Physics 216

Lecture II - Ideal Fluids
(Read Landau)

- Equations
- Basic Concepts, especially Kelvin's Theorem
- Induced Mass

I.) Euler Equations / Ideal Fluids

Ideal - "The Flow of Any Water" blob
(Feynman)

\[ \frac{\partial}{\partial t} \int_V \rho \, dV = \int_V \rho \, \nabla \cdot \mathbf{v} \, dV \]

- argue macroscopically but really derive from Boltzmann Equation
- viscosity brings additional time scale.

= mass conservation

\[ \frac{\partial m}{\partial t} = \frac{\partial}{\partial t} \int \rho \, dV = -\int \rho \, \nabla \cdot \mathbf{v} \, dV \]

= \int \rho \, \nabla \cdot \mathbf{v} \cdot (dV) \]
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \]

\[ \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \nabla \mathbf{p} = \mathbf{f} \]

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**Momentum Conservation**

\[ \mathbf{f} = -\nabla \mathbf{p} + \mathbf{f}_{\text{body}} \]

*blob/element* 

\[ \mathbf{f}_{\text{body force}} \]

*body force*

\[ \nabla \mathbf{p} \]

*pressure gradient*

\[ \rho \mathbf{g} \]

*body force*

\[ \mathbf{I} \times \mathbf{B}/c \]

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In terms of:

\[ \mathbf{a} = -\nabla \mathbf{p} + \mathbf{f} \]

*acceleration.*

\[ \mathbf{g} = \frac{\partial \mathbf{u}}{\partial t} \rightarrow \text{"substantive derivative"} \]

\[ \mathbf{d} \left( \mathbf{u} \times \mathbf{d} \right) \]

*local acceleration*

\[ \mathbf{d} \left( \mathbf{u} \times \mathbf{d} \right) \]

*displacement*

\[ \mathbf{d} \mathbf{p} \rightarrow \mathbf{d} \mathbf{u} \]

*particle move in inhomogeneous velocity field*
\[
\frac{d\rho}{dt} + \rho \frac{du}{dt} \cdot \nabla \cdot \mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}
\]

Euler Eqn.

\[\rho \frac{d\rho}{dt} = \rho \left( \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho \right) = -\nabla \cdot \mathbf{F} \]

Momentum Flux

We'll show:

\[
\frac{d}{dt} (\rho \mathbf{v}) = -\nabla \cdot \mathbf{T}^\mathbf{v} \]

\[
\frac{d}{dt} (\rho \mathbf{v}) = \nabla \cdot \left( \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right)
\]

\[
= -\nabla \left( \rho (\mathbf{v} \cdot \nabla \mathbf{v}) + \mathbf{v} \cdot \nabla \rho \right)
\]

\[
+ \rho \left( -\mathbf{v} \cdot \nabla \mathbf{v} - \frac{\partial \mathbf{v}}{\partial t} \rho \right)
\]

\[
= -\left( \rho \left[ \nabla (\mathbf{v} \cdot \nabla \mathbf{v}) + \mathbf{v} \cdot \nabla \mathbf{v} \right] + \nabla (\mathbf{v} \cdot \nabla \rho) \right) - \nabla \cdot \mathbf{F} \]
\[ \frac{\partial}{\partial t} \rho \mathbf{v} = - \mathbf{\nabla} \cdot (\rho \mathbf{v} \mathbf{v} + \mathbf{\tau}) \]

\[ \mathbf{\tau} \rightarrow \text{identity} \]

\[ \mathbf{\tau} \rightarrow \text{Reynolds stress tensor} \]

\[ (\text{analogue to Maxwell stress tensor}) \]

\[ \Pi_{ik} = \partial_{jk} v_k + \partial_{jk} P \]

\[ \text{momentum flux} \]

\[ \frac{\partial}{\partial t} \int d^3x \ (\rho \mathbf{v}) = \frac{d}{dt} \int d^3x \ (P) = -\int dS \cdot (\rho \mathbf{v} \mathbf{v} + \mathbf{\tau}) \]

\[ \text{change in momentum of blob} \]

\[ \Pi_{in} dS_n \equiv \text{momentum flux in } i\text{th direction} \]

Beyond Euler, viscous stress appears due to momentum flux from collisions interacting with microscopic flow gradients.

For incompressible flow \((\rho \cdot \mathbf{v} = 0)\), continuity and Euler/Navier-Stokes describe flow.
Mass, Momentum and Energy

In ideal fluid, no heat exchanged between fluid elements = motion adiabatic - i.e. entropy conserved along trajectories.

\[ \frac{dS}{dt} = 0 \]

\[ S = \frac{\text{entropy}}{\text{mass}} \]

\[ \frac{\partial S}{\partial t} + \nabla \cdot (\mathbf{V} S) = 0 \] → adiabatic equation for fluid

For energy flux

\[ E = \left( \frac{\rho v^2}{2} \right) + \rho e \]

\[ E \] total energy density
\[ \rho \] fluid density
\[ v \] kinetic energy density
\[ e \] internal energy density (i.e. thermal).

Then use dynamics + thermodynamics to derive total energy balance equation.
\[ \partial_t \left( \frac{\rho u^2}{2} + p e \right) + \nabla \cdot \left( \rho u \left( \frac{u^2}{2} + w \right) \right) = 0 \]

\[ w = e + \frac{p}{\rho} \]

**Enthalpy**

\[ \int \delta s \cdot \left( \frac{\rho u^2}{2} + p e \right) = -\int \delta s \cdot \left[ \rho u \left( \frac{u^2}{2} + w \right) \right] \]

What does this mean?

\[ \frac{\partial j}{\partial t} = \rho u \left( \frac{u^2}{2} + w \right) \]

Energy flux density

- What does it mean?

\[ w = e + \frac{p}{\rho} \]

\[ \delta s \cdot j = \delta s \cdot \rho u \left( \frac{u^2}{2} + e \right) \]

\[ + \int \delta s \cdot \rho u \frac{p}{\rho} \]
\[ \mathbf{a} = \int_{\partial S} \mathbf{v} \cdot \mathbf{P} \]
\[ = \int \mathbf{(v \cdot ds)} \mathbf{P} \rightarrow PdV \text{ work by pressure on fluid in blob} \]

\[ \mathbf{a} = \text{transport of energy thru the surface of the blob} \]

Rate change of energy density
\[ = \mathbf{a} + \mathbf{b} \]

To show:
\[ \frac{\partial E}{\partial t} = \frac{\partial}{\partial t} \left( \frac{\rho v^2}{2} + p E \right) \]
\[ \mathbf{a} = \frac{v^2}{2} \frac{\partial p}{\partial t} + \rho v \cdot \frac{\partial v}{\partial t} \]
\[ = -\frac{\nu^2}{2} \, \frac{\partial}{\partial t} (\rho \mathbf{u}) - \mathbf{u} \cdot \nabla \rho - \rho \mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \]

**continuity**  
**mom. balance**

But

\[ \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \times \mathbf{u} + \frac{D (\nu^2)}{2} \]

**\( \mathbf{u} = \nabla \times \mathbf{u} \rightarrow \text{vorticity} \)**

\[ (\partial \nu) (\mathbf{u} \cdot \nabla \mathbf{u}) = \partial \nu \left( -\nabla \times \mathbf{u} + \frac{D \nu^2}{2} \right) \]

\[ = \rho \mathbf{u} \cdot \nabla \frac{\nu^2}{2} \]

**To deal with pressure**:

\[ d\omega = dE + d(\rho v) \]

**Enthalpy**

\[ = T dS - p dV + v dP + \rho dV \]

\[ = T dS + \frac{dP}{\rho} \]

\[ \frac{dP}{d\rho} = \rho d\omega - \rho T dS \]
thus:

\[ 1 = \frac{\partial}{\partial t} \left( \text{conservative force} \right) = -\frac{1}{2} \mathbf{v} \cdot \nabla \left( \frac{1}{2} \mathbf{v} \right) - \mathbf{p} \cdot \mathbf{V} \left( \frac{1}{2} \mathbf{v}^2 + \mathbf{w} \right) + \partial T \mathbf{V} \cdot \mathbf{V} \]

\[ 2 \quad \frac{\partial}{\partial t} (\rho \mathbf{e}) = \ldots \]

Useful to note:

\[ d\mathbf{e} = \, d\mathbf{e} - \mathbf{p} \, d\mathbf{V} = T \, dS - \mathbf{p} \, d\mathbf{V} \]

\[ \mathbf{v} = \frac{1}{\rho}, \quad \mathbf{u} = -\frac{d\mathbf{e}}{\rho^2} \]

\[ d\mathbf{e} = T \, dS + \frac{\mathbf{p}}{\rho^2} \, d\rho \]

\[ \mathbf{e} \quad d(\rho \mathbf{e}) = \left( \rho \mathbf{e} + 6 \rho \mathbf{\theta} \right) \mathbf{d} \rho + \rho T \mathbf{d} S \]
\[ w = e + pV = e + \rho / \rho_0 \]
\[ d(e) = \omega d\rho + \partial T d\delta \]

and
\[ \dot{\omega} = \frac{\partial}{\partial t} (\rho \dot{\omega}) = \omega \frac{\partial \rho}{\partial t} + \partial T \frac{\partial \delta}{\partial t} \]

\[ = -\omega \partial \cdot (\rho V) - \partial T V \cdot \partial S \]

and, combining \( \dot{\omega}, \dot{\omega} \)

\[ \dot{\omega} \left( \frac{\rho V^2}{2} + \dot{\omega} \rho \epsilon \right) = -\left( \frac{V^2}{2} + \omega \right) \partial \cdot (\rho V) \]

\[ -\omega \partial \cdot \left( \frac{V^2}{2} + \omega \right) \]

\[ = -\partial \cdot \left( \rho V \left( \frac{V^2}{2} + \omega \right) \right) \]

and
\[ \dot{\omega} \left( \frac{\rho V^2}{2} + \dot{\omega} \rho \epsilon \right) + \partial \cdot \left( \rho V \left( \frac{V^2}{2} + \omega \right) \right) = 0 \]
Basic Laws and Concepts

What about vorticity \( \omega = \nabla \times \mathbf{v} \)?

Convenient to note:

\[
dE = \frac{\partial Q}{\partial t} - p dV = T dS - p dV
\]

\( W = E + p V \) \( \rightarrow \) enthalpy

then

\[
dW = T dS + V dP = T dS + \frac{dP}{\rho}
\]

and for isentropic flow (\( dS = 0 \))

\[
\frac{dP}{\rho} = dW
\]

thus can write (in isentropic case) RHS of Euler as perfect derivative

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{\partial W}{\partial t}
\]

Then consider circulation
\[ \Gamma = \oint \mathbf{F} \cdot d\mathbf{r} \]

then

\[ \frac{d}{dt} \oint \mathbf{V} \cdot d\mathbf{r} = \oint \frac{\partial \mathbf{V}}{\partial t} \cdot d\mathbf{r} + \oint \mathbf{V} \cdot \frac{d}{dt} \mathbf{r} \]

\[ = \oint (-\nabla \mathbf{W}) \cdot d\mathbf{r} + \oint \mathbf{V} \cdot d\mathbf{V} \]

\[ = 0 \]

so

\[ \Gamma = \oint \mathbf{V} \cdot d\mathbf{r} = \text{const.} \]

for ideally isentropic fluid.

Kelvin's Thm.

\[ \text{Circulation conserved} \]

\[ \text{N.B.: broken by viscosity} \]

\[ \nabla \cdot \mathbf{V} \neq 0 \text{ irrelevant.} \]

- Analogy in mechanics is Poincare - Cartan invariant

\[ I = \oint \mathbf{P} \cdot d\mathbf{r} \]

\[ \frac{dI}{dt} = 0 \text{ for Hamiltonian system.} \]
and elementary vector calculus:

\[ \mathbf{\Gamma} = \oint \mathbf{v} \cdot d\mathbf{A} = \int_{\text{enclosed area}} \mathbf{\omega} \cdot d\mathbf{A} \]

\[ \mathbf{A} \]

\[ \mathbf{0} \times \mathbf{V} = \mathbf{W} \]

**What is vorticity:**
- describes rotation of fluid element
- \( \mathbf{W} \) is effective local angular velocity of the fluid

\[ \mathbf{\omega} = (\mathbf{\omega} \times \mathbf{r}) / 2 \]

*Vorticity is the non-trivial element in fluid dynamics.*

Vorticity is central to all interesting topics.

How evolve vorticity?
\[
\frac{\partial y}{\partial t} + u \cdot \frac{\partial y}{\partial x} = -\nabla W
\]

\[
\frac{\partial u}{\partial t} = -\nabla \times (D \times u) + \frac{D}{2} \frac{\partial u}{\partial t} = -\nabla \times u + \frac{D}{2} \frac{\partial u}{\partial t}
\]

\[
\frac{\partial v}{\partial t} - u \times w = -D \left( u + \frac{u^2}{2} \right)
\]

Magnus Force

then \( D \times \)

\[
\frac{\partial \omega}{\partial t} = D \times (u \times \omega) \rightarrow \text{induction equation}
\]

\[
= -v \cdot D \omega + \omega \cdot D v - \omega \cdot D \cdot v
\]

\[
\frac{\partial \omega}{\partial t} = \omega \cdot D v - \omega \left( D \cdot v \right)
\]

and with continuity:

\[
\frac{1}{\rho} \frac{\partial (\rho \omega)}{\partial t} = \frac{\rho D v}{\rho} \rightarrow \frac{\omega}{\rho} \text{ "Frozen-in"}
\]

Can derive Kelvin's Thm from induction eqn.
Potential Flow

C1 -> C2 -> C3

streamline

Consider streamlines

Fluid flows along these, so

\[
\frac{dx}{V_x} = \frac{dy}{V_y} = \frac{dz}{V_z}
\]

If \( \omega = 0 \) at any point on streamline, Kelvin's then \( \omega = 0 \) everywhere on line,

\( \oint \mathbf{v} \cdot d\mathbf{r} = \oint \omega \cdot ds = 0 \quad \Rightarrow \quad \forall \text{ all } C_n, \text{ along line} \)
- Flow with $\omega = 0$ everywhere is potential or irrotational flow.

Important: Fails for separation

\[\text{e.g. consider flow around sphere}\]

- Streamlines separate from the body
- Surface of tangential discontinuity appears in velocity component line of separation

\[\text{e.g. surface tangential discontinuity}\]

- Cannot infer $\oint v \cdot dl$ from $\oint v \cdot dl$, due to separation induced tangential discontinuity
- Viscosity important in boundary layer. (No slip B.C.)

Now, for isentropic fluids:

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla W
\]

Potential flow

if \( \mathbf{W} = 0 \), \( \mathbf{u} = \nabla \phi \)

Stream function

\[
\mathbf{u} \cdot \nabla \mathbf{u} = -\mathbf{u} \times \mathbf{u} + \nabla (\nabla^2 u)
\]

\[= \nabla (\nabla^2 u)\]

\[
\frac{\partial \mathbf{u}}{\partial t} + \nabla (\nabla^2 u) = -\nabla W
\]

\[
\nabla \left( \frac{\partial \phi}{\partial t} + \frac{(\nabla \phi)^2}{2} + W \right) = 0
\]
have dynamical equation for potential flow:

$$\frac{\partial \phi}{\partial t} + (D\phi)^2 + \nu = f(t)$$

defined for each stream line

- $\frac{\partial \phi}{\partial t} = 0$, recover $(ds = 0)$

- $p + \frac{v^2}{2} = \text{const. (Bernoulli Law)}$

- Potential not uniquely defined, as $v = \nabla \phi$.

Consider incompressible potential flow:

- $v = \nabla \phi$, $\nabla \cdot v = 0$

- $\nabla^2 \phi = 0$

- $\frac{\partial \phi}{\partial t} + \frac{(D\phi)^2 + p}{\rho} = f(t)$
For static flow, with gravity:

\[
\frac{V^2}{2} + \frac{P}{\rho} + gZ = \text{const}
\]

\[\text{N.B.: In potential flow, streamlines must be open}\]

\[\oint \mathbf{u} \cdot d\mathbf{l} = \oint \mathbf{u} \cdot d\mathbf{s} = 0\]
\[\mathbf{u} = 0 \text{ along line}\]

but then

\[\oint \mathbf{w} \cdot d\mathbf{s} = 0\]
\[A\]

but

\[= \oint \mathbf{v} \cdot d\mathbf{l}\]

but \(\oint \mathbf{v} \cdot d\mathbf{l} \neq 0\) \(\rightarrow\) Fluid flow

\[\Rightarrow \text{contradiction.} \Rightarrow \text{Streamlines must be open.}\]
Also, streamlines (for potential flow) should not intersect boundaries.

Generally, potential flow problems apply to infinite media, some distance from surfaces.

\[ \text{d.e.} \]

Sphere in \( V = V_0 \) flow, far locations away from sphere is typically a flow problem potential.

Aside: What does "incompressibility" mean? When is \( \nabla \cdot \mathbf{v} = 0 \) a good approximation?

\[ |V| \ll C_S \quad C_S^2 = \frac{dp}{d\rho} \]

\[ \left( \frac{\ell}{T} \right)^2 \ll C_S^2 \]

\[ \rightarrow \text{length, time scale ratio} \]
We compare terms in continuity equation:

\[
\frac{dP}{dt} = -\rho \mathbf{v} \cdot \mathbf{v}
\]

\[
\frac{d\rho}{dt} = \frac{\rho \mathbf{v}}{l}
\]

For \( \mathbf{v} \):

\[
\frac{d\mathbf{v}}{dt} = -\frac{dP}{\rho}
\]

\[
\frac{\mathbf{v}}{l} = \frac{c^2 \mathbf{A} \rho}{\epsilon l}
\]

So,

\[
\frac{dP}{\rho} = \frac{\rho \mathbf{v} \frac{c^2 \mathbf{A} \rho}{l \epsilon}}{l}
\]

Now, \( |D \cdot \mathbf{v}| \gg \left| \frac{\mathbf{v}}{e \frac{dP}{dt}} \right| \)

means \( D \cdot \mathbf{v} = 0 \), to good approximation.
So, incompressible if:

\[
\frac{\gamma c^2}{\rho} \Delta P \gg \frac{\rho \Delta \xi}{T}
\]

\[\Rightarrow \quad c_s^2 \gg \frac{u^2}{\gamma \rho}\]

- criteria on terms length
- time scales of flow.

\[\Rightarrow \quad c_s^2 \gg \frac{u^2}{\gamma \rho}\]

Note: Long time favors incompressible.

\[\Rightarrow \quad \nabla \cdot \mathbf{u} = 0 \quad \text{if}\]

- Flow speeds SUBSONIC
- Times slow compared to time to traverse a spatial scale at acoustic speed.
\[ \nabla = \left( \frac{\partial \psi}{\partial x} - i \frac{\partial \psi}{\partial y} \right) \nabla = i \left( -\psi \nabla^2 \psi \right) \]

\[ \frac{d \nabla}{dt} = 0 \Rightarrow \left\{ \begin{array}{l}
\psi + i \nabla \psi \times \nabla \cdot \nabla \psi = 0 \\
\text{2D incompressible fluid eqn.}
\end{array} \right. \]

iv) Problems in Potential Flow

a.) Incompressible Potential Flow Around Sphere

Consider sphere in motion at \( U \) in infinite fluid

Flow Pattern ?

Now:

- Intuitively expect:

i.e. equivalent to

\[ \begin{cases} 
\text{sphere at rest} \\
\nabla \text{fluid} = -U \\
\end{cases} \]
Electrostatic analogy: Conducting sphere in uniform electric field

\[ E_0 \]

\[ \phi = -E_0 \cdot \vec{r} + \phi_{\text{sphere}} \]

\( \phi_{\text{sphere}} \) is dipole field.

Dipole moment determined by b.c.,
i.e. \( \phi = \text{const} = 0 \) on sphere surface

Now, for potential flow (incompressible):

\[ \nabla^2 \phi = 0 \quad V = \nabla \phi \]

\[ V_n = \vec{v} \cdot \hat{n} = u_n \bigg|_{\text{surface}} \]

(i.e. normal velocity = sphere velocity on surface)

By analogy with electrostatics, can solve via:
- multipole expansion
- b.c. to determine effective "charge" distribution
Recall c.s. \( \nabla^2 \phi = -4\pi G \rho \)

\[ \phi = \int d^3x' \frac{\rho(x')}{|x-x'|} \]

For \( x \) outside region \( \rho \):

\[ \phi(x) = \int d^3x' \frac{\rho(x')}{|x-x'|} \]

\[ = \int d^3x' \frac{\rho(x')}{|x-x'|} - \int d^3x' x' \cdot \nabla \left( \frac{1}{|x|} \right) + \ldots \]

\[ = 0 \quad \text{b, dipole} \quad \text{b, quadrupole} \]

Thus we can write down general solution for potential flow streamlines around body as multipole expansion.

\( Q = 0 \) (no sources, sinks)

... in general dipole dominates
In 2D, some story with $\ln |x-x'| \Rightarrow \frac{1}{|x-x'|}$

Here:
\[ y = u \frac{\varPhi}{R} \quad \text{(spherical symmetry)} \]
\[ V_n = V_r = u \hat{\varPhi} \cdot \hat{n} = u \cos \theta \quad \text{(body velocity)} \]

\[ \frac{V_n}{R} = \frac{V_r}{R} = u \hat{\varPhi} \cdot \hat{n} = u \cos \theta \rightarrow \text{boundary condition} \]

Now:
\[ \phi(x) = A \cdot \frac{1}{r} \]
\[ A = A \hat{\varPhi} \quad \text{(dipole moment in } \hat{z} \text{ direction)} \]

\[ \phi = -A \frac{\cos \theta}{r^2} \]

\[ V_r = 2A \cos \theta / r^3 \]

\[ 2A \cos \theta = u \cos \theta \]

\[ A = \frac{R^3}{2} u \]

\[ \phi = -u \frac{R^3 \cos \theta}{2 r^2} \quad \text{determines general flow field} \]

\[ \mathbf{V} = \nabla \phi \]
Note:
- can recover from \( \Phi = \frac{8r}{\pi^2} \left( r + \frac{b}{r} \right) + \int Pe \cos \theta \) expansion and b.c.'s.
- if sphere in uniform field:
  \[ \Phi = U \cos \theta + \Phi_{\text{sphere}} \]
  determine from \( V_n = 0 \)
- to determine pressure distribution on sphere,
  recall:
  \[ \rho \frac{\partial \Phi}{\partial t} + \frac{1}{2} \rho v^2 + \rho = \rho_0 \]
  Bernoulli's Eqn.
  \[ \text{incompressible} \]
  \[ \text{pressure at } \Phi \]

Thus, can immediately write:
\[ \Phi(x) = \rho_0 - \frac{1}{2} \rho \nabla \Phi \cdot \nabla \Phi - \rho \frac{\partial \Phi}{\partial t} \]
\[ \Phi(x) \] determined a/q' above via \( \nabla^2 \Phi = 0 \)
and b.c.'s.
As a sphere in motion (but uniform):  
\[ \dot{\varphi} = -u \cdot \varphi + \frac{\partial \varphi}{\partial y} \]

So
\[ p(x) = p_0 - \frac{\rho}{2} \frac{\partial \varphi}{\partial x} \frac{\partial \varphi}{\partial y} - u \cdot \nabla \varphi \]

Generally, leads to concept of stagnation point

d.e. For Bernoulli Eqn. for incompressible fluid:
\[ \frac{p + \frac{1}{2} \rho u^2}{\rho} = \text{const.} = p_0^1 \]

Now, consider fixed body in fluid with \( \begin{cases} v_{\infty} = u_0 \\ p_{\infty} = p_0 \end{cases} \)

As \( u = 0 \) on surface body:
\[ p_{\max} = p_1 = p_0 + \frac{1}{2} \rho u^2 \]

- Stagnation point \((u=0)\) on body is point of maximal pressure
- Maximal pressure determined by \( \frac{p_0}{\text{speed}} \)
Fish skeleton strongest on front face, weakest elsewhere

Front face is point of maximal pressure (head)

Eye lens adjusts to allow for speed-induced pressure changes.

b.) Drag Force and Induced Mass

Heuristics: Consider rigid body in water.

Slow body motion \(\Rightarrow\) potential flow around sphere

\(\Rightarrow\) energy in fluid motion, too!

Thus, for \(\text{F}_{\text{ext}}\) to move body in fluid, need work against inertia of body (obvious)

\(-\) inertia of fluid, excited into potential flow

Thus, for body in water, need interpret Newton's 2nd law as:
\[
F_{ext} = \frac{\text{Meff}}{\text{dt}}
\]

\[
\text{Meff} = M + M_{\text{induced}}
\]

\[
M_{\text{induced}} = \text{induced mass of fluid in potential flow around body}
\]

\[
\text{mass of fluid flow which "dresses" the body}
\]

To calculate induced mass:

1. Calculate energy in potential flow around rigid body in uniform motion in fluid

2. Use \( \text{dE} = \text{dp}_y \) to determine momentum in fluid

as \( p = p(y) \) \( \Rightarrow \) \( \text{p}_y = \text{mi}_k \text{u}_k \)

\( \text{i} \). \( \text{mi}_k \) is induced mass tensor!

→ Calculation: Consider rigid body moving in fluid

i.e.

Now, for flow field outside body multiple expansion solution to \( \nabla^2 \Phi = 0 \) yields
\[ \phi = \frac{A}{r} + \nabla \cdot \mathbf{D} \left( \frac{1}{r^3} \right) \]

Monopole \hspace{1cm} Dipole

(Dominant multipole \hspace{1cm} at large radius)

Dipole moment: \[ A = C R^3 \hat{n} \]

(\( C = \frac{1}{2} \), sphere)

\[ \phi = A \cdot \nabla \cdot \mathbf{D} \left( \frac{1}{r^3} \right) \]

\[ = -A \cdot \nabla \frac{1}{r^3} = -A \cdot \mathbf{n} \frac{1}{r^2} \]

\[ V = \nabla \phi = A \cdot \nabla \cdot \mathbf{D} \left( \frac{1}{r^3} \right) \]

\[ = (A \cdot \nabla) \left( -\frac{1}{r^3} \right) \]

\[ V = \left( 3(A \cdot \hat{n}) \frac{1}{r^3} - A \right) / r^3 \]

\[ \Phi = -A \frac{\cos \theta}{r^2} \]

\[ \omega_r = 2A \cos \theta \frac{1}{r^3} \]

\[ \omega_r = \frac{2A}{R^3} \cos \theta \frac{1}{r^3} \]

\[ A = \frac{4}{3} \pi \]

Now, for energy seek calculate fluid energy in volume \( V \) enclosed within radius \( R \) around body. Take \( R^2 \gg V_0 \equiv \text{volume of body} \).

Thus:

\[ E = \frac{1}{2} \rho \int dV |\nabla \phi|^2 \]

\[ = \frac{1}{2} \rho \int dV (\omega_r^2 + 1 \mathbf{v}^2 - U^2) \]
\[ M^2 - u^2 = (v + y) \cdot (v - y) \]
\[ = \nabla \cdot (\phi + y \cdot \nabla) \cdot (v - y) \]
\[ = \nabla \cdot \left[ (\phi + y \cdot \nabla) (v - y) \right] \]

As \( y = \text{const} \)
\[ \nabla \cdot v = 0 \]
\[ \nabla \cdot y = 0 \]

\[ E = \frac{1}{2} \int \int d^3 \mathbf{x} \left[ u^2 + \nabla \cdot \left[ (\phi + y \cdot \nabla) (v - y) \right] \right] \]

\[ = \frac{1}{2} \int d^3 \mathbf{u} (v - \mathbf{u}) \]
\[ + \frac{1}{2} \int \int dS \cdot \left[ (\phi + y \cdot \nabla) (v - y) \right] \]

Volume or body
\[ V = \frac{4\pi}{3} R^3 \]
\[ \int_{\text{outer surface}} (v - y) \cdot dS = 0 \]

Now, \( dS = \hat{n} R^2 dS \), on outer surface

\[ E = \frac{1}{2} \int d^3 \mathbf{u} (v - \mathbf{u}) \]
\[ + \frac{1}{2} \int \int R^2 dS \left[ (\nabla \cdot v - \nabla \cdot y) \cdot (\phi + y \cdot \nabla) \right] \]
\[ E = \frac{1}{2} p u^2 (V-V_0) \]

\[ + \frac{1}{2} p \int R^2 d\Omega \left[ \left( \frac{2 (A \cdot \hat{n})}{R^3} \right) \left( -\frac{A \cdot \hat{n}}{R^2} + R (y \cdot \hat{n}) \right) \right] \]

\[ = \frac{1}{2} p u^2 (V-V_0) + \frac{1}{2} p \int R^2 d\Omega \left[ -2 \frac{(A \cdot \hat{n})^2}{R^5} \right. \]

\[ + \frac{(y \cdot \hat{n})(A \cdot \hat{n})}{R^2} + \frac{2(A \cdot \hat{n})(y \cdot \hat{n})}{R^2} = R (y \cdot \hat{n})^2 \left. \right] \]

Thus finally,

\[ E = \frac{1}{2} p u^2 (V-V_0) + \frac{1}{2} p \int R^2 d\Omega \left[ \frac{3(y \cdot \hat{n})(A \cdot \hat{n})}{R^2} - R^3 (y \cdot \hat{n})^2 \right] \]

\[ d\Omega = d\phi \sin \Theta d\Theta \]

\[ \text{if} \quad \int d\Omega (\_\_\_\_\_\_\_\_) = <(1)> \]

\[ \Rightarrow \quad <(A \cdot \hat{n})(B \cdot \hat{n})> = \frac{1}{2} \delta_{ij} \quad A \cdot B = \frac{1}{3} A \cdot B \]
\[ E = \frac{1}{2} \rho u^2 (v-v_0) + \frac{1}{2} \rho \left[ 4\pi A \cdot \mathbf{y} - \frac{4\pi}{3} R^3 u^2 \right] \]

\[ = \frac{4}{3} \rho \left[ 4\pi A \cdot \mathbf{y} - u^2 V_0 \right] \]

Thus finally, \[ E = \frac{\rho}{2} \left[ 4\pi A \cdot \mathbf{y} - u^2 V_0 \right] \]

\( \begin{align*} \text{energy in} & \quad \text{potential flow and body} \\ \text{flow and body} & \quad \text{energy in} \\ \text{flow and body} & \quad \text{potential flow and body} \end{align*} \]

Now, \( A = A(u) \) \( \Rightarrow \)

\[ E = \frac{4}{3} \rho \left( u \cdot u \right) \]

\( \begin{align*} \text{defines induced mass} & \quad \text{tensor} \\ \text{tensor} & \quad \text{momentum in} \\ \text{momentum in} & \quad \text{potential flow} \end{align*} \]

\[ dE = u \cdot dP \]

\[ \Rightarrow \quad P = P \left[ 4\pi A - V_0 u \right] \]
Now consider external forces acting on system where system = body + fluid (in potential flow)

\[ f_{ext} = dP_{\text{fluid}} \frac{dy}{dt} + m_{\text{body}} \frac{dy}{dt} \]

\[ f_r = (M_{\text{circ}} + m_{\text{inc}}) \frac{dy}{dt} \]

The effective mass of "system" is sum of

- body mass
- induced mass of fluid in potential flow around body

Note induced mass is determined purely by body shape (i.e. via volume and dipole moment)

i.e. for sphere \[ A = \frac{R_0^3}{2} \]

\[ P = \rho \left[ 4\pi \frac{R_0^3}{2} y - 4\pi \frac{R_0^3}{3} \right] \]

\[ = \rho \frac{2}{3} \pi R_0^3 y \]

\[ m_{\text{induced}} = \rho \frac{2}{3} \pi R_0^3 \]
In general, \( M_{\text{induced}} \sim \# \rho R^3 \)

\[ \sim \# \rho \frac{V}{a} \]

\( \rho \rightarrow \) displaced mass, numerical fluid factor shape dependent

\( \rightarrow \) Example of "renormalization" in classical physics, "dressing field" in continuum e.g. Debye, etc.

c.f. in quantum electrodynamics \( \rightarrow \) electron polarized vacuum

\[ \frac{-e}{c^2} \]

\[ \rightarrow m_e = m_{e, \text{bare}} + m_{e, V/P} \]

\( E = mc^2 \)

in classical potential flow \( \rightarrow \) moving a sphere in \( H_2O \) requires that some energy go into surrounding media (the water?)

(skip)

\( \rightarrow \) Enhanced inertia due induced mass may alternatively, be viewed as drag force on body mom. transmitted to fluid (careful of phase)

c.f. \[ F_{\text{ext}} = \frac{dP_{\text{fluid}}}{dt} + M\frac{dy}{dt} \]
\[ \frac{d}{dt} \left( \frac{M}{dy} \right) = \text{ext} - \frac{dP}{dt} \text{fluid} \]
\[ = \text{ext} + \text{drag, lift} \]
\[ \text{drag} = -\frac{dP}{dt} \text{fluid} \text{ along direction motion.} \]
\[ \text{lift} = -\frac{dP}{dt} \text{ fluid} \text{ in direction of motion.} \]

**Note:** if body is uniform motion in ideal (fantasy) fluid, \( \text{drag} = \text{lift} = 0 \) \( \Rightarrow \) D'Alembert's Paradox

\( \Rightarrow \) need external force to maintain uniform motion as \( \Rightarrow \) no dissipation (ideal fluid)
\( \Rightarrow \) no loss of energy to \( \infty \) \( (V \sim 1/R^3) \)

\( \Rightarrow \) but if body near surface

\( \text{Kelvin wave} \)

Body will radiate surface waves to \( \infty \)

\( \Rightarrow \) wave drag endured energy loss!
example: obtain:

a) eqn. of motion for sphere in fluid
b) sphere in oscillating fluid

a) for sphere $A = \frac{1}{2} R^2 y$

for oscillating sphere

$$F_{ext} = ma_{sphere} + (mv')_{induced}$$

acceleration of dressing

$$mv' = M \text{ind} \cdot U$$

$$M_{v'p} = \frac{2}{3} \pi R^3 \rho$$

virtual mass

$$F_{ext} = \frac{4\pi}{3} R^2 (P_{\text{dyn}} + \frac{\rho y \kappa}{2}) \frac{dy}{dt}$$
Related Problem:

- Consider body in fluid which is set in motion by external agent.

Relate $\mathbf{u}$ body to $\mathbf{v}$ fluid?

Now $\mathbf{v} = \text{velocity of unperturbed flow}$

\[
\frac{\|\mathbf{v}\|}{R_0} \ll 1 \Rightarrow \mathbf{v} \sim \text{const over scale of body (potential flow valid)}
\]

So if body fully carried along by fluid ($\mathbf{v} = \mathbf{y}$), then force on it would equal force on volume of displaced fluid.

\[
\frac{d}{dt} (M \mathbf{u}) = p \mathbf{V}_0 \frac{d\mathbf{y}}{dt}
\]

but body moves relative to fluid, so that fluid acquires momentum $\Rightarrow$ drag due to relative motion

\[
\frac{d}{dt} (M \mathbf{u}_{\text{fluid}}) = -M \cdot \frac{d}{dt} \left[ \mathbf{u} - \mathbf{v} \right]
\]
\[ \frac{d}{dt} (Mu) = pV_0 \frac{dV}{dt} - m \cdot \frac{d}{dt} (y - v) \]

\[ \frac{d}{dt} (Mu_i) = pV_0 \frac{dV_i}{dt} - m \cdot \frac{d}{dt} (u_i - v_i) \]

\[ Mu_i = pV_0 V_i - m \cdot \frac{d}{dt} (u_i - v_i) \]

\[ (Mo_{\text{fr}} + m \cdot \frac{d}{dt}) u_k = (pV_0 d_{\text{fr}} + m \cdot \frac{d}{dt}) V_k \]

\[ U_k = \left( \frac{pV_0 d_{\text{fr}} + m \cdot \frac{d}{dt}}{Mo_{\text{fr}} + m \cdot \frac{d}{dt}} \right) V_k \]

Note:

- \( pV_0 < M \) (body heavier than displaced fluid) \( \rightarrow \) body legs
- \( pV_0 > M \) \( \rightarrow \) body leads
- \( pV_0 = M \) \( \Rightarrow U_k = V_k \)
Thus

\[
\frac{M \, du}{dt} = \rho_s \, \frac{dV}{dt} - \frac{m \cdot d}{dt} [u - v]
\]

\[
(M \, \delta \, + \, m \text{'}) \frac{du}{dt} = M \rho \, \delta \, + \, m \text{'} \, \frac{dv}{dt}
\]

\[
u = \left[ \frac{(M \rho \, \delta \, + \, m \text{'})}{(M \rho \, \delta \, + \, m \text{'})} \right] v
\]

\[M_0 = \rho_s \, V_0\]

\[M = \rho \, V_0\]

\[\Rightarrow \quad u = v \quad \text{if} \quad \rho_s = \rho\]

\[u < v \quad \text{if} \quad \rho_s < \rho \quad \Rightarrow \text{heavy object}\]

\[\rho_s = \text{fluid density}\]

\[\rho = \text{body density}\]

\[u > v \quad \text{if} \quad \rho_s > \rho \quad \Rightarrow \text{light object}\]

C.) Potential Flow - General Skender Body

Till now, have considered simple body potential flows, i.e., sphere, cylinder.
Here consider general body from surface of revolution.
i.e. generally axially symmetric slender body
- slender \( \Rightarrow \) \( W/L \ll 1 \)

Now, observe analogy with electrostatics again,

i.e. e.s. \( \Rightarrow \phi(x) = \int d^3x' \rho(x') / |x-x'| \)

potential flow \( (A \sim uV) \)
\[ \phi(x) = \frac{1}{4\pi} \int d^3x' \left( \frac{\rho(x')}{\rho_0} \right) / |x-x'| \]

\( \rho(x') / \rho_0 \) = normalized density of fluid flowing across cross-section of body

\( \Rightarrow \) yields \( A \sim V_0 u \) etc.

\[ \phi(x) = \frac{1}{4\pi |x|^2} \int d^3x' \frac{\rho(x')}{\rho_0} x' + \text{h.o.t.} \]

dipole term dominated
- body slender $\Rightarrow \frac{r}{L} < 1 \Rightarrow \rho \ll \rho_0$

- $V \cdot U = 0$ and axial symmetry $\Rightarrow$

$$\frac{\partial V_x}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} \left( r V_r \right) = 0$$

- $V_r \sim \frac{A r}{V_x} \sim \frac{V}{L}$

$\Rightarrow$ need only consider $x$ fluid motion

1. To compute dipole moment need $\rho x / \rho_0$

For fluid flow across body

$$\frac{\partial}{\partial x} \Rightarrow$$

$A \rightarrow A + dA$

($A =$ face area) $u \frac{u}{x}$

Net

$$\frac{\partial}{\partial x} = u \left[ A + dA \right] - A = u \frac{\partial A}{\partial x} dx$$
\[ \frac{\rho(x)}{\rho_0} = u \frac{\partial A}{\partial x} \]

\[ \phi(x) = \frac{1}{4\pi V} \int dx' \frac{dx' u}{x'} \frac{\partial A(x')}{\partial x'} \]

\[ = -\frac{u}{4\pi V} \sqrt{\bar{x}} A(x') \]

\[ = -\frac{u}{4\pi V} V \]

\[ V = \text{volume of body} = \int dx' A(x') \]

\[ \Rightarrow \text{yields intuitive result:} \]

\[ \phi(x) = -u \frac{V_{\text{body}}}{4\pi r^2} \]

effective dipole moment for slender body.