Beyond the Landau Problem

$\boxed{\text{Physics 210b}}$

→ Recall Landau Problem:

- $\boxed{\text{linear response problem for Vlasov - Poisson system}}$

- n.b. → classic, well-posed example

  * → example of collisionless damping

  → $\frac{dS}{dt} = 0$ as $df/dt = 0$
  but $\tilde{n}, \tilde{E}$ $\left(\int d\Gamma\, \tilde{f}\right)$ decay.

  → analytical continuation to damped case, phase mixing
  ⇒ see notes.

→ $\boxed{\text{Physics : Resonant particle heating i.e. } \langle \underline{E} \cdot \underline{J} \rangle \neq 0 .}$

i.e. beam example.

2.) Now,

- Vlasov-Poisson is <u>nonlinear</u>

  i.e. $E\frac{\partial f}{\partial v}$ with $-\partial_x^2\phi = \int 4\pi n_0 q f$

- So natural to ask of nonlinear evolution.

- Observe can have $\left\{\begin{array}{l}\text{instability}\\ \text{multiple modes}\end{array}\right.$

i.e. Bump-on-Tail (Plasma + Beam)

$$\frac{\partial \langle f \rangle}{\partial v} > 0 \Rightarrow \mathrm{Im}\,\epsilon < 0$$

unstable

? How will system evolve ?

3.) Linear ⇒ Nonlinear story of Landau Problem?

Landau Problem

- → coherent – single wave; phase mixing / coarse graining
- → stochastic – multiple waves

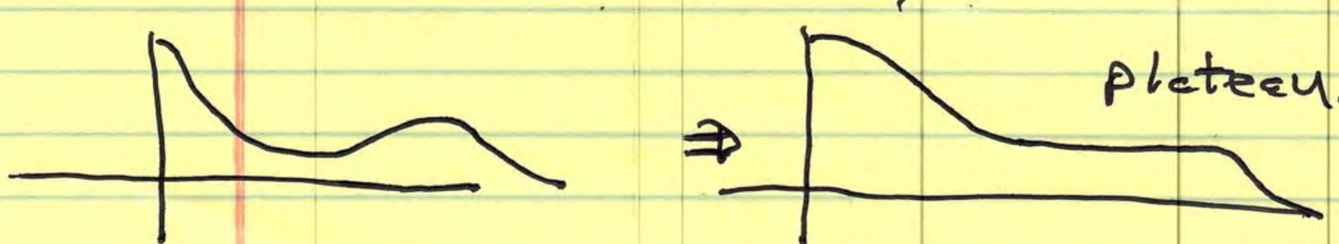
⇒ Quasilinear Theory

⇕

* $\boxed{\text{Using linear response in mean field theory}}$

[Q.L.T. answers question of how $\langle f \rangle$ evolves due to effects of multiple resonant waves.]

Obviously, relieving $\partial \langle f \rangle / \partial v$ saturates instability. Rate? ⇔ <u>saturation amplitude</u>



→ Why?

- Linear response
- statistical (multi-wave)
- mean field $\langle f \rangle$

on target!

c.f. – $\boxed{\text{Vedenov, Velikov, Sagdeev} \quad \text{1960's}}$

⇒

- also Drummond, Pines
- Pedagogical Treatments
  - Sagdeev, Galeev "Nonlinear Plasma Theory"
  - P.D., Itoh, Itoh "Phys Kin. of Plasmas" 2010, CUP.

see next

Cmnt Landau.

$\partial_t |A|^2 = 2\gamma_0 |A|^2 - \sigma |A|^4$

$\nu_{\sigma} \propto \Delta \bar{v}' \rightarrow -\sigma |A|^2$

$\Delta \bar{v}' \rightarrow \frac{\partial \langle f \rangle}{\partial v}$

Quasilinear Theory - Vlasov Plasma

i) Motivation and Overview

$\Rightarrow$ linear theory determines 'instantaneous stability' of plasma

i.e. $\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k} \int dv \, \frac{\partial \langle f \rangle / \partial v}{\omega - kv}$

[Landau Problem] via linear response

$\Rightarrow$ growth/damping rate $\gamma_k = \gamma_k[\langle f \rangle]$

but $\langle f \rangle$ evolves... If $\langle f \rangle$ evolves slowly:

i.e. "slowly" $\Rightarrow \frac{1}{\langle f \rangle} \frac{\partial \langle f \rangle}{\partial t} < \gamma_k$

can consider: $\gamma_k = \gamma_k[\langle f(t) \rangle] \Rightarrow$ {evolution driven by instabilities

$\Rightarrow$ Physics: mean distribution evolution ... driven by relaxation.

$$\boxed{\text{quasilinear theory is concerned with describing and understanding the slow evolution of } \langle f \rangle \ldots}$$

$\langle \; \rangle$ = space/time

To connect to <u>Landau Theory</u>:

in Landau theory, for weakly nonlinear evolution:

$$\partial_t A^2 = 2\gamma_0 A^2 - \sigma A^4$$

$\gamma_0 \simeq V_0'$ (i.e. shear inst.)

but

$$\partial_t \langle v \rangle = -\partial_x \langle \tilde{v}_x \tilde{v}_y \rangle$$

so: $V_0' \to V_0' + \Delta V'$

$$\Delta V' \sim A^2 \text{, necessarily}$$

write $\Delta V' \sim -\sigma' A^2$

$\hookrightarrow$ [QL-like feedback on driving profile.

$$\partial_t A^2 = 2\gamma A^2$$
$$= 2(\gamma_0 + \Delta\gamma) A^2$$

but

$$\Delta\gamma \simeq \Delta V' \simeq -\sigma' A^2$$

<u>so</u>

$$\partial_t A^2 = 2\gamma_0 A^2 - \sigma A^4$$
$$= (2\gamma_0 - \sigma A^2) A^2$$

③ quasilinear theory as "mindless mean field theory", i.e.

$\langle f \rangle \equiv \langle f(v,t) \rangle$ where → $\langle \ \rangle$ eliminates spatial dependence
→ t understood "slow"

so if:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$$

then Q.L. equation is simply: (upon avg.)

$$\frac{\partial \langle f \rangle}{\partial t} + \frac{\partial}{\partial v}\left\langle \frac{q}{m}\tilde{E}\tilde{f} \right\rangle = 0$$

i.e. generic mean field equation (for $\langle f \rangle$) for mean of conserved order parameter ($\langle f \rangle$)

$$\frac{\partial \langle f \rangle}{\partial t} + \frac{\partial}{\partial v} J_v = 0 \rightarrow \boxed{\text{phase space continuity equation}}$$

$$J_v = \Gamma_v = \left\langle \frac{q}{m} E f \right\rangle$$

$$= \frac{q}{m}\langle \tilde{E}\tilde{f} \rangle$$

for: $E = \tilde{E}$

$f = \langle f \rangle + \tilde{f}$

elementary closure problem
i.e. relate $\langle f \rangle$ to $\langle \tilde{E}\tilde{f} \rangle$ → hierarchy!

How close?

simplest example of moment closure

then Q.L.T. simply takes form:

(f) $\tilde{f} \rightarrow \tilde{f}_{\text{linear}}$ (i.e. linear response of f)
plug in linear response

i.e. $\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$ Vlasov Eqn.

$\Rightarrow -i(\omega - kv)\,\tilde{f}_{k,\omega} = -\frac{q}{m}\tilde{E}_{k,\omega}\,\frac{\partial \langle f \rangle}{\partial v}$

so $\Gamma_v = -\frac{q^2}{m^2}\sum_{k,\omega} |\tilde{E}_{k,\omega}|^2 \frac{i}{(\omega - kv)}\,\frac{\partial \langle f \rangle}{\partial v}$

and with $\omega = \omega(k)$ only (i.e. spectrum of eigenmodes, only) — i.e. contrast approach to criticality in usual phase transitions (2nd order) * → here growth of order parameter in broken symmetric phase ... not noise driven *

Q.L. equation is:

$$\boxed{\begin{aligned}\frac{\partial \langle f \rangle}{\partial t} &= \frac{\partial}{\partial v} D \frac{\partial \langle f \rangle}{\partial v} \\ D &= \frac{q^2}{m^2}\sum_k |\tilde{E}_k|^2 \frac{i}{\omega - kv + i\cancel{0}}\end{aligned}}$$

"Q.L. equation"

"Diffusion will fill in", as expected

i.e. mindless mean field theory ...

→ with $\epsilon(k,\omega) = 0$
→ $\partial_t |E_k|^2 = 2\gamma_k |E_k|^2$
→ advance fields.

i.e. describes how mode spectrum evolves mean $\langle f \rangle$.

But,

Surprisingly: Q.L.T. works quite well!

Key issue: why?

N.B.: In contrast to critical phenomena, external noise ignored → instability driven ... deterministic

④ Some questions to keep in mind:

→ (i) why is QL equation a diffusion equation? When is this valid?

↔ nature of "irreversibility" ...

→ (ii) can Q.L. equation be derived from Fokker-Planck theory?

↔ also "irreversibility" related ...
what of drift?

→ (iii) how does Q.L. equation balance the energy-momentum budgets?

→ (iv) when / how does Q.L. theory fail? : $\frac{\langle \tilde{N}^2 \rangle}{\langle N \rangle^2}$

↔ related (i) ..... What is "Ginzburg Criterion" for Q.L.T.? Can such a criterion be formulated?

→ v.) what is dynamics of quasilinear relaxation?

i.e. physics?

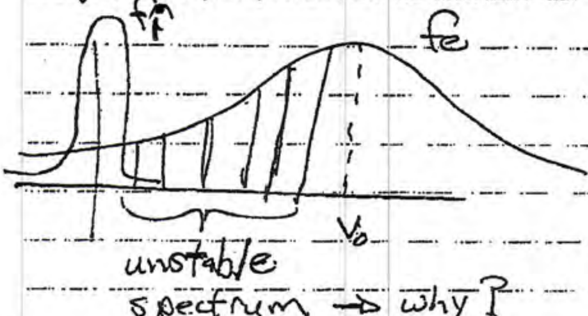
iii) Basic Scales / Regime Definition

① → Generally, Q.L.T. concerned with

i) 'broad' spectrum of:
→ how broad?

ii) unstable waves

i.e. for current-driven ion-acoustic (C.D.I-A.) turbulence:



or B-O-T

②
→ In finite system, k quantized, i.e.

$k_m = m\pi/L$, etc.

- so, have spectrum of phase velocities

$\omega_m/k_m = \omega(k_m)/k_m = V_{ph\,m}$

- wave-particle resonance occurs when

$V = V_{ph,m}$

then Sir Isaac ⇒

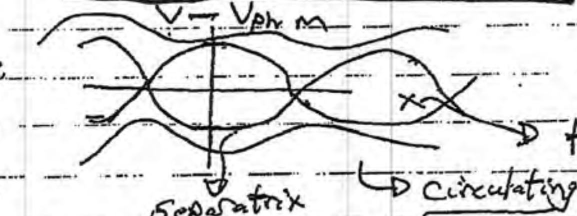
$m\ddot{x} = \sum_m qE_m \cos(k_m x - \omega_m t)$ — n.b. $\left\{ \begin{array}{l} \text{deterministic,} \\ \text{no RPA} \end{array} \right.$

and 1 resonance dominant ⇒

$m\ddot{x} \cong qE_{m_0} \cos(k_m x_0 + (k_m v - \omega_m)t)$

⇒ each resonant velocity defines a $\boxed{\text{phase space island}}$

i.e.

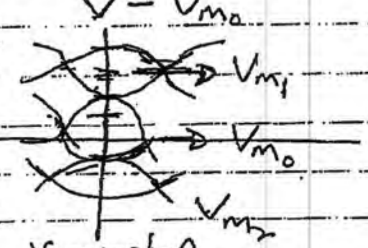


$\Delta v \sim \left(\frac{q}{m}\phi_0\right)^{1/2}$

∴ QLT is concerned with the case of:

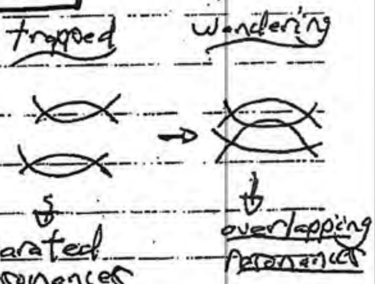
⇒ multiple, $\boxed{\text{overlapping resonances}}$ → $\left\{ \begin{array}{l} \text{separatrix} \\ \text{proximity} \Rightarrow \\ \text{destruction} \end{array} \right.$

→ i.e.



Stochasticity

if overlap, hop



∴ particle can wander stochastically from resonance - to - resonance, i.e. hopping

⇒ diffusion in v / $D_v \sim \frac{(\Delta v)^2}{\tau_{ac}}$ — Δv ~ resonance width; τ_{ac} ~ pattern time ↳ what is it?

ergodicity → mixing

— See Phys. 200B Notes 2014, 2015 — Google directly.

— Or, Supplementary Material

overlap $\Leftrightarrow \exists$ 1 $h_k > 0$. (Positive Lyapunov exp.)

Overlap condition (B.V. Chirikov):

$$\boxed{\frac{1}{2}\left(\Delta V_m + \Delta V_{m+1}\right) \gtrsim V_{ph_{m+1}} - V_{ph_m}.}$$

$\Delta V_T \sim \sqrt{\frac{eE_0}{m}}$

underlies diffusion eqn. - QL

$\therefore \rightarrow$ particle motion stochastic $\Leftrightarrow$ irreversibility

$\rightarrow$ fundamental irreversibility $\Rightarrow$ orbit stochasticity (not dissipation, Landau damping) $\Leftrightarrow$ contrast critical phenomena).

$\rightarrow$ underpinning of diffusion equation. (noise)

③ $\rightarrow$ But, a swindle !? $\rightarrow$ [use of un-perturbed orbit is estimate]!

Why does linearization work?

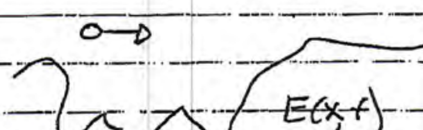
i.e. is $x \rightarrow x_0 + vt$ valid ?

Consider: linear, un-perturbed orbit ?! ?

have: $E(x,t) = \sum_k E_k \exp[i(kx - \omega_k t)]$

$\therefore$ particle "sees" instantaneous pattern of electric field, from modal superposition

i.e.



∴ relevant comparison is:

τ_L → life time of 'instantaneous' pattern

τ_b → 'bounce time' of particle in pattern

obviously, ① $\boxed{\tau_L \ll \tau_b}$ → unperturbed orbit is satisfactory approximation

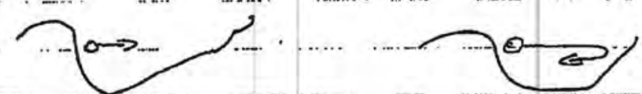
(pattern changes prior ↔ bouncing)

② $\tau_L \gg \tau_b$ → particle bounces prior pattern changes

so must consider orbit perturbation ...

i.e.

vs.



∴ quasilinear theory relevant to evolution when:

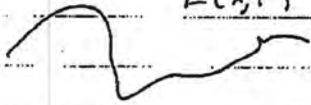
① → orbits stochastic (Chirikov condition satisfied)

② → $\tau_{Life} < \tau_{bounce}$ → unperturbed orbits ~ valid.

3)

But, how relate $\tau_{lifetime}$, τ_{bounce} to physical quantities?

$$\boxed{\text{Key point: Superposition patterns } \underline{\text{disperse}}!}$$



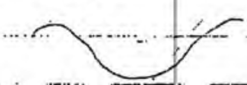
$$E(x,t) \Leftrightarrow \sum_{k} E_k \, e^{i(kx-\omega_k t)}$$

$$= \sum_{k} E_k \exp\left[i\left(k\left[x-\underbrace{\omega_k/k}_{v_{ph}(k)}\,t\right]\right)\right]$$

$\therefore$ $\Delta(\omega_k/k) \equiv$ spread in phase velocities

sets dispersal speed.

so dispersal rate is $(\text{time})^{-1}$ to disperse by one wavelength $\rightarrow$ FOM



$$1/\tau_{life} = k\,\Delta(\omega_k/k) \Big| \quad \underline{\tau_{ac}}$$

$$= k\left(\frac{d\omega_k}{dk}\frac{\Delta k}{k} - \frac{\omega_k}{k^2}\Delta k\right) \Big|$$

$$\Big|(v_g - v_{ph})\Big|$$

$$\boxed{1/\tau_{life} = \left(\frac{d\omega_k}{dk} - \frac{\omega_k}{k}\right)\Delta k} = \Big|(v_g(k) - v_{ph}(k))\Big|\Delta k$$

n.b. $\tau_{life} \rightarrow \infty$ for non-dispersive waves!

Generally, QLT / weak turbulence encounters trouble for {non dispersive, weakly dispersive} waves.

$1/\tau_L \sim 1/\tau_{ac}$ for resonant particle.

How systematize? — E-field correlation Fctn

Consider: $\langle E(x_1, t_1)\, E(x_2, t_2) \rangle_{x,t} = C$

electric field correlation function

$C = C(x_-, T)$, for $\left\{ \begin{matrix} \text{homogeneous} \\ \text{stationary} \end{matrix} \right\}$ fluctuations

relative coords, space/time

$x_1 = x_+ + x_-$ $\quad t_1 = t_+ + t_-$
$x_2 = x_+ - x_-$ $\quad t_2 = t_+ - t_-$

$\langle\ \rangle_{x,t} = \langle\ \rangle_{\underline{x_+, t_+}}$ $\left\{ \text{what the bracket means} \right.$

so

$$C(x_-, T) = \Big\langle \sum_{k,k'} E_k E_{k'} e^{i(k+k')x_+} e^{-i(\omega_k + \omega_{k'})t_+} \, e^{i(k-k')x_-} e^{-i(\omega_k - \omega_{k'})t_-} \Big\rangle_{x_+, t_+}$$

x_+, t_+ average $\Rightarrow k = -k'$ $\quad \omega_k = -\omega_{k'}$

so

$$C(x_-, T) = \sum_k |E_k|^2 e^{ikx_-} e^{-i\omega_k t_-}$$

Now:

→ assume continuous spectrum — i.e. post-overlap

→ for simplicity, take model

$$|E_k|^2 = E_0^2 \Big/ \underbrace{\left[\left(\frac{k-k_0}{\Delta k}\right)^2+1\right]}_{\text{width}}$$ (trial form)

→ evaluate on u.p.o.

$$x_- = x_{0-} + V\tau$$

∴

$$\langle E^2\rangle = \int dk \frac{E_0^2}{\left[\frac{(k-k_0)^2}{\Delta k^2}+1\right]} e^{ikx_0} e^{i(kV-\omega_k)\tau}$$

integrating:

phase info. – irrelevant

$$\sim E_0^2 e^{ik_0x} e^{-|\Delta k x_0|} *$$

$$e^{i(k_0V-\omega_{k_0})\tau} e^{-|\Delta(kV-\omega_k)|\tau}$$

oscillation (→ 0 on resonance)

correlation decay due dispersion and its interplay with resonance.

n.b.: note that spread in doppler-shifted ω is critical

13.

now $\Delta(kV - \omega_k) = V\Delta k - V_{gr}\Delta k$

$$= |(V - V_{gr})\Delta k|$$

$V_{gr} = \dfrac{d\omega}{dk}$

so
$\langle E^2 \rangle = C(x, \tau)$

$$= E_0^2\, e^{ik_0 x}\, e^{i(k_0 V - \omega_{k_0})\tau}\, e^{-|\Delta k\, x_0|}$$

$$* \exp\left[|(V - V_{gr})\Delta k|\,\tau\right]$$

sets lifetime

$$\boxed{1/\tau_L = |(V - V_{gr}(k))\Delta k| \equiv (\text{Autocorrelation Time})^{-1}} \equiv 1/\tau_{ac}.$$

Note:

- for resonant particles, $V = \omega_k/k$

$$\boxed{1/\tau_L = |(V_{ph} - V_{gr})\Delta k|}$$

→ recovers earlier!

- can think: $|V\Delta k| \rightarrow 1/\tau_{ac}^{\text{wave-particle}}$

$|V_{gr}\Delta k| \rightarrow 1/\tau_{ac}^{\text{wave}}$ packet dispersal

14.

generally, shorter time dominates, except for non-dispersive waves.

So, $\boxed{\text{can enumerate key time scales}}$

→ $\tilde{\tau}_{ac} = |\Delta k(v_{ph} - v_{gr})|^{-1}$

≡ persistance of E pattern ($\langle E^2 \rangle$ autocorrelation) for resonant particles

→ γ^{-1} ≡ growth/damping time

→ $\tilde{\tau}_{Tr} = \left(k\sqrt{q\phi/m}\right)^{-1}$ ≡ trapping time / bounce

→ $\tilde{\tau}_{relax} = \left(\frac{1}{\langle f \rangle}\frac{\partial \langle f \rangle}{\partial t}\right)^{-1}$ ≡ avg. distribution relaxation time

So

$\tilde{\tau}_{ac} < \tilde{\tau}_{Tr}$ → u.p.o. valid

$\tilde{\tau}_{ac} < \tilde{\tau}_{relax}$, γ^{-1} → $\langle f \rangle$ closure meaningful.

$\tilde{\tau}_{ac} < \gamma^{-1} < \tilde{\tau}_{relax}$ → Q.L.T. valid.

iii) Energy – Momentum Budgets

→ Key Point: There are two ways of implementing the book-keeping and accounting

i.e. $\begin{cases} \text{resonant particles vs. 'waves' / Quasi-particles} \\ \text{or} \\ \text{particles vs. fields} \end{cases}$

keep in mind: Wave = Field + Non-resonant Particles

i.e. for plasma oscillation, $\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$

$$\text{Wave Energy} = W = \frac{\partial}{\partial\omega}(\omega\epsilon)\Big|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$= \omega\frac{\partial\epsilon}{\partial\omega}\Big|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$= 2 \cdot \frac{|E|^2}{8\pi}$$

field non-resonant particle

(show!)

→ Resonant Particles vs. Waves ?

$$\frac{\partial \langle f \rangle}{\partial t} = -\frac{\partial}{\partial v} \frac{q}{m} \langle \tilde{E} \tilde{f} \rangle$$

$$\frac{\partial}{\partial t} \int dv \, \frac{m v^2}{2} \langle f \rangle = -\int dv \, \frac{m v^2}{2} \frac{\partial}{\partial v} \frac{q}{m} \langle \tilde{E} \tilde{f} \rangle$$

$$= \int dv \, \cancel{m} v \frac{q}{\cancel{m}} \langle \tilde{E} \tilde{f} \rangle$$

plugging in $\tilde{f}_k^{\text{Linear}}$ for $\tilde{f}$:

Resonant

$$\frac{\partial}{\partial t} \mathcal{E}_{kin} = -i \int dv \, \frac{v q^2}{m} \sum_k |E_k|^2 \left(\frac{P}{\omega - kv} - i\pi \delta(\omega - kv) \right) \frac{\partial \langle f \rangle}{\partial v}$$

resonant

$$\frac{\partial}{\partial t} \mathcal{E}_{kin}^{res} = -\int dv \, \frac{\pi q^2}{m} \sum_k \frac{\omega}{k|k|} \delta(\omega/k - v) \frac{\partial \langle f \rangle}{\partial v} |E_k|^2$$

resonant only

$$= -\frac{\pi q^2}{m} \sum_k \frac{\omega}{k|k|} \left. \frac{\partial \langle f \rangle}{\partial v} \right|_{\omega/k} |E_k|^2$$

As resonant particles stabilize/destabilize wave, expect resonant particles conserve energy ~~against waves.~~

For wave energy evolution:

Recall: $\epsilon = 1 + \frac{\omega_p^2}{k}\int dV \frac{\partial\langle f\rangle/\partial V}{\omega - kV}$

$$\epsilon^r(\omega_k + i\gamma_k) + i\epsilon^{IM} = 0$$

$$i\gamma_k = \frac{-i\epsilon^{IM}}{\partial\epsilon^r/\partial\omega} \quad ; \quad \cancel{i}\gamma_k = -\frac{\cancel{i}\,\epsilon_{IM}}{\partial\epsilon^r/\partial\omega}$$

$$= -\epsilon_{IM}/\partial\epsilon^r/\partial\omega$$

Now, $W \equiv$ Wave Energy Density

action density

$$W = \sum_k \frac{\partial}{\partial\omega}(\omega\epsilon)\frac{|E_k|^2}{8\pi} = \sum_k \omega_{\underline{k}} N_{\underline{k}}$$

quasiparticle density

$$= \sum_k \omega_k \underbrace{\left.\frac{\partial\epsilon^r}{\partial\omega}\right|_{\omega_k} \frac{|E_k|^2}{8\pi}}$$

$$|E_k|^2 = |E_k^0|^2 e^{2\gamma_k t}$$

$$\frac{\partial W}{\partial t} = \sum_{\underline{k}} 2\gamma_{\underline{k}}\,\omega_k \left.\frac{\partial\epsilon^r}{\partial\omega}\right|_{\omega_{\underline{k}}} \frac{|E_k|^2}{8\pi}$$

$$= \sum_{\underline{k}} 2\left(\frac{-\epsilon_{IM}}{\partial\epsilon^r/\partial\omega}\right)\omega_k \frac{\partial\epsilon^r}{\partial\omega}\frac{|E_k|^2}{8\pi}$$

$$= \sum_{\underline{k}} -\epsilon_{IM}(k,\omega_k)\,\omega_k\left(\frac{|E_k|^2}{4\pi}\right)$$

$$i\epsilon_{IM} = \frac{\omega_p^2}{k}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}\frac{(-i\pi)}{|k|}$$

$(n_0 = 1)$

$$\therefore \frac{\partial W}{\partial t} = \sum_k \frac{\pi q^2}{m}\frac{\omega_k \cancel{4\pi}}{k|k|}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}\frac{|E_k|^2}{\cancel{4\pi}}$$

$$= +\frac{\pi q^2}{m}\sum_k \frac{\omega}{k|k|}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}|E_k|^2$$

$$\Rightarrow \boxed{\partial_t \mathcal{E}^{\text{resonant}}_{\text{kinetic}} + \partial_t W = 0}$$

it works

$$\partial_t\Big[\sum_k \omega_k N_k + \mathcal{E}^{Res}_K\Big] = 0$$

$$\cancel{\partial_t \int dv \frac{1}{2} m v^2 f = q\langle E f\rangle}$$

Note:

— this is essentially a re-write of the Poynting theorem for plasma waves, i.e.

$$\frac{\partial}{\partial t} W + \nabla\cdot S + Q = 0$$

$Q \to \langle E\cdot J\rangle$ coupling

$\frac{\partial}{\partial t}W$: wave energy

$\nabla\cdot S$: divergence of wave energy density flux

$\langle E\cdot J\rangle$: resonant particle heating

$$\frac{\partial}{\partial t}\int (N\omega) + \frac{\partial}{\partial t} RPKED = 0$$

For homogeneous system: $\underline{\nabla}\cdot\underline{S} = 0$

so $\frac{\partial W}{\partial t} + \underbrace{Q}_{\langle E\cdot \underline{J}\rangle \text{ mediated by resonant particles (DC field)}} = 0$

$$\Leftrightarrow \boxed{\frac{\partial W}{\partial t} + \frac{\partial}{\partial t}\underbrace{(RPKED)}_{\text{resonant particle kinetic energy density}} = 0}$$

Energy Thm I

Waves and Resonant particles conserve energy!

?? What is the fate of RPKED for saturated waves. What must happen ??

→ Now, can observe:

$$W = \underbrace{NRPKED}_{\text{non-resonant particle kinetic energy density}} + \underbrace{FED}_{\text{field energy density}}$$

so, simply re-grouping terms:

$$\boxed{\frac{\partial}{\partial t}(FED) + \frac{\partial}{\partial t}\underbrace{(RPKED + NRPKED)}_{\underline{PKED} \to \underline{total}} = 0}$$

so $$\boxed{\frac{\partial}{\partial t} FED + \frac{\partial}{\partial t}(PKED) = 0}$$

Energy Thm. II

i.e. fields and particles conserve energy.

What is the Physics of all this ? !

$$D = \mathrm{Re} \sum_k \frac{q^2}{m^2} |E_k|^2 \left(i/\omega - kv \right)$$

QL diffusion

for general, weakly non-stationary state ---

$$= \sum_k \frac{q^2}{m^2} |E_k|^2 \left(|\gamma_k| \Big/ \left((\omega-kv)^2 + |\gamma_k|^2\right) \right)$$

n.b. causality ⇒ no negative diffusion for damped waves

$$\boxed{D \cong \sum_k \frac{q^2}{m^2} |E_k|^2 \Big\{ \underbrace{\pi \delta(\omega - kv)}_{\text{resonant diffusion}} + \underbrace{\frac{|\gamma_k|}{\omega_k^2}}_{\text{non-resonant diffusion}} \Big\}}$$

Resonant Diffusion → irreversible – resonance overlap is underpinning

→ rooted in particle stochasticity

- Relation to Fokker-Planck ?

$D_{QL}(v) \equiv$ Res. + Non-Res.
$\qquad\qquad D_R \qquad\qquad D_{NR}$

$D_R \Rightarrow$ stochasticity / chaos
- irreversible
- short τ_{ac}

$\Rightarrow$ F-P.

$D_{NR} \Rightarrow$ No. (sloshing)

Now, as Langevin Eqn. here is:

$$\frac{dv}{dt} = \frac{q}{m}\tilde{E} \quad , \quad \frac{dx}{dt} = v$$

(Hamiltonian / no drag)

then F-P is:

(for res.)

$$\partial_t \langle f \rangle = -\frac{\partial}{\partial v}\Big[\underbrace{F\langle f\rangle}_{\text{drift/friction}} + \underbrace{\frac{1}{2}\frac{\partial}{\partial v^2} D\langle f\rangle}_{\text{diffusion}}\Big]$$

but

$$\partial_t \langle f \rangle = \partial_v D(v)\, \partial_v \langle f \rangle$$

order of derivatives!

To reconcile, for resonant particles:

$$\boxed{F = \frac{1}{2}\frac{\partial}{\partial v} D(v)}$$

(cf Landau '37)

– drift, diffusion partially cancel for Hamiltonian systems

To show:

$$v(t+\Delta t) = v(t) + \dot{v}\,\Delta t + \frac{1}{2}\ddot{v}\,(\Delta t)^2$$

$$\dot{v} = -\partial H/\partial x \qquad (m=1)$$

$$\ddot{v} = -\frac{\partial^2 H}{\partial x^2}\dot{x} - \frac{\partial^2 H}{\partial x \partial v}\dot{v} - \frac{\partial^2 H}{\partial x \partial t}$$

$$= -\frac{\partial^2 H}{\partial x^2}\frac{\partial H}{\partial v} + \left(\frac{\partial^2 H}{\partial x \partial v}\right)\frac{\partial H}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial t}\right)$$

partial cancel.

$$= -\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}\frac{\partial H}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial H}{\partial x}\right)^2 - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial t}\right)$$

then

$$\Delta v = -\frac{\partial H}{\partial x}\Delta t + \frac{1}{2}(\Delta t)^2\left[\frac{\partial}{\partial v}\left(\frac{\partial H}{\partial x}\right)^2 - \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}\frac{\partial H}{\partial u} + \frac{\partial H}{\partial t}\right)\right]$$

so $\langle \Delta v \rangle = \frac{1}{2}(\Delta t)^2 \frac{\partial}{\partial v}\left\langle \left(\frac{\partial H}{\partial x}\right)^2 \right\rangle$ (1)

$\langle \; \rangle = \int \frac{dx}{L}$

and $\Delta v \Delta v = \dot{v}^2 (\Delta t)^2 = \left(\frac{\partial H}{\partial x}\right)^2 (\Delta t)^2$

$$\langle \Delta v \Delta v \rangle = \left\langle \left(\frac{\partial H}{\partial x}\right)^2 \right\rangle \Delta t^2 \qquad (2)$$

Comparing (1), (2)

$$\frac{1}{2}\frac{\partial}{\partial v} D = F \quad \checkmark$$

Drag – Diffusion cancellation for 1D QL ⟺ Hamiltonian

① Beware order of derivatives!

② Resonant QL diffusion – Fokker-Planck

→ Linear Response – Review

– Linear response fundamental to transport, collective resonance

– Lin. resp th. → formulated at fundamental kin Eqn. (conservative) ⇔ Liouville (Vlasov)

– Linear Response Theory:

- Transport → Kubo
  Coeff ~ F.T. [correlation]
  classical, QM.
- Response/Stability → Landau
  Plasma Dielectric
  Collisionless damping
  Mean Field Evoln. (QL).
  also
  (Resonance Broadening – Notes)

see also:

$$\boxed{\text{Kadanoff + Martin, posted "Hydrodynamic Equations and Correlation Functions"}}$$