\, (x_1, t_1)] = \int dx_2\, P(x_3, t_3 \mid x_2, t_2)\, P(x_2, t_2 \mid x_1, t_1)}$$

Chapman – Kolmogorov Eqn.

Now, re-write for small step Δx (how small is small ?) in short/small time increment τ; (how short ... ?)

Transition Probability

Take: $P(x_2, t_2 \mid x_1, t_1) = T(x, \Delta x, \tau)$

at x — step Δx — in time

$t_2 - t_1 \to \tau$

$x_2 - x_1 \to \Delta x$

then

$$P(x, t+\tau) = \int d(\Delta x)\, P(x-\Delta x, t)\, T(x, \Delta x, \tau)$$

1 step away, at t.

Now expand → lowest odd, even:
(Kramers - Moyal Expansion)

↓

truncation → C. L. T.

vector form

Order of Derivatives Matters!

$$P(x,t) + \tau \frac{\partial P(x,t)}{\partial t}$$

$$= \int d(\Delta x) \Big\{ \overset{①}{P(x,t)\, T(x,\Delta x,\tau)}$$

$$\overset{②}{- \frac{\partial}{\partial \underline{x}} \cdot \left(\underline{\Delta x}\, P(x,t)\, T(x,\Delta x,\tau) \right)}$$

$$\overset{③}{+ \frac{1}{2} \frac{\partial}{\partial \underline{x}} \cdot \frac{\partial}{\partial \underline{x}} \left(\underline{\Delta x}\, \underline{\Delta x} \right) P(x,t)\, T(x,\Delta x,\tau)} \Big\}$$

$$+ \cancel{h.o.}$$

① Normalization:

$$\int d(\Delta x)\, T(x, \Delta x, \tau) = 1$$

T must be normalizable

$$② - \int d(\Delta x)\, \frac{\partial}{\partial \underline{x}} \cdot \left(\underline{\Delta x}\, P(x,t)\, T(x,\Delta x,\tau) \right)$$

$$= - \frac{\partial}{\partial \underline{x}} \cdot \left(\langle \underline{\Delta x} \rangle P(x,t) \right)$$

$$\langle \underline{\Delta x} \rangle = \int d(\underline{\Delta x}) \; \underline{\Delta x} \; T(x, \underline{\Delta x}, t)$$

Mean step → drift

$$③ \int d(\underline{\Delta x}) \; \frac{\partial}{\partial \underline{x}} \cdot \frac{\partial}{\partial \underline{x}} \cdot \left(\frac{\underline{\Delta x}\,\underline{\Delta x}}{2} \right) P(\underline{x}, t) \; T(x, \underline{\Delta x}, \tau)$$

$$= \frac{\partial}{\partial \underline{x}} \cdot \frac{\partial}{\partial \underline{x}} \cdot \left[\frac{\langle \underline{\Delta x}\,\underline{\Delta x} \rangle}{2} P(\underline{x}, t) \right]$$

$$\langle \underline{\Delta x}\,\underline{\Delta x} \rangle = \int d(\underline{\Delta x}) \; \underline{\Delta x} \; \underline{\Delta x} \; T(x, \underline{\Delta x}, \tau)$$

→ mean square step

→ tensor → cross terms !

⇒ will become diffusion.

then dividing thru by τ:

$$\frac{\partial P(\underline{x},t)}{\partial t} = -\frac{\partial}{\partial \underline{x}} \cdot \left\{ \frac{\langle \underline{\Delta x} \rangle}{\tau} P(\underline{x},t) - \frac{\partial}{\partial \underline{x}} \cdot \left(\frac{\langle \underline{\Delta x}\, \underline{\Delta x} \rangle}{2\tau} P(\underline{x},t) \right) \right\}$$

$$= -\frac{\partial}{\partial \underline{x}} \cdot \left\{ \underline{V}\, P(\underline{x},t) - \frac{\partial}{\partial \underline{x}} \cdot \left(\underline{\underline{D}}\, P(\underline{x},t) \right) \right\}$$

$\underline{V} = \frac{\langle \underline{\Delta x} \rangle}{\tau}$ → drift velocity

$\underline{\underline{D}} = \frac{\langle \underline{\Delta x}\, \underline{\Delta x} \rangle}{2}$ → diffusion (tensor)

Welcome to $\boxed{\text{Fokker-Planck Equation}}$!

(N.B. : Diffusion is special case).

Note: Observations

- F.-P. based on assumption of Markov process

- coarse grains on scales $\tilde{\tau}$, Δx (minimum resolvable)

$\tilde{\tau} \leftrightarrow \tau_{ac}$ in random process

$\Delta x \sim \int dt\, \tilde{v}$

$\Delta x^2 \sim 2D\,t$

$$D = \int_0^\infty \langle \tilde{v}(0)\, \tilde{v}(\tau) \rangle \rightarrow |\tilde{v}|^2 \tau_{ac}$$

$\Delta x \sim \tilde{v}\, \tau_{ac}$

- can write in form:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \underline{x}} \cdot \underbrace{\underline{\Gamma}_P}_{\text{probability flux}}$$

manifestly conserved probability

write as divergence

$$\Gamma_p = \underline{V} P(\underline{x},t) - \frac{\partial}{\partial \underline{x}} \cdot \left(\underline{\underline{D}}\, P(\underline{x},t) \right)$$

Probability flux

- describes dynamics where macro evolution set by accumulation of many small steps.

Buried Bodies

① - no long time, long range correlations

②* - assumes $\langle \Delta x^2 \rangle = \int d(\Delta x)\, (\Delta x)^2\, T$

$< \infty$!

not guaranteed by T normalization.

→ ☺ near equilibrium
Gaussian

$\Rightarrow \int (\Delta x)^2 T < \infty$ ✓

→ exponential ✓ (Gamma)

→ Power Law ? → self-similar processes → scaling → reality

$$T \sim 1 \Big/ \Big[\underbrace{1}_{\text{infared control}} + \left(\frac{\Delta x}{L}\right)^{\alpha} \Big]$$

⇒ need $\alpha > 3$, or

[expansion fails, and C. L. Th. violated.

③ → τ not uniform ?!

i.e. assume: | | | | | |

clock-like regularity

but | ||| | ||

sticking

②, ③ ⇒ $\left\{ \begin{array}{l} \text{Fractional Kinetics} \\ \text{CTRW} \end{array} \right.$ > $\left\{ \begin{array}{l} \text{Levy} \\ \text{Walk} \end{array} \right.$

Central Limit Thm. violated ⇒
Levy Distribution, not Gaussian.

Will discuss.

Light Reading:

- R. Lowenstein, "When Genius Failed"
  → LTCM

- B. Mandelbrot, "The (Mis) Behavior of Markets".

Examples:

- trivial → Brownian Motion
- Aside: Stochastic Liouville
- Useful → Schmoluchowski and Sedimentation
  Kramers 1

- Problem → Logistics with Multiplicative Noise

(a.) Brownian Motion

bath at T.

- What is $f(v,t)$?
$t \to \infty$

Obviously a Gaussian, with variance v_{Th}^2 !

How show ?

→ Fokker-Planck Eqn.

→ stationarity : $\partial f/\partial t \to 0$

$\Rightarrow \Gamma_v \to 0$.

Of course, have:

$$m \frac{\partial v}{\partial t} = -\beta v + \tilde{f}$$

$$\langle \tilde{f}(t)\tilde{f}(t')\rangle = |\tilde{f}|^2 \tau_{ac}\,\delta(t'-t)$$

So, can set up F-P Eqn, in v ?

$$f(\underline{v}, t+\tau) = \int d(\underline{\Delta v})\, f(\underline{v}-\underline{\Delta v}, t)\, T(\underline{\Delta v}, t)$$

crank as before gives:

$$\frac{\partial f(v,t)}{\partial t} = -\frac{\partial}{\partial \underline{v}}\cdot \underline{\Gamma}_v$$

$$\underline{\Gamma}_v = \left\{ \frac{\langle \underline{\Delta v}\rangle}{\tau} f(\underline{v},t) - \frac{\partial}{\partial \underline{v}}\cdot\left(\frac{\langle \underline{\Delta v}\,\underline{\Delta v}\rangle}{2\tau} f(\underline{v},t)\right)\right\}$$

$$= \left\{ \underline{V} f(v,t) - \frac{\partial}{\partial \underline{v}}\cdot \underline{\underline{D}}\, f(v,t) \right\}$$

To calculate drift, diffusion:

use Langevin Eqn.

$$\frac{\partial \underline{v}}{\partial t} = -\frac{\beta}{m}\underline{v} + \tilde{a}(t)$$

then avg:

$$\left\langle \frac{\Delta \underline{v}}{T} \right\rangle = \bar{V} = -\frac{\beta}{m}\underline{v}$$

→ drift opposes motion, $\sim \beta/m$

For velocity diffusion:

$$D_v = \left\langle \frac{\Delta v\, \Delta v}{2T} \right\rangle = \int_0^\infty \langle \tilde{a}(0)\, \tilde{a}(\tau) \rangle \, d\tau$$

$$= \frac{|\tilde{f}_0|^2}{m^2} \tilde{\tau}_{ac}$$

but FDT:

$$|\tilde{f}_0|^2 \tilde{\tau}_{ac} = \beta m \langle \tilde{v}^2 \rangle = \frac{\beta}{\cancel{m}} m^2 \frac{T}{\cancel{m}}$$

$$D_v = \beta \frac{T}{m^2} = \left(\frac{\beta}{m}\right) v_{Th}^2 \quad \checkmark$$

(dims. / expected)

$$D_v = \frac{\beta}{m} v_{Th}^2 \left[\underline{\underline{I}} \right]$$

$$\underline{V} = -\frac{\beta}{M} \underline{v}$$

1D

$$\Gamma_v = V f - \frac{\partial}{\partial v}\left(D_v f \right)$$

$$= -\frac{\beta}{M} \underline{v} f - \frac{\partial}{\partial v}\left(\frac{\beta}{M} v_{Th}^2 f \right)$$

$$\Gamma_v = 0$$

$$f = \# \exp\left[- v^2 / 2 v_{Th}^2 \right]$$

$$= \# \exp\left[-\frac{\beta}{M} v^2 / 2 D_v \right].$$

$$D_v = \frac{\beta}{M} v_{Th}^2$$

Gaussian results from balance of drag / drift with diffusion