Lecture VII  Waves: Stability and Structure

Recall D'Alembert's Paradox

\[ F_d = \int_{A_1} \left( P_1 + \rho U_1^2 \right) \, dA_1 - \int_{A_2} \left( P_2 + \rho U_2^2 \right) \, dA_2 \]

- For an ideal fluid, upstream/downstream symmetry of flow
  - i.e. \( P_1 = P_2 \)
  - \( U_1 = U_2 \)

so \( F_d = 0 \)

i.e., effect of flow is enhanced inertia (induced mass)
- For viscous fluid (i.e. no slip boundary condition) → upstream/downstream asymmetry

\[ \rightarrow \text{wake} \]

\[ \rightarrow \text{Drag occurs} \]

Aim is to understand structure and dynamics of wake.

\[ \rightarrow \text{Reali: No-slip B.C.'s} \]

\[ \rightarrow \text{Separation occurs} \]

\[ \rightarrow \text{potential flow outside wake} \]

\[ \rightarrow \text{hole fills in to form wake} \]

\[ \rightarrow \text{line of separation} \]
Separation is unstable

Kinetic - Helmholtz Instability

$\text{Re} = \text{Re}^{*}$

$V_x = V$

$\partial V / \partial x = 0$

$\Omega = \text{D} \times \text{B} = 0$

$\text{except at interface}$

$\text{Interface}$

$\text{ripples} = \text{dynamic b.c.}$

$\text{Can treat as potential flow}$

$\text{in region 0}$

$\text{and match}$

$\text{at}$

$\text{interface}$
Physical ideas:

1. High $u$, low $p$
2. Zero $u$, high $p$

A perturbation at the ripple surface:
$p + \frac{\beta u^2}{2} = \text{const.}$

$\sigma v < 0$ - Ripple brings in slow fluid
- Flow in region of drops

$\sigma v > 0$ - Ripple brings in fast fluid
- Flow in region of bubbles

But $\frac{\partial p}{\partial x} > 0$ → Bernoulli pressure increases
- Pressure decreases
\[ \sigma > 0 \Rightarrow \sigma V < 0 \text{ Further } \]
\[ \sigma < 0 \Rightarrow \sigma V > 0 \text{ Further } \]
reinforcing thermal perturbation!

N.B. \( k-H \) instability driven viscous, mixing via turbulence, mixing billows, etc.

To calculate:

\[ \frac{1}{\rho_1} \frac{d}{dx} \left( \frac{1}{\rho_1} \frac{d}{dx} \right) \rho_1 + \rho_1 \frac{V^2}{2} = \text{const} \]

\[ \rho_2 \]

\[ \frac{d}{dx} V = 0 \]
\[ V = \nabla \phi \quad \omega = 0, \text{ except interface} \]
\[ \nabla^2 \phi = 0 \quad (\omega^2 - k^2) \phi = 0 \text{ wave along interface (symmetry)} \]

\[ \phi = \sum \phi_n e^{ikx} e^{-i\omega t} e^{-ixt} \quad \text{decay away from interface} \]
match the conditions

→ pressure continuity

\[ \bar{p}_1 (x, y, z) = \bar{p}_2 (x, y, z) \]

→ \( \phi \) continuity

\[ \phi (x, y, z) = \phi (x, y, z) \]

\[ \text{e.g. } - \rho = \rho \frac{\partial \phi}{\partial x} + \rho \left( \frac{\partial \phi}{\partial y} \right)^2 \]

→

\[ \frac{\partial \phi}{\partial z} \bigg|_0 \bigg| = \frac{\partial \phi}{\partial z} \bigg|_0 \bigg| \]

\[ \text{e.g. } \frac{\partial^2 \phi}{\partial z^2} - k^2 \phi = 0 \]

and, \( \int_{-a}^{a} \frac{\partial^2 \phi}{\partial z^2} \bigg| \bigg. = 0 \Rightarrow \frac{\partial \phi}{\partial z} \bigg|_0 - \frac{\partial \phi}{\partial z} \bigg|_0 = 0 \)

Now,

\[ \nabla \cdot \vec{V} + \nabla \cdot \vec{V} = \frac{-\partial \rho}{\partial \rho} \]

\[ \rho_1 \left( \frac{\partial \vec{V}_1}{\partial x} + \vec{V} \cdot \frac{\partial \vec{V}_1}{\partial x} \right) = - \frac{\partial \rho}{\partial z} \]

\[ \rho_2 \left( \frac{\partial \vec{V}_2}{\partial x} \right) = - \frac{\partial \rho}{\partial z} \]
\[ \vec{u}_{21} = -ie^{-jk} \rho_1 \]
\[ \vec{u}_{22} = ie^{-jk} \rho_2 \]
\[ \nabla \rho \nabla \omega \]

Now, dynamic boundary:
\[ M = M(x,t) \quad \text{displacement} \]

\[ \frac{d\vec{u}}{dt} = \vec{u}_{21} \]

And
\[ \partial_t \vec{\eta} + V \partial_x \vec{\eta} = \frac{d\vec{\eta}}{dt} \]

\[-ie^{-jk} \omega - ku \vec{\eta} = \vec{u}_{22}, \quad n \]

\[ -ie^{-jk} \omega - ku \nabla \vec{\eta} = -ie^{-jk} \frac{\rho_1}{\rho_1 (\omega - ku - \omega)} \]

\[ \vec{\rho}_1 = -ie^{-jk} \frac{\omega}{k} \]
\[ p_2 = \frac{\sigma_1 \omega^2}{k} \]

and \[ p_1 = p_2 \Rightarrow \]

\[ \frac{- \sigma_1 (kv - \omega)^2}{k} = \frac{\sigma_2 \omega^2}{k} \]

and finally,

\[ \omega = kv \left( \frac{\sigma_1 + i (\sigma_1 - \sigma_2)}{\sigma_1 + \sigma_2} \right)^{1/2} \]

\[ \Rightarrow \sigma \sim kv (\sigma_1 \sigma_2) \]

\[ \frac{\sigma_1}{\sigma_1 + \sigma_2} \]

\[ \Rightarrow \omega_{\text{real}} \sim kv \left[ \frac{\sigma_1}{(\sigma_1 + \sigma_2)} \right] \]

\[ \Rightarrow \text{no exchange of stability here.} \]

\[ \Rightarrow \sigma_1 = \sigma_2, \quad \gamma = kv \]
what happens?

vortex roll-up, billows

(see Felskovitch E=2.3 F2.4)

vortex streets etc.

N.B. Vorticity concentrated on layer of interface.

More generally:

\[
\gamma^2 = \frac{54h}{(\rho+\rho_0)^2} \left( \frac{v_2-v_0}{2} \right)^2 + \frac{\rho_1-a}{\rho+\rho_0} \left( \frac{\rho_1-a}{\rho+\rho_0} \right) \]

\[
- \frac{7h^3}{(\rho+\rho_0)^2} \quad \text{Rayleigh-Taylor}
\]

\[
\text{Surface tension}
\]

\[
\text{Capillarity}
\]

Threshold wind to excite waves
Surface Tension

- An important property of interface is surface tension.
  i.e. - force due to decrease in surface area of interface.

- Familiar from droplets, capillary waves, etc.

\[
\delta F = -p_1 \, dV - p_2 \, (-dV) + \gamma \, dA
\]

\[\text{change in free energy}\]
\[ dV = dA \, dm \]

\[ dA = \int dx \, dy \left( 1 + \left( \frac{\partial A}{\partial x} \right)^2 + \left( \frac{\partial A}{\partial y} \right)^2 \right)^{1/2} \]

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\[ = \int dx \, dy \]

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\[ \text{i.e. - displacement expands area} \]

\[ x, y \text{ parametrize surface} \]

\[ \text{so, for small step:} \]

\[ dA = \int dx \, dy \pm \left[ \frac{\partial A}{\partial x} \right]^2 + \left( \frac{\partial A}{\partial y} \right]^2 \]

\[ = \int dx \, dy \left( \mp \frac{\partial A}{\partial y} \right) \, dm \]

\[ dF = \int dx \, dy \, dm \left[ \mathbf{F}_n - \mathbf{F}_1 \right] \]
An criterion for equilibrium:

\[ p_2 - p_1 = \sqrt{\mathbf{D}^2} \]

\[ \delta F = (p_2 - p_1) \, dA \, dM + \nabla \cdot \mathbf{D} \]

More generally:

Now consider (i.e. not "weakly curved" interface)

\[ d\sigma = (R_0 + dM) \, d\theta \]

\[ = d\sigma_0 \left( 1 + \frac{dM}{R_1} \right) \]

Radial curvature of interface as shown.

In general, surface parametrized by 2 radii' curvature, \( R_1, R_2 \) (Gauss):

\[ dA = \int d\theta_1 \, d\theta_2 \left( 1 + \frac{dM}{R_1} \right) \left( 1 + \frac{dM}{R_2} \right) \]

\[ = \int d\theta_1 \, d\theta_2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \]
\[
dF = \int \text{d}y \left[ (p_2 - p_1) + \gamma \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \right] \text{d}y
\]

For equilibrium with interface (general)

\[
\sqrt{\frac{1}{R_1} + \frac{1}{R_2}} = p_1 - p_2
\]

Laplace's Law

N.B.

Given 2-phase equilibria (separate domains), can use Laplace Law to estimate droplet size for immiscible liquids

\[ R = \sqrt{\frac{p_1 - p_2}{\gamma}} \]

For SW, \( R = \frac{\rho}{\rho - \rho_l} \frac{d}{g} \frac{\Delta h^2}{\rho_h} \)

\[ \omega^2 \approx \nu \left( \frac{\rho_h - \rho_l}{\rho_h + \rho_l} \right) + \frac{\Delta h^3}{(\rho_h + \rho_l)} \]
For $\rho_1 > \rho_2$

$$\omega^2 = gh + \frac{\sigma}{\rho} \frac{h^2}{\rho} \rightarrow \text{Gravity wave}$$

$$\rightarrow \text{Capillary wave}$$

- Crossover at few cm.
- Capillary effects important at \( \leq 5 \text{ cm} \).