Lecture 7 (Oct. 26)

We now turn to the subject of small oscillations. We assume that the kinetic energy is homogeneous of degree two in the generalized velocities: $T = \frac{1}{2} T_{\sigma\sigma'}(q_1,\ldots,q_n)\,\dot{q}_\sigma \dot{q}_{\sigma'}$, and that the potential $U(q_1,\ldots,q_n)$ is degree zero in the $\{\dot{q}_\sigma\}$. The equations of motion are then obtained as follows:

$$L = T - U \Rightarrow \begin{cases} p_\sigma = \dfrac{\partial L}{\partial \dot{q}_\sigma} = T_{\sigma\sigma'}(q)\,\dot{q}_{\sigma'} \\[2ex] F_\sigma = \dfrac{\partial L}{\partial q_\sigma} = \dfrac{1}{2}\dfrac{\partial T_{\sigma'\sigma''}(q)}{\partial q_\sigma}\,\dot{q}_{\sigma'}\dot{q}_{\sigma''} - \dfrac{\partial U(q)}{\partial q_\sigma} \end{cases}$$

Thus, $\dot{p}_\sigma = F_\sigma$ says

$$T_{\sigma\sigma'}\ddot{q}_{\sigma'} + \left(\frac{\partial T_{\sigma\sigma'}}{\partial q_{\sigma''}} - \frac{1}{2}\frac{\partial T_{\sigma'\sigma''}}{\partial q_\sigma}\right)\dot{q}_{\sigma'}\dot{q}_{\sigma''} = -\frac{\partial U}{\partial q_\sigma}$$

This may be written as

$$T_{\sigma\alpha}\ddot{q}_\alpha + \frac{1}{2}\left(\frac{\partial T_{\sigma\mu}}{\partial q_\nu} + \frac{\partial T_{\sigma\nu}}{\partial q_\mu} - \frac{\partial T_{\mu\nu}}{\partial q_\sigma}\right)\dot{q}_\mu \dot{q}_\nu = -\frac{\partial U}{\partial q_\sigma}$$

multiply by $T^{-1}_{\lambda\sigma}$

$$\ddot{q}_\lambda + \Gamma^\lambda_{\mu\nu}\,\dot{q}_\mu \dot{q}_\nu = A_\lambda \quad , \quad \text{with}$$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} T^{-1}_{\lambda\sigma}\left(\frac{\partial T_{\sigma\mu}}{\partial q_\nu} + \frac{\partial T_{\sigma\nu}}{\partial q_\mu} - \frac{\partial T_{\mu\nu}}{\partial q_\sigma}\right) \leftarrow \text{Christoffel symbols}$$

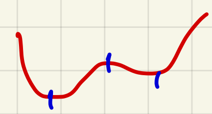
$$A_\lambda = -T^{-1}_{\lambda\sigma}\frac{\partial U}{\partial q_\sigma}$$

- Static equilibrium: $\dot{q}_\sigma = 0 \ \forall \sigma \in \{1, \ldots, n\} \Rightarrow$

$\frac{\partial U}{\partial q_\sigma} = 0 \ \forall \sigma$; n equations in n unknowns $\{q_1, \ldots, q_n\}$

Generically this has pointlike solutions, $\{\bar{q}_1, \ldots, \bar{q}_n\}$.

Let's write $q_\sigma = \bar{q}_\sigma + \eta_\sigma$ and expand the Lagrangian to quadratic order in the q_σ and $\dot{q}_\sigma$:



$$L = \frac{1}{2} T_{\sigma\sigma'} \dot{\eta}_\sigma \dot{\eta}_{\sigma'} - \frac{1}{2} V_{\sigma\sigma'} \eta_\sigma \eta_{\sigma'} + \ldots$$

where

$$\left.\begin{aligned} T_{\sigma\sigma'} &= T_{\sigma\sigma'}(\bar{q}) = \left.\frac{\partial^2 T}{\partial \dot{q}_\sigma \partial \dot{q}_{\sigma'}}\right|_{\bar{q}} \\ V_{\sigma\sigma'} &= \left.\frac{\partial^2 U}{\partial q_\sigma \partial q_{\sigma'}}\right|_{\bar{q}} \end{aligned}\right\}$$

T and V are constant, real, symmetric, $n \times n$ matrices

So to quadratic order, $L = \frac{1}{2} \dot{\eta}^t T \dot{\eta} - \frac{1}{2} \eta^t V \eta$

- Method of small oscillations

The idea here is to express the η_σ in terms of normal modes, ξ_i, which diagonalize the equations of motion,

$$T_{\sigma\sigma'} \ddot{\eta}_{\sigma'} = - V_{\sigma\sigma'} \eta_\sigma$$

This being a linear problem, we write $\eta_\sigma = A_{\sigma i} \xi_i$ and demand

$\uparrow$ $n \times n$ real matrix

$$A^t T A = \mathbb{1}$$
$$A^t V A = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2)$$

The vector form of the linearized EL eqns is

$$T\ddot{\vec{\eta}} = -V\vec{\eta}$$

so

$$TA\ddot{\vec{\xi}} = -VA\vec{\xi}$$

Multiplying on the left by A^t, we then have

$$\underbrace{(A^tTA)}_{\equiv \mathbb{1}}\ddot{\vec{\xi}} = -\underbrace{(A^tVA)}_{\equiv \mathrm{diag}(\omega_1^2,\ldots,\omega_n^2)}\vec{\xi}$$

Thus we have n decoupled second order ODEs:

$$\ddot{\xi}_i = -\omega_i^2\xi_i$$

with solutions

$$\xi_i(t) = C_i\cos(\omega_i t) + D_i\sin(\omega_i t)$$

with $2n$ constants of integration $\{C_i, D_i\}$ with $i\in\{1,\ldots,n\}$.

Note $\vec{\eta} = A\vec{\xi}$ yields $\vec{\xi} = A^{-1}\vec{\eta} = A^tT\vec{\eta}$, thus

$$\eta_\sigma(t) = \sum_i A_{\sigma i}\left[C_i\cos(\omega_i t) + D_i\sin(\omega_i t)\right]$$

Multiplying on the left by A^tT, we obtain

$$C_i\cos(\omega_i t) + D_i\sin(\omega_i t) = A^t_{i\sigma}T_{\sigma\sigma'}\eta_{\sigma'}(t)$$

and thus

$$C_i = A^t_{i\sigma}T_{\sigma\sigma'}\eta_{\sigma'}(0)$$

$$D_i = \omega_i^{-1}A^t_{i\sigma}T_{\sigma\sigma'}\dot{\eta}_{\sigma'}(0) \qquad \text{(no sum on } i\text{)}$$

At this point, we have the complete solution to the problem for arbitrary initial conditions $\{\eta_\sigma(0), \dot{\eta}_\sigma(0)\}$. The matrix $A_{\sigma i}$ is called the modal matrix. If all the generalized coordinates have dimensions $[q_\sigma] = L$,

$$[T_{\sigma\sigma'}] = M \quad , \quad [V_{\sigma\sigma'}] = \frac{E}{L^2} = \frac{M}{T^2}$$

$$[A_{\sigma i}] = M^{-1/2} \quad , \quad [\xi_i] = M^{1/2} L$$

- Why can we demand $A^t T A = \mathbb{1}$ and $A^t V A = \mathrm{diag}(\omega_1^2, \dots, \omega_n^2)$?

Proof by construction:

(i) Since $T_{\sigma\sigma'}$ is symmetric, there exists $O_1 \in O(n)$ such that $O_1^t T O_1 = T_d$, where T_d is diagonal. Additionally, the entries of T_d are all positive because the kinetic energy is in general positive (only zero if $\dot{q}_\sigma = 0 \ \forall \sigma$).

(ii) T_d being positive definite, we may construct its square root $T_d^{1/2}$ simply by taking the square root of each diagonal entry. Note then that

$$T_d^{-1/2} O_1^t T O_1 T_d^{-1/2} = T_d^{-1/2} T_d T_d^{-1/2} = \mathbb{1}$$

(iii) The matrix $T_d^{-1/2} O_1^t V O_1 T_d^{-1/2}$ is symmetric, and hence diagonalized by some $O_2 \in O(n)$. Thus, we have two matrices $\mathcal{O}_1$ and $\mathcal{O}_2$ such that

$$\mathcal{O}_2^t T_d^{-1/2} \mathcal{O}_1^t T \mathcal{O}_1 T_d^{-1/2} \mathcal{O}_2 = \mathbb{1}$$

$$\mathcal{O}_2^t T_d^{-1/2} \mathcal{O}_1^t V \mathcal{O}_1 T_d^{-1/2} \mathcal{O}_2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2)$$

Therefore the modal matrix is

$$A = \mathcal{O}_1 T_d^{-1/2} \mathcal{O}_2 \qquad \text{(NB: } A \text{ not orthogonal!)}$$

We can see that it is in general not possible to simultaneously diagonal three symmetric matrices. Two is the limit!

- How to find the modal matrix

(i) Assume $\eta_\sigma(t) = \mathrm{Re}\, \psi_\sigma e^{-i\omega t}$. Then from the EL eqn $T\ddot{\vec{\eta}} = -V\vec{\eta}$ we have $(\omega^2 T - V)_{\sigma\sigma'} \psi_{\sigma'} = 0$.

In order to have nontrivial solutions, we demand

$$\det(\omega^2 T - V) = 0$$

This yields an n^{th} order polynomial equation in ω^2. Its n roots are the n normal mode frequencies, ω_i^2.

(ii) Next, find the eigenvectors $\psi_\sigma^{(i)}$ by demanding

$$\sum_{\sigma'} (\omega_i^2 T_{\sigma\sigma'} - V_{\sigma\sigma'}) \psi_{\sigma'}^{(i)} = 0$$

Since $\omega_i^2 T - V$ is defective, these equations are $(n-1)$ inhomogeneous linear equations for $\{\psi_2^{(i)}, \ldots, \psi_n^{(i)}\}$ yielding the ratios $\{\psi_2^{(i)}/\psi_1^{(i)}, \ldots, \psi_n^{(i)}/\psi_1^{(i)}\}$. It then follows (see §5.3.3) that $\psi_\sigma^{(i)} T_{\sigma\sigma'} \psi_{\sigma'}^{(j)} = 0$ if $i \neq j$. In fact, this is only guaranteed if $\omega_i^2 \neq \omega_j^2$, but for degenerate eigenvalues $\omega_i^2 = \omega_j^2$, we may still choose the eigenvectors to be orthogonal (wrt T) via the Gram–Schmidt process. Finally, we may choose to normalize each eigenvector, so that

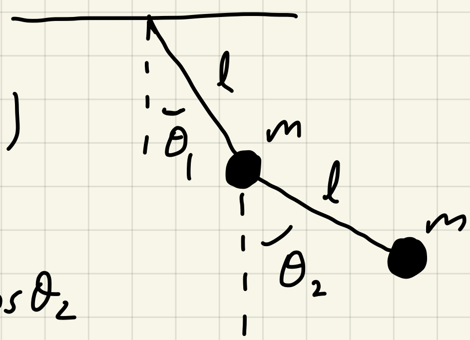
$$\langle \psi^{(i)} | \psi^{(j)} \rangle \equiv \psi_\sigma^{(i)} T_{\sigma\sigma'} \psi_{\sigma'}^{(j)} = \delta_{ij}$$

(iii) The modal matrix is then given by $A_{\sigma i} = \psi_\sigma^{(i)}$.

(iv) Since $\vec{\eta} = A\vec{\xi}$ and $A^t T A = \mathbb{1}$, $A^{-1} = A^t T$ and $\vec{\xi} = A^t T \vec{\eta}$.

- Example: the double pendulum

(For simplicity, choose $\ell_1 = \ell_2 = \ell$, $m_1 = m_2 = m$)



$x_1 = \ell \sin\theta_1 \,, \quad y_1 = -\ell\cos\theta_1$

$x_2 = \ell\sin\theta_1 + \ell\sin\theta_2 \,, \quad y_2 = -\ell\cos\theta_1 - \ell\cos\theta_2$

$$T = \frac{1}{2} m (\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) = \frac{1}{2} m\ell^2 (2\dot{\theta}_1^2 + 2\cos(\theta_1 - \theta_2)\,\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2)$$

$$V = -mg\ell(2\cos\theta_1 + \cos\theta_2) \,; \quad \text{equilibrium @ } \theta_1 = \theta_2 = 0$$

$$T = \begin{pmatrix} 2m\ell^2 & m\ell^2 \\ m\ell^2 & m\ell^2 \end{pmatrix} \,, \quad V = \begin{pmatrix} 2mg\ell & 0 \\ 0 & mg\ell \end{pmatrix}$$

Let $\omega_0^2 \equiv g/\ell$. Then

$$\omega^2 T - V = m\ell^2 \begin{pmatrix} 2\omega^2 - 2\omega_0^2 & \omega^2 \\ \omega^2 & \omega^2 - \omega_0^2 \end{pmatrix}$$

$$\det(\omega^2 T - V) = (m\ell^2)^2 \cdot \left\{ 2(\omega^2 - \omega_0^2)^2 - \omega^4 \right\}$$

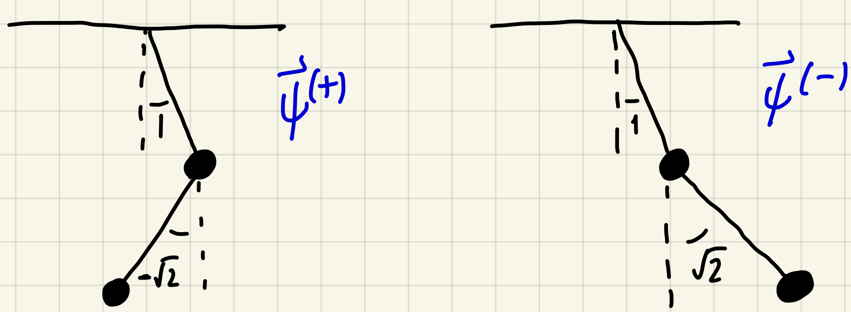
Setting $\det(\omega^2 T - V) = 0$ then yields $\omega_\pm^2 = (2 \pm \sqrt{2})\,\omega_0^2$.

Find

$$A = \begin{pmatrix} \psi_1^{(+)} & \psi_1^{(-)} \\ \psi_2^{(+)} & \psi_2^{(-)} \end{pmatrix} = \frac{1}{2\sqrt{m\ell^2}} \begin{array}{c} \begin{array}{cc} i=+ & i=- \end{array} \\ \begin{pmatrix} \sqrt{2+\sqrt{2}} & \sqrt{2-\sqrt{2}} \\ -\sqrt{2}\cdot\sqrt{2+\sqrt{2}} & \sqrt{2}\cdot\sqrt{2-\sqrt{2}} \end{pmatrix} \end{array} \begin{array}{l} \\ \sigma = 1 \\ \sigma = 2 \end{array}$$

Note that $\vec{\psi}^{(+)} \propto \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$ and $\vec{\psi}^{(-)} \propto \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$

Normal mode shapes:



In the low frequency normal mode, the two masses oscillate in phase, while in the high frequency normal mode, they are π out of phase.

- Zero modes

Recall that to each continuous one-parameter family of coordinate transformations

$$q_\sigma \to \tilde{q}_\sigma(q,\zeta) \quad , \quad \tilde{q}_\sigma(q,\zeta=0) = q_\sigma$$

leaving L invariant corresponds a conserved "charge",

$$\Lambda = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} \quad , \quad \frac{d\Lambda}{dt} = 0$$

Let us label the various one-parameter invariances with a label k. For small oscillations,

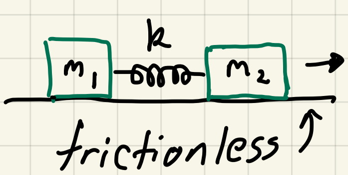
$$\frac{\partial L}{\partial \dot{q}_\sigma} = \frac{\partial L}{\partial \dot{\eta}_\sigma} = T_{\sigma\sigma'}\,\dot{\eta}_{\sigma'}$$

which says $C_{k\sigma} = \sum_{\sigma'} T_{\sigma\sigma'} \frac{\partial \tilde{q}_\sigma}{\partial \zeta_k}\bigg|_{\vec{\zeta}=0}$ so that

$$\xi_k = \sum_\sigma C_{k\sigma}\,\eta_\sigma$$

is a zero mode, satisfying $\ddot{\xi}_k = 0$. (As written it is unnormalized. Thus, in systems with continuous symmetries, associated with each such symmetry is a zero mode of the corresponding small oscillations problem.

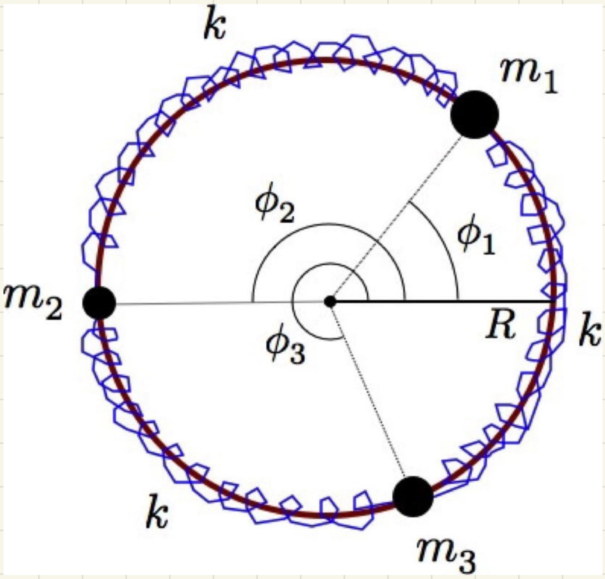
Example 1 : $L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k(x_2 - x_1 - a)^2$



$$= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\mu\dot{x}^2 - \frac{1}{2}k(x-a)^2 \Rightarrow X \text{ (CM) is a ZM}$$

$$X = \tfrac{1}{2}(x_1 + x_2) \quad , \quad x = x_2 - x_1$$

Example 2



Consider the system to the right, for which

$$T = \frac{1}{2}R^2\left(m_1\dot{\phi}_1^2 + m_2\dot{\phi}_2^2 + m_3\dot{\phi}_3^2\right)$$

and

$$U = \frac{1}{2}kR^2\left[(\phi_2-\phi_1-\chi)^2 + (\phi_3-\phi_2-\chi)^2 + (2\pi+\phi_1-\phi_3-\chi)^2\right]$$

where $\phi_3 - 2\pi < \phi_1 < \phi_2 < \phi_3 < \phi_1 + 2\pi$, and where $R\chi \equiv a$ is the unstretched length of each spring. The equilibrium configuration is

$$\phi_1^0 = \zeta \quad , \quad \phi_2^0 = \zeta + \frac{2\pi}{3} \quad , \quad \phi_3^0 = \zeta + \frac{4\pi}{3}$$

where ζ is an arbitrary continuous parameter, corresponding to the continuous translational symmetry that is present. Find

$$T = \begin{pmatrix} m_1R^2 & 0 & 0 \\ 0 & m_2R^2 & 0 \\ 0 & 0 & m_3R^2 \end{pmatrix} \quad , \quad V = \begin{pmatrix} 2kR^2 & -kR^2 & -kR^2 \\ -kR^2 & 2kR^2 & -kR^2 \\ -kR^2 & -kR^2 & 2kR^2 \end{pmatrix}$$

and

$$\omega^2 T - V = kR^2 \begin{pmatrix} \frac{\omega^2}{\nu_1^2} - 2 & 1 & 1 \\ 1 & \frac{\omega^2}{\nu_2^2} - 2 & 1 \\ 1 & 1 & \frac{\omega^2}{\nu_3^2} - 2 \end{pmatrix} \quad , \quad \nu_j^2 \equiv \frac{k}{m_j}$$

The characteristic polynomial is

$$P(\omega^2) = \det(\omega^2 T - V) \equiv (kR^2)^3 \cdot \tilde{P}(\omega^2)$$

$$\tilde{P}(\omega^2) = \frac{\omega^6}{\nu_1^2\nu_2^2\nu_3^2} - 2\left(\frac{1}{\nu_1^2\nu_2^2} + \frac{1}{\nu_2^2\nu_3^2} + \frac{1}{\nu_3^2\nu_1^2}\right)\omega^4 + 3\left(\frac{1}{\nu_1^2} + \frac{1}{\nu_2^2} + \frac{1}{\nu_3^2}\right)\omega^2$$

This is cubic in ω^2, but since there is no $(\omega^2)^0$ term, ω^2 divides $\tilde{P}(\omega^2)$, i.e. $\tilde{P}(\omega^2) = \omega^2\,\tilde{Q}(\omega^2)$, where $\tilde{Q}(\omega^2)$ is a quadratic function of its argument. Thus the normal mode frequencies are

$$\omega_1^2 = 0$$

$$\omega_{2,3}^2 = \nu_1^2 + \nu_2^2 + \nu_3^2 \pm \frac{1}{4}\sqrt{(\nu_1^2-\nu_2^2)^2 + (\nu_2^2-\nu_3^2) + (\nu_3^2-\nu_1^2)^2}$$

To find the modal matrix, set $(\omega_j^2 T - V)\,\psi^{(j)} = 0$:

$$\begin{pmatrix} \frac{\omega_j^2}{\nu_1^2} - 2 & 1 & 1 \\ 1 & \frac{\omega_j^2}{\nu_2^2} - 2 & 1 \\ 1 & 1 & \frac{\omega_j^2}{\nu_3^2} - 2 \end{pmatrix} \begin{pmatrix} \psi_1^{(j)} \\ \psi_2^{(j)} \\ \psi_3^{(j)} \end{pmatrix} = 0$$

which yields $\psi_\sigma^{(j)} = C_j \Big/ \left(3 - \frac{\omega_j^2}{\nu_\sigma^2}\right)$, where

$$C_j = \left[\sum_{\sigma=1}^{3} m_\sigma \left(3 - \frac{\omega_j^2}{\nu_\sigma^2}\right)^{-2}\right]^{-1/2} \quad \text{for normalization.}$$

Note for the zero mode ($j=1$) we have

$$A_{\sigma 1} = \psi_\sigma^{(1)} = \frac{C_1}{3} = (m_1+m_2+m_3)^{-1/2} \quad \forall\, \sigma \in \{1,2,3\}$$

Thus,

$$\begin{aligned} \xi_1 &= A_{1\sigma} T_{\sigma\sigma'} \eta_{\sigma'} \\ &= (m_1+m_2+m_3)^{-1/2} R^2 (m_1 \eta_1 + m_2 \eta_2 + m_3 \eta_3) \end{aligned}$$

is the normalized zero mode. This is consistent with Noether's theorem, which says

$$\Lambda = \sum_{\sigma=1}^{3} \frac{\partial L}{\partial \dot\phi_\sigma} \frac{\partial \tilde\phi_\sigma}{\partial \zeta} = R^2 (m_1 \dot\phi_1 + m_2 \dot\phi_2 + m_3 \dot\phi_3)$$

with $\dot\Lambda = 0$. Note that $\dot\Lambda = 0$ always, and not only in the limit of small deviations from static equilibrium.

- Chain of identical masses and springs

tension

$$L = \frac{1}{2} m \sum_\sigma \dot x_\sigma^2 - \frac{1}{2} k \sum (x_{\sigma+1} - x_\sigma - a)^2 + \tau \sum_n (x_{\sigma+1} - x_\sigma)$$

Clearly $p_\sigma = \frac{\partial L}{\partial \dot x_\sigma} = m \dot x_\sigma$. If the chain is finite, with n running from 1 to N, then

$$F_1 = \frac{\partial L}{\partial x_1} = k(x_2 - x_1 - a) - \tau$$

$$F_N = \frac{\partial L}{\partial x_N} = -k(x_N - x_{N-1} - a) + \tau$$

$$F_\sigma = \frac{\partial L}{\partial x_\sigma} = k(x_{\sigma+1} + x_{\sigma-1} - 2x_j) \qquad \sigma \in \{2, \ldots, N-1\}$$

The last equation says that $F_\sigma = 0\ \forall\ \sigma \in \{1,\ldots,N\}$ if

$$x_{\sigma+1} - x_\sigma = b \quad , \quad \sigma \in \{1,\ldots,N-1\}$$

where b is a constant. Plugging this into the first equations then yields $b = a + k^{-1}\tau$.

If the chain is a periodic ring with $x_{N+1} \equiv x_1 + C$, then $b = C/N$ is the only solution. We'll solve the problem in this case of periodic boundary conditions (PBCs). In the limit $N \to \infty$, the bulk behavior wont differ between the two cases. Writing

$$x_\sigma = \sigma b + u_\sigma + \xi \qquad \sigma \in \{1,\ldots,N\}$$

we have

$u_{N+1} \equiv u_1$

$$L = \frac{1}{2} m \sum_{\sigma=1}^{N} \dot{u}_\sigma^2 - \frac{1}{2} k \sum_{\sigma=1}^{N} (u_{\sigma+1} - u_\sigma)^2 - k(b-a)C - \frac{1}{2} N k (b-a)^2$$

The last two terms arise when $b \neq a$ due to the fact that the springs are all (equally) stretched in the static equilibrium configuration. These terms are both constants which we henceforth drop. The EL equations are then

$$m\ddot{u}_\sigma = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_\sigma}\right) = \frac{\partial L}{\partial u_\sigma} = k(u_{\sigma+1} + u_{\sigma-1} - 2u_\sigma)$$

with $u_{N+1} \equiv u_1$. These N coupled ODEs may easily be solved

$$k(u_{\sigma+1} - u_\sigma) - k(u_\sigma - u_{\sigma-1})$$

$\sigma \longrightarrow \qquad \longleftarrow \sigma$

by transforming to Fourier space coordinates, viz.

$$u_\sigma = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{2\pi i j\sigma/N} \hat{u}_j \iff \hat{u}_j = \frac{1}{\sqrt{N}} \sum_{\sigma=1}^{N} e^{-2\pi i j\sigma/N} u_\sigma$$

Note that $\hat{u}_j$ is complex, with

$$\hat{u}_{N-j} = \frac{1}{\sqrt{N}} \sum^{N} e^{2\pi i j\sigma/N} u_\sigma = \hat{u}_j^*$$

Let's count degrees of freedom. The set $\{u_1, \ldots, u_N\}$ constitutes N real degrees of freedom. For N even, $\hat{u}_N$ and $\hat{u}_{N/2}$ are real, while $\hat{u}_j$ for $j \in \{1, \ldots, \frac{1}{2}N-1\}$ are complex and satisfy $\mathrm{Re}\,\hat{u}_{N-j} = \mathrm{Re}\,\hat{u}_j$ and $\mathrm{Im}\,\hat{u}_{N-j} = -\mathrm{Im}\,\hat{u}_j$. The number of real degrees of freedom is then

$$\mathrm{DOF} = 2 + 2\times\left(\tfrac{1}{2}N - 1\right) = N \quad \checkmark$$

If N is odd, then $\hat{u}_N$ is again real, but there is no mode $\hat{u}_j$ with $j = \frac{1}{2}N$. We again have $\hat{u}_{N-j} = \hat{u}_j^*$, this time for $j \in \{1, \ldots, \frac{1}{2}(N-1)\}$. The number of real degrees of freedom is

$$\mathrm{DOF} = 1 + 2 \times \tfrac{1}{2}(N-1) = N \quad \checkmark$$

We now have

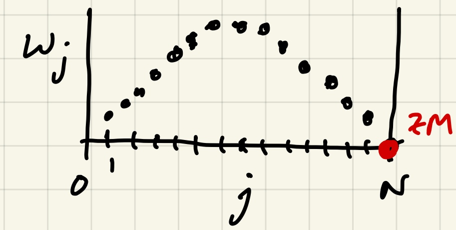
$$m \frac{1}{\sqrt{N}} \sum_{\sigma=1}^{N} e^{-2\pi i j\sigma/N} \ddot{u}_\sigma = k \frac{1}{\sqrt{N}} \sum_{\sigma=1}^{N} e^{-2\pi i j\sigma/N} (u_{\sigma+1} + u_{\sigma-1} - 2u_\sigma)$$

$$m\ddot{\hat{u}}_j = -2k\left[1 - \cos\left(\frac{2\pi j}{N}\right)\right] \hat{u}_j$$

Thus we may write $\ddot{\hat{u}}_j = -\omega_j^2 \hat{u}_j$ with

$$\omega_j = 2\sqrt{\frac{k}{m}}\,\left|\sin\left(\frac{\pi j}{N}\right)\right|$$

$j = N$ is ZM



The solution for each normal mode is

$$\hat{u}_j(t) = C_j\, e^{-i\omega_j t}\, e^{i\delta_j}$$

where $C_{N-j} = C_j$ and $\delta_{N-j} = -\delta_j$ for all $j \notin \{\frac{N}{2}, N\}$, and $\delta_{N/2} = \delta_N = 0$. The $\{C_j, \delta_j\}$ are all real constants

The modal matrix is then $A_{\sigma j} = \frac{1}{\sqrt{Nm}}\, e^{2\pi i j \sigma / N}$, where we have now included the $m^{-1/2}$ factor. Note

$$T_{\sigma\sigma'} = m\,\delta_{\sigma\sigma'}$$
$$V_{\sigma\sigma'} = 2k\,\delta_{\sigma\sigma'} - k\,\delta_{\sigma',\sigma+1} - k\,\delta_{\sigma',\sigma-1}$$

the Kronecker deltas are understood to be modulo N, i.e.

$$\delta_{\sigma\sigma'} = \begin{cases} 1 & \text{if } \sigma' = \sigma \bmod N \\ 0 & \text{otherwise} \end{cases}$$

Thus, the matrix forms of T and V are

$$T = \begin{pmatrix} m & 0 & & & \\ 0 & m & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & m & 0 \\ & & & 0 & m \end{pmatrix}, \quad V = \begin{pmatrix} 2k & -k & 0 & \cdots & 0 & -k \\ -k & 2k & -k & 0 & \cdots & 0 \\ 0 & -k & 2k & -k & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & & -k & 2k & -k \\ -k & 0 & \cdots & 0 & -k & 2k \end{pmatrix}$$

Using the equation $\frac{1}{N}\sum_{\sigma=1}^{N} e^{2\pi i(j-j')\sigma/N} = \delta_{jj'}$ we can prove that $A^t T A = \mathbb{1}$ and $A^t V A = \mathrm{diag}(\omega_1^2, \dots, \omega_N^2)$.

Continuum limit: We take

$$u_\sigma(t) \to u(x=\sigma b, t)$$

and

$$u_{\sigma+1} - u_\sigma = u(x+b) - u(x) = b\frac{\partial u}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 u}{\partial x^2} + \dots$$

Thus,

$$T = \frac{1}{2}m\sum_\sigma \dot{u}_\sigma^2 \longrightarrow \frac{1}{2}m\int\frac{dx}{b}\left(\frac{\partial u}{\partial t}\right)^2$$

$$V = \frac{1}{2}k\sum_\sigma (u_{\sigma+1}-u_\sigma)^2 \longrightarrow \frac{1}{2}k\int\frac{dx}{b}\left(b\frac{\partial u}{\partial x}\right)^2 + \dots$$

and we may write

$$S = \int dt\, L(\{u_\sigma\}, \{\dot{u}_\sigma\}, t) = \int dt \int dx\, \mathcal{L}(u, \partial_x u, \partial_t u, t)$$

where

$$\mathcal{L}(u, \partial_x u, \partial_t u, t) = \frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau\left(\frac{\partial u}{\partial x}\right)^2$$

with $\rho = m/b$ = mass density and $\tau = kb$ = "tension" is the **Lagrangian density**. Suppose the Lagrangian is of the form

$$L = \sum_\sigma L_\sigma\Big(u_\sigma, \dot{u}_\sigma, \overbrace{\frac{u_{\sigma+1}-u_\sigma}{b}}^{\equiv u'_\sigma}, t\Big)$$

We have

$$L = \sum_\sigma L_\sigma\Big(u_\sigma, \dot{u}_\sigma, \overbrace{\frac{u_{\sigma+1}-u_\sigma}{b}}^{\equiv u'_\sigma}, t\Big)$$

The EL eqns are then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_\sigma}\right) = \frac{\partial L}{\partial u_\sigma} = \frac{\partial L_\sigma}{\partial u_\sigma} + \frac{1}{b}\frac{\partial L_{\sigma-1}}{\partial u'_\sigma} - \frac{1}{b}\frac{\partial L_\sigma}{\partial u'_\sigma}$$

Now

$$\frac{(\partial L_\sigma/\partial u'_\sigma) - (\partial L_{\sigma-1}/\partial u'_\sigma)}{b} = \frac{\partial}{\partial x}\frac{\partial L_\sigma}{\partial u'_\sigma} + \dots$$

and writing

$$L_\sigma\Big(u_\sigma, \dot{u}_\sigma, \frac{u_{\sigma+1}-u_\sigma}{b}, t\Big) \equiv \frac{1}{b}\mathcal{L}\Big(u_\sigma, \dot{u}_\sigma, \overbrace{\frac{u_{\sigma+1}-u_\sigma}{b}}^{\equiv u'_\sigma}, \overbrace{\sigma b}^{x}, t\Big)$$

$$= \frac{1}{b}\mathcal{L}(u, \partial_t u, \partial_x u, x, t)$$

we have

$$S = \int dt \int dx\, \mathcal{L}(u, \partial_t u, \partial_x u, x, t)$$

and the equations of motion

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \partial_t u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \partial_x u}\right) = \frac{\partial \mathcal{L}}{\partial u}$$

More about this in chapter 9 of the lecture notes.