**PHYSICS 200A : CLASSICAL MECHANICS**
**MIDTERM EXAMINATION SOLUTIONS**
Normative time limit: four hours (consecutive!)
*You are allowed to consult the online PHYS 200A course materials.*
All problems are worth a total of 50 points each.

[**1**] A uniformly dense ladder of mass $m$ and length $2\ell$ leans against a block of mass $M$, as shown in Fig. 1. Choose as generalized coordinates the horizontal position $X$ of the right end of the block, the angle $\theta$ the ladder makes with respect to the floor, and the coordinates $(x, y)$ of the ladder's center-of-mass. These four generalized coordinates are not all independent, but instead are related by a certain set of constraints.

Recall that the kinetic energy of the ladder is $T_{\rm CM} + T_{\rm rot}$, where $T_{\rm CM} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ is the kinetic energy of the center-of-mass motion, and $T_{\rm rot} = \frac{1}{2}I\dot{\theta}^2$, where $I$ is the moment of inertial. For a uniformly dense ladder of length $2\ell$, the moment of inertia is $I = \frac{1}{3}m\ell^2$.

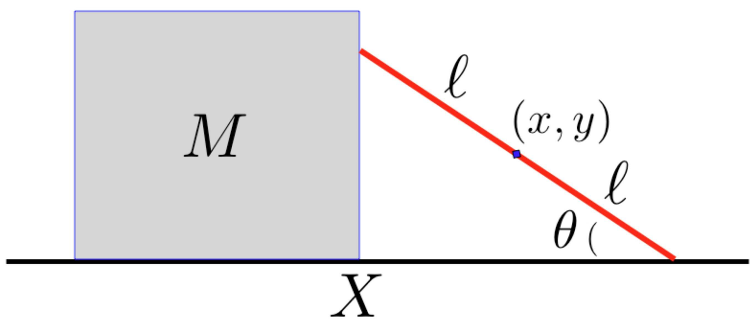


Figure 1: A ladder of length $2\ell$ leaning against a massive block. All surfaces are frictionless.

(a) Write down the Lagrangian for this system in terms of the coordinates $X$, $\theta$, $x$, $y$, and their time derivatives. `[10 points]`

We have $L = T - U$, hence

$$L = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\dot{\theta}^2 - mgy \quad .$$

(b) Write down all the equations of constraint. `[10 points]`

There are two constraints, corresponding to contact between the ladder and the block, and contact between the ladder and the horizontal surface:

$$G_1(X,\theta,x,y) = x - \ell\cos\theta - X = 0$$
$$G_2(X,\theta,x,y) = y - \ell\sin\theta = 0 \quad .$$

(c) Write down all the equations of motion. `[10 points]`

Two Lagrange multipliers, $\lambda_1$ and $\lambda_2$, are introduced to effect the constraints. We have for each generalized coordinate $q_\sigma$,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial q_\sigma} \equiv Q_\sigma \quad ,$$

where there are $k = 2$ constraints. We therefore have

$$\begin{aligned} M\ddot{X} &= -\lambda_1 \\ m\ddot{x} &= +\lambda_1 \\ m\ddot{y} &= -mg + \lambda_2 \\ I\ddot{\theta} &= \ell \sin\theta \, \lambda_1 - \ell \cos\theta \, \lambda_2 \quad . \end{aligned}$$

These four equations of motion are supplemented by the two constraint equations, yielding six equations in the six unknowns $\{X, \theta, x, y, \lambda_1, \lambda_2\}$.

(d) Find all conserved quantities. `[10 points]`

The Lagrangian and all the constraints are invariant under the transformation

$$X \to X + \zeta \quad , \quad x \to x + \zeta \quad , \quad y \to y \quad , \quad \theta \to \theta \quad .$$

The associated conserved 'charge' is

$$\Lambda = \frac{\partial L}{\partial \dot{q}_\sigma} \, \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} = M\dot{X} + m\dot{x} \quad .$$

Using the first constraint to eliminate $x$ in terms of $X$ and $\theta$, we may write this as

$$\Lambda = (M + m)\dot{X} - m\ell \sin\theta \, \dot{\theta} \quad .$$

The second conserved quantity is the total energy $E$. This follows because the Lagrangian and all the constraints are independent of $t$, and because the kinetic energy is homogeneous of degree two in the generalized velocities. Thus,

$$\begin{aligned} E &= \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\dot{\theta}^2 + mgy \\ &= \frac{\Lambda^2}{2(M+m)} + \tfrac{1}{2}\Big(I + m\ell^2 - \tfrac{m}{M+m}\, m\ell^2 \sin^2\!\theta\Big)\,\dot{\theta}^2 + mg\ell \sin\theta \quad , \end{aligned}$$

where the second line is obtained by using the constraint equations to eliminate $x$ and $y$ in terms of $X$ and $\theta$.

(e) Find an equation relating the angle $\theta^*$ at which the ladder detaches from the block and the initial angle of inclination $\theta_0$. Your equation should only include $\theta^*$, $\theta_0$, and the dimensionless ratios $M/m$ and $I/m\ell^2$, but not $\dot{\theta}$ or $\ddot{\theta}$. *Hint: Find the energy of the system at the moment of detachment.* `[10 points]`

The condition for detachment from the block is simply $\lambda_1 = 0$, *i.e.* the normal force vanishes. If we eliminate $x$ and $y$ in terms of $X$ and $\theta$, we find

$$
\begin{aligned}
x &= X + \ell\cos\theta & y &= \ell\sin\theta \\
\dot{x} &= \dot{X} - \ell\sin\theta\,\dot{\theta} & \dot{y} &= \ell\cos\theta\,\dot{\theta} \\
\ddot{x} &= \ddot{X} - \ell\sin\theta\,\ddot{\theta} - \ell\cos\theta\,\dot{\theta}^2 & \ddot{y} &= \ell\cos\theta\,\ddot{\theta} - \ell\sin\theta\,\dot{\theta}^2 \quad .
\end{aligned}
$$

We can now write

$$\lambda_1 = m\ddot{x} = m\ddot{X} - m\ell\sin\theta\,\ddot{\theta} - m\ell\cos\theta\,\dot{\theta}^2 = -M\ddot{X} \quad ,$$

which gives

$$(M+m)\ddot{X} = m\ell\big(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\big) \quad ,$$

and hence

$$Q_x = \lambda_1 = -\frac{Mm}{m+m}\,\ell\,\big(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\big) \quad .$$

We also have

$$
\begin{aligned}
Q_y = \lambda_2 &= mg + m\ddot{y} \\
&= mg + m\ell\big(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\big) \quad .
\end{aligned}
$$

We now need an equation relating $\ddot{\theta}$ and $\dot{\theta}$. This comes from the last of the equations of motion:

$$
\begin{aligned}
I\ddot{\theta} &= \ell\sin\theta\,\lambda_1 - \ell\cos\theta\lambda_2 \\
&= -\tfrac{Mm}{M+m}\,\ell^2\big(\sin^2\!\theta\,\ddot{\theta} + \sin\theta\cos\theta\,\dot{\theta}^2\big) - mg\ell\cos\theta - m\ell^2\big(\cos^2\!\theta\,\ddot{\theta} - \sin\theta\cos\theta\,\dot{\theta}^2\big) \\
&= -mg\ell\cos\theta - m\ell^2\Big(1 - \tfrac{m}{M+m}\,\sin^2\!\theta\Big)\,\ddot{\theta} + \tfrac{m}{M+m}\,m\ell^2\,\sin\theta\cos\theta\,\dot{\theta}^2 \quad .
\end{aligned}
$$

Collecting terms proportional to $\ddot{\theta}$, we obtain

$$\big(I + m\ell^2 - \tfrac{m}{M+m}\,\sin^2\!\theta\big)\,\ddot{\theta} = \tfrac{m}{M+m}\,m\ell^2\,\sin\theta\cos\theta\,\dot{\theta}^2 - mg\ell\,\cos\theta \quad .$$

We are now ready to demand $Q_x = \lambda_1 = 0$, which entails

$$\ddot{\theta} = -\frac{\cos\theta}{\sin\theta}\,\dot{\theta}^2 \quad .$$

Substituting this into the previous equation, we obtain

$$\big(I + m\ell^2\big)\,\dot{\theta}^2 = mg\ell\,\sin\theta \quad .$$

Finally, we substitute this into the equation for the total energy $E$, we obtain the following relation between the detachment angle, $\theta^*$, and the initial angle of inclination, $\theta_0$:

$$E - \frac{\Lambda^2}{2(M+m)} = mg\ell\sin\theta_0 = \left(3 - \frac{m}{M+m}\cdot\frac{m\ell^2}{I+m\ell^2}\,\sin^2\!\theta^*\right)\cdot\tfrac{1}{2}mg\ell\,\sin\theta^* \quad .$$

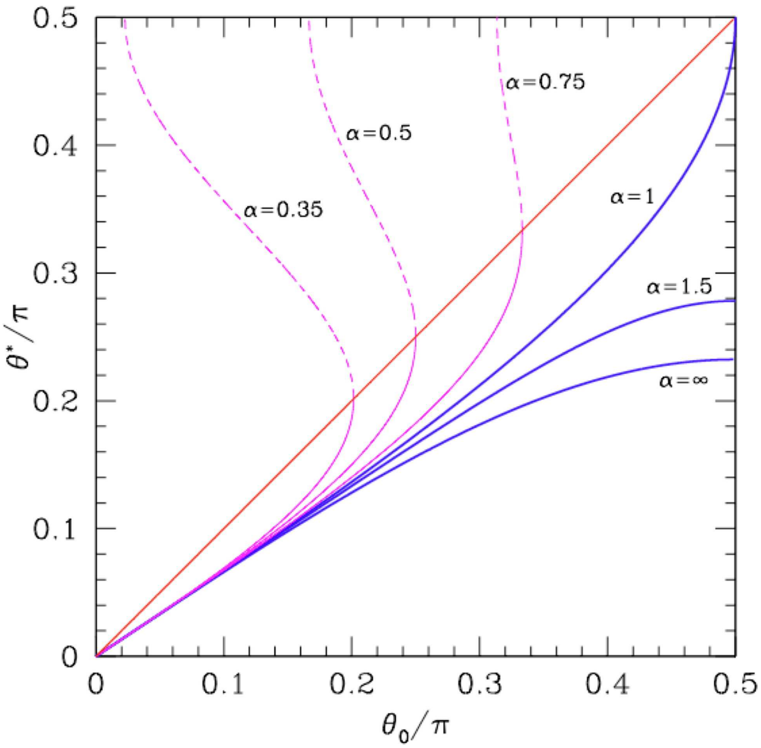


Figure 2: Plot of $\theta^*$ *versus* $\theta_0$ for the ladder-block problem. Allowed solutions, shown in blue, have $\alpha \geq 1$, and thus $\theta^* \leq \theta_0$. Unphysical solutions, with $\alpha < 1$, are shown in magenta. The line $\theta^* = \theta_0$ is shown in red.

Thus, we have

$$\sin\theta_0 = \tfrac{3}{2}\sin\theta^* - \tfrac{1}{2}\left(\frac{m}{M+m}\right)\left(\frac{m\ell^2}{I+m\ell^2}\right)\sin^3\!\theta^* = \tfrac{3}{2}\sin\theta^* - \tfrac{1}{2}\,\alpha^{-1}\sin^3\!\theta^* \quad ,$$

where

$$\alpha \equiv \left(1+\frac{M}{m}\right)\left(1+\frac{I}{m\ell^2}\right) \quad .$$

Note that $\alpha \geq 1$, and that when $M/m = \infty$,[1] we recover $\theta^* = \sin^{-1}\!\big(\tfrac{2}{3}\sin\theta_0\big)$, which is the angle of detachment relation for a ladder slipping against a rigid wall. For finite $\alpha$, the ladder detaches at a larger value of $\theta^*$. A sketch of $\theta^*$ *versus* $\theta_0$ is provided in Fig. 2. Note that, provided $\alpha \geq 1$, detachment always occurs for some unique value $\theta^*$ for each $\theta_0$.

**[2]** Two identical semi-infinite lengths of string are joined at a point of mass $m$ which moves vertically along a thin wire, as depicted in fig. 3. The mass moves with friction coefficient $\gamma$, *i.e.* its equation of motion is

$$m\ddot{z} + \gamma\dot{z} = F \quad ,$$

[1] The ladder's rotational inertia must satisfy $I \leq m\ell^2$.

where $z$ is the vertical displacement of the mass, and $F$ is the force on the mass due to the string segments on either side. In this problem, gravity is to be neglected. It may be convenient to define $K \equiv 2\tau/mc^2$ and $Q \equiv \gamma/mc$.

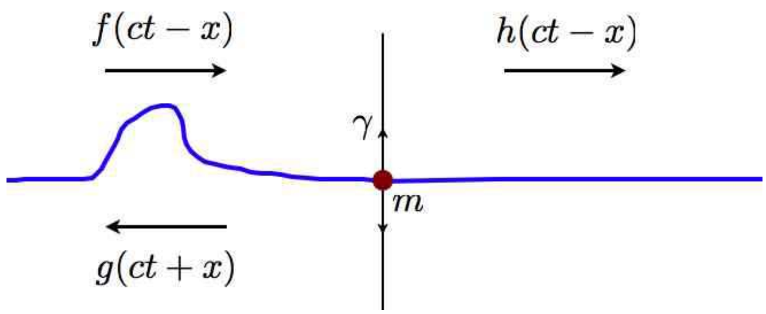


Figure 3: A point mass $m$ joining two semi-infinite lengths of identical string moves vertically along a thin wire with friction coefficient $\gamma$.

(a) The general solution with an incident wave from the left is written

$$y(x,t) = \begin{cases} f(ct-x) + g(ct+x) & (x<0) \\ h(ct-x) & (x>0) \end{cases} \quad .$$

Find two equations relating the functions $f(\xi)$, $g(\xi)$, and $h(\xi)$. `[15 points]`

The first equation is continuity at $x=0$:

$$f(\xi) = g(\xi) + h(\xi) \quad ,$$

where $\xi = ct$ ranges over the real line $[-\infty, \infty]$. The second equation comes from Newton's 2nd law $F = ma$ applied to the mass point:

$$m\,\ddot{y}(0,t) + \gamma\,\dot{y}(0,t) = \tau\,y'(0^+,t) - \tau\,y'(0^-,t) \quad .$$

Expressed in terms of the functions $f(\xi)$, $g(\xi)$, and $h(\xi)$, and dividing through by $mc^2$, this gives

$$f''(\xi) + g''(\xi) + Q\,f'(\xi) + Q\,g'(\xi) = -\tfrac{1}{2}\,K\,h'(\xi) + \tfrac{1}{2}\,K\,f'(\xi) - \tfrac{1}{2}\,K\,g'(\xi).$$

Integrating once, and invoking $h = f + g$, this second equation becomes

$$f'(\xi) + Q\,f(\xi) = -g'(\xi) - (K+Q)\,g(\xi) \quad .$$

(b) Solve for the reflection amplitude $r(k) = \hat{g}(k)/\hat{f}(k)$ and the transmission amplitude $t(k) = \hat{h}(k)/\hat{f}(k)$. Recall that

$$f(\xi) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\,\hat{f}(k)\,e^{ik\xi} \qquad \Longleftrightarrow \qquad \hat{f}(k) = \int\limits_{-\infty}^{\infty} d\xi\,f(\xi)\,e^{-ik\xi} \quad ,$$

*et cetera* for the Fourier transforms. Also compute the sum of the reflection and transmission coefficients, $\big|r(k)\big|^2 + \big|t(k)\big|^2$. Show that this sum is always less than or equal to unity, and interpret this fact. `[15 points]`

Using $d/d\xi \longrightarrow ik$, we have

$$(Q + ik)\,\hat{f}(k) = -(K + Q + ik)\,\hat{g}(k) \quad .$$

Therefore,

$$r(k) = \frac{\hat{g}(k)}{\hat{f}(k)} = -\frac{Q + ik}{Q + K + ik} \quad .$$

To find the transmission amplitude, we invoke $h(\xi) = f(\xi) + g(\xi)$, in which case

$$t(k) = \frac{\hat{h}(k)}{\hat{f}(k)} = -\frac{K}{Q + K + ik} \quad .$$

The sum of reflection and transmission coefficients is

$$\big|r(k)\big|^2 + \big|t(k)\big|^2 = \frac{Q^2 + K^2 + k^2}{(Q+K)^2 + k^2} \quad .$$

Clearly the RHS of this equation is bounded from above by unity, since both $Q$ and $K$ are nonnegative.

(c) Find an expression which is a functional of $f(x)$ or $\hat{f}(k)$, for the total energy change $\Delta E$ of the string due to the friction acting on the mass point. *Hint: You can compute $\Delta E$ by computing the net outgoing energy current at $x = 0^{\pm}$ and then integrating over time.* `[10 points]`

Recall the formulae for the energy density in a string,

$$\mathcal{E}(x,t) = \tfrac{1}{2}\,\mu\,\dot{y}^2(x,t) + \tfrac{1}{2}\,\tau\,y'^2(x,t)$$

and

$$j_{\mathcal{E}}(x,t) = -\tau\,\dot{y}(x,t)\,y'(x,t) \quad .$$

The energy continuity equation is $\partial_t\,\mathcal{E} + \partial_x\,j_{\mathcal{E}} = 0$. Assuming $j_{\mathcal{E}}(\pm\infty,t) = 0$, we have

$$\begin{aligned}\frac{dE}{dt} &= \int\limits_{-\infty}^{0^-}\!dx\,\frac{\partial\mathcal{E}}{\partial t} + \int\limits_{0^+}^{\infty}\!dx\,\frac{\partial\mathcal{E}}{\partial t}\\ &= -j_{\mathcal{E}}(\infty,t) + j_{\mathcal{E}}(0^+,t) + j_{\mathcal{E}}(-\infty,t) - j_{\mathcal{E}}(0^-,t) \quad .\end{aligned}$$

Thus,

$$\frac{dE}{dt} = c\tau\Big(\big[g'(ct)\big]^2 + \big[h'(ct)\big]^2 - \big[f'(ct)\big]^2\Big)$$

is the rate at which the string loses energy. We now integrate over all time, obtaining the total energy change in the string:

$$\begin{aligned}\Delta E &= \tau \int\limits_{-\infty}^{\infty} d\xi \left( \big[g'(\xi)\big]^2 + \big[h'(\xi)\big]^2 - \big[f'(\xi)\big]^2 \right) \\ &= -\tau \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi} \, \frac{2QK\,k^2}{(Q+K)^2 + k^2} \, \big|\hat{f}(k)\big|^2 \quad .\end{aligned}$$

(d) For an incident wave whose characteristic wavelength $\lambda$ satisfies $K\lambda \gg 1$ and $Q\lambda \gg 1$, find the ratio $|\Delta E|/E_0$, where $E_0$ is the initial energy in the string. `[10 points]`

Note that the initial energy in the string, at time $t = -\infty$, is

$$E_0 = \tau \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi} \, k^2 \, \big|\hat{f}(k)\big|^2 \quad .$$

If the incident wave packet is very broad, say described by a Gaussian $f(\xi) = A\,\exp(-x^2/2\sigma^2)$ with $\sigma K \gg 1$ and $\sigma Q \gg 1$, then $k^2$ may be neglected in the denominator of the integrand for $\Delta E$, in which case

$$\frac{|\Delta E|}{E_0} \approx \frac{2QK}{(Q+K)^2} \le \tfrac{1}{2}E \quad .$$

For the Lorentzian,

$$\hat{f}(k) = \frac{2\beta}{k^2 + \beta^2} \qquad \Longleftrightarrow \qquad f(\xi) = \exp\big(-\beta|\xi|\big) \quad ,$$

we have the exact results

$$E_0 = \tfrac{1}{2}\beta\tau \qquad , \qquad \Delta E = -\frac{KQ\beta\tau}{(K+Q+\beta)^2} \quad .$$

**[3]** Consider the map

$$\begin{aligned} q_{n+1} &= q_n + f(q_n, p_{n+1}) \\ p_{n+1} &= p_n + g(q_n, p_{n+1}) \quad . \end{aligned}$$

(a) Under what conditions does this map generate a canonical transformation $(q_n, p_n) \to (q_{n+1}, p_{n+1})$? `[10 points]`

According to §16.1.2, the conditions are

$$\frac{\partial f}{\partial q_n} = -\frac{\partial g}{\partial p_{n+1}} \quad .$$

To see this explicitly, take the differentials:

$$dq_{n+1} = dq_n + \frac{\partial f}{\partial q_n}\, dq_n + \frac{\partial f}{\partial p_{n+1}}\, dp_{n+1}$$
$$dp_{n+1} = dp_n + \frac{\partial g}{\partial q_n}\, dq_n + \frac{\partial g}{\partial p_{n+1}}\, dp_{n+1} \quad .$$

Bringing all differentials with iteration subscript $n+1$ to the left and all with subscript $n$ to the right, we have

$$\begin{pmatrix} 1 & -\frac{\partial f}{\partial p_{n+1}} \\ 0 & 1 - \frac{\partial g}{\partial p_{n+1}} \end{pmatrix} \begin{pmatrix} dq_{n+1} \\ dp_{n+1} \end{pmatrix} = \begin{pmatrix} 1 + \frac{\partial f}{\partial q_n} & 0 \\ \frac{\partial g}{\partial q_n} & 1 \end{pmatrix} \begin{pmatrix} dq_n \\ dp_n \end{pmatrix} \quad .$$

Thus

$$\begin{pmatrix} dq_{n+1} \\ dp_{n+1} \end{pmatrix} = \frac{1}{1-g_p} \begin{pmatrix} \big\{(1+f_q)(1-g_p) + g_q\, f_p\big\} & f_p \\ g_q & 1 \end{pmatrix} \begin{pmatrix} dq_n \\ dp_n \end{pmatrix} \equiv \begin{pmatrix} dq_n \\ dp_n \end{pmatrix} \quad .$$

The Poisson bracket we seek is

$$\big\{q_{n+1}, p_{n+1}\big\}_{\{q_n, p_n\}} = \frac{\partial q_{n+1}}{\partial q_n}\frac{\partial p_{n+1}}{\partial p_n} - \frac{\partial q_{n+1}}{\partial p_n}\frac{\partial p_{n+1}}{\partial q_n} = \det M = 1 \quad ,$$

and thus the map is canonical.

(b) Show that the conditions in part (a) are satisfied if $f$ and $g$ are expressed as first (partial) derivatives of a function $R(q_n, p_{n+1})$. `[10 points]`

If

$$f(q_n, p_{n+1}) = \frac{\partial R(q_n, p_{n+1})}{\partial p_{n+1}} \quad , \qquad g(q_n, p_{n+1}) = -\frac{\partial R(q_n, p_{n+1})}{\partial q_n} \quad ,$$

then

$$\frac{\partial f}{\partial q_n} = -\frac{\partial g}{\partial p_{n+1}} = \frac{\partial^2 R(q_n, p_{n+1})}{\partial q_n\, \partial p_{n+1}} \quad .$$

(c) For the map

$$q_{n+1} = q_n + b\, q_n + c\, p_{n+1}$$
$$p_{n+1} = p_n - a\, q_n - b\, p_{n+1} \quad ,$$

where $a$, $b$, and $c$ are constants, what is the function $R(q_n, p_{n+1})$ from part (b)? `[10 points]`

For

$$f(q_n, p_{n+1}) = b\, q_n + c\, p_{n+1}$$
$$g(q_n, p_{n+1}) = -a\, q_n - b\, p_{n+1}$$

we have

$$R(q_n, p_{n+1}) = \tfrac{1}{2} a\, q_n^2 + b\, q_n\, p_{n+1} + \tfrac{1}{2} c\, p_{n+1}^2 \quad .$$

(d) Express the map in part (c) as $\boldsymbol{\varphi}_{n+1} = \hat{T}\boldsymbol{\varphi}_n$, where $\boldsymbol{\varphi}_n = \begin{pmatrix} q_n \\ p_n \end{pmatrix}$. Find an explicit expression for $\hat{T}$. `[10 points]`

From the equation $p_{n+1} = p_n - a\, q_n - b\, p_{n+1}$ we obtain

$$p_{n+1} = \frac{p_n - a\, q_n}{1+b} \quad .$$

Substitute this into $q_{n+1} = q_n + b\, q_n + c\, p_{n+1}$ to obtain the linear map

$$\begin{pmatrix} q_{n+1} \\ p_{n+1} \end{pmatrix} = \overbrace{\begin{pmatrix} 1 + b - \frac{ac}{1+b} & \frac{c}{1+b} \\ -\frac{a}{1+b} & \frac{1}{1+b} \end{pmatrix}}^{T} \begin{pmatrix} q_n \\ p_n \end{pmatrix} \quad .$$

Thus the action of the map $\hat{T}$ is the action of the $2 \times 2$ matrix $T$ on the vector of phase space coordinates; the map is linear.

(e) For fixed $b > 0$, plot the phase diagram in the $(a, c)$ plane, identifying regions where $|\hat{T}^n \boldsymbol{\varphi}_0|$ grows exponentially with $n$ (for generic initial conditions $\boldsymbol{\varphi}_0$), and regions where it is bounded. Sketch your results. `[10 points]`

The characteristic polynomial of a $2 \times 2$ matrix such as $T$ is

$$P(\lambda) = \det(\lambda \mathbb{I} - T) = \lambda^2 - \tau\lambda + \Delta \quad ,$$

where $\tau = \mathsf{Tr}\, T$ and $\Delta = \det T$. Since $\Delta = 1$ as the map is canonical, the eigenvalues of $T$ are

$$\lambda_\pm = \tfrac{1}{2}\tau \pm \tfrac{1}{2}\sqrt{\tau^2 - 4} \quad ,$$

where $\tau = \mathsf{Tr}\, T$. Provided $|\tau| < 2$, the eigenvalues are phases: $\lambda_\pm = \exp\big[\pm\, i \cos^{-1}(\tau/2)\big]$. In this case, there is no growth of the iterates $\boldsymbol{\varphi}_n$. When $|\tau| > 2$ both eigenvalues are real, with $\lambda_\pm = \exp\big[\pm \cosh^{-1}(\tau/2)\big] \operatorname{sgn}\tau$, and this if the initial vector $\boldsymbol{\varphi}_{n=0}$ has any overlap with the eigenvector corresponding to $\lambda_+$, $|\boldsymbol{\varphi}_n|$ will grow exponentially for large $n$ as $|\lambda_+|^n$. In our case,

$$\tau = \mathsf{Tr}\, T = 1 + b + \frac{1 - ac}{1+b} \quad .$$

Setting $\tau = 2$ then yields the condition

$$\tau = 1 + b + \frac{1 - ac}{1+b} = 2 \quad \Longrightarrow \quad ac = b^2 \quad .$$

Setting $\tau = -2$ yields

$$\tau = 1 + b + \frac{1 - ac}{1+b} = -2 \quad \Longrightarrow \quad ac = (b+2)^2 \quad .$$

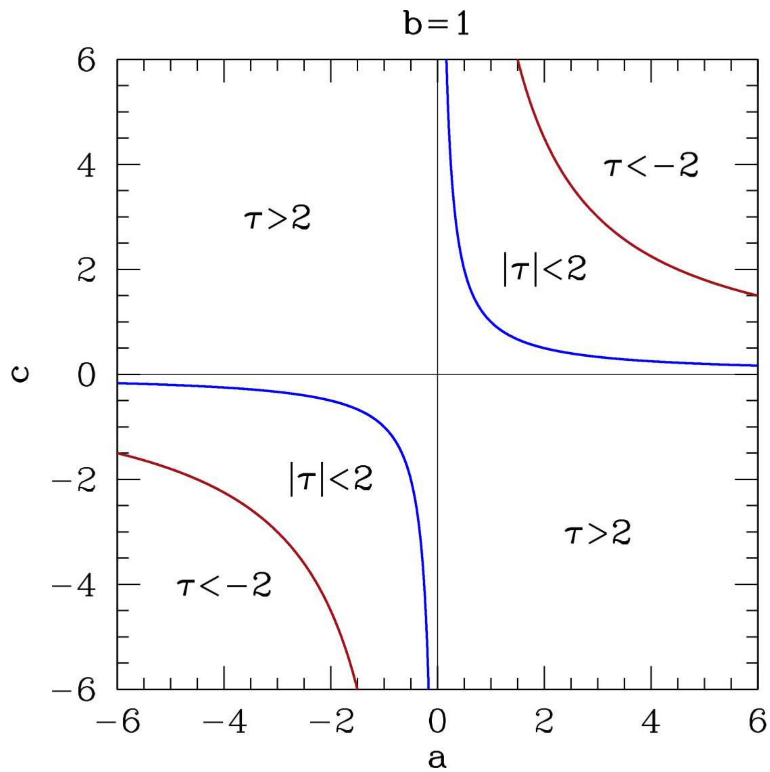


Figure 4: Solution to problem 3(e).

Thus the "no growth" region in which $|\lambda_\pm| = 1$ lies between the hyperbolae $ac = b^2$ and $ac = (b+2)^2$.

**[4]** Consider the Hamiltonian for one-dimensional particle motion in a gravitational field,

$$H(z,p) = \overbrace{\frac{p^2}{2m} + mgz}^{H_0} + \overbrace{\varepsilon\alpha z^3}^{\varepsilon H_1} \quad ,$$

where $\varepsilon$ is small. The particle is constrained such that $z \geq 0$. It msy be useful to consult §15.5.5 of the Lecture Notes.

(a) Find the unperturbed Hamiltonian $\widetilde{H}_0(J_0)$ and the unperturbed frequency $\nu_0(J_0)$. `[15 points]`

We have

$$H_0 = \frac{1}{2m}\left(\frac{\partial W}{\partial z}\right)^2 + mgz \equiv E$$

from which we obtain

$$p = \frac{\partial W}{\partial z} = \pm\sqrt{2m(E - mgz)} \quad \Rightarrow \quad W(z) = \text{const.} \mp \frac{\sqrt{8}}{3\sqrt{m}\,g}\big(E - mgz\big)^{3/2} \quad .$$

The amplitude of the oscillations is $h = E/mg$. Thus, the action is

$$J_0 = \frac{1}{\pi}\int\limits_0^h dz\,\sqrt{2m(E - mgz)} = \frac{\sqrt{8}\,E^{3/2}}{3\pi\sqrt{m}\,g} \quad .$$

Thus,

$$\widetilde{H}_0(J_0) = E = \tfrac{1}{2}\big(3\pi g\sqrt{m}\,\big)^{2/3}\,J_0^{2/3} \quad .$$

The frequencies $\nu_0(J_0)$ are given by

$$\nu_0(J_0) = \tfrac{1}{3}\big(3\pi g\sqrt{m}\big)^{2/3}J_0^{-1/3} \quad .$$

(b) Find the unperturbed frequencies $\nu_0(h)$, where $h$ is the amplitude of the $z$ motion. Your result should look familiar. `[15 points]`

To express in terms of the amplitude $h$, we note

$$h(J_0) = \frac{E}{mg} = \frac{(3\pi)^{2/3}}{2\,m^{2/3}g^{1/3}}\,J_0^{2/3}$$

and therefore

$$\nu_0(h) = \pi\sqrt{\frac{g}{2h}} = \frac{\pi}{T} \quad ,$$

where $T = \sqrt{2h/g}$ is the time to fall from $h$.

(c) Find the energy $E(J)$ to lowest nontrivial order in $\varepsilon$. `[20 points]`

To find the perturbed frequencies, we must express $H_1 = \alpha z^3$ in terms of $(J_0, \phi_0)$. The first order of business, then, is to obtain $\phi_0 = \partial F_2(J_0, z)/\partial J_0$, where

$$F_2(J_0, z) = W(J_0, z) = \mp \pi J_0 \left(1 - \frac{z}{h(J_0)}\right)^{3/2} \quad .$$

The top sign corresponds to the part of the motion where $\dot{z} > 0$ and the bottom sign when $\dot{z} < 0$. We obtain

$$\phi_0 = \mp\pi\left(1 - \frac{z}{h(J_0)}\right)^{1/2} \quad .$$

Note that $\phi_0$ advances from $-\pi$ to $0$ to $+\pi$ as $z$ moves from $z = 0$ to $z = h$ and back down to $z = 0$. Thus,

$$z = h\left(1 - \frac{\phi_0^2}{\pi^2}\right) \quad ,$$

and

$$\langle z^3 \rangle = h^3 \int\limits_0^\pi \frac{d\phi_0}{\pi}\left(1 - \frac{\phi_0^2}{\pi^2}\right)^3 = \tfrac{16}{35}\,h^3 \quad .$$

Therefore,

$$\left\langle \widetilde{H}_1(J,\phi_0)\right\rangle_{\phi_0} = \tfrac{16}{35}\,\alpha h^3(J)$$

and, to first order in $\varepsilon$,

$$E(J) = mgh(J) + \tfrac{16}{35}\varepsilon\,\alpha h^3(J) + \mathcal{O}(\varepsilon^2) \quad ,$$

where the function $h(J)$ is as above: $h(J) = \frac{1}{2}\big(3\pi/m\sqrt{g}\big)^{2/3} J^{2/3}$. While the above expression yields $E(J)$ to $\mathcal{O}(\varepsilon^1)$, this particular relation between the amplitude $h$ and the action $J$ is valid only to $\mathcal{O}(\varepsilon^0)$.