Fluids in Flatland - A Short Introduction

apologies to Edwin Abbott

A Quick Look:
(Sneak Preview)
what physical process underpins
K41 cascade, etc. 

\[ \text{vortex tube stretching} \]

i.e.

\[ \partial_t \omega = \nabla \times (u \times \omega) + \nu \nabla^2 \omega \]

\[ \partial_t \omega + u \cdot \nabla \omega = \nu \nabla^2 \omega = \omega \cdot \nabla u \]

\[ \nabla \rightarrow \omega_1 \rightarrow \omega_2 \]

\[ \text{2D } \omega \cdot \nabla u = 0 \]

\[ \frac{\partial}{\partial t} \omega + u \cdot \nabla \omega = \nu \nabla^2 \omega + \dot{r} \]

\[ \frac{\partial}{\partial t} \Phi + \nabla \times \omega \cdot \nabla \Phi = \nu \nabla^2 \Phi + \dot{r} \]

two universal quadratic invariants:

\[ \int \frac{u^2}{2} \rightarrow \text{energy} \]

\[ \int \frac{\omega^2}{2} \rightarrow \text{enstrophy (mean square vorticity)} \]
Refs:
- R. Salmon: Notes on GFD
- G. Vallis: Atmospheric and Oceanic Fluids

+ Posted Notes from GFD Module
+ Posted References (both Module & Lectures)

N.B. Butterfield and Ecke review especially recommended.
Key element in 2D turbulence is constraint imposed on dynamics by dual conservation law.

**Upshot:** Dual Cascade (Kraichnan '67)

inverse energy cascade

forward enstrophy cascade

forcing

dual self-similarity ranges

\[ \partial_t \omega + \nabla \cdot (\omega \mathbf{u}) + \nu \Delta \omega = \nabla \cdot \mathbf{f} \]

Why 2D?

\[ \Rightarrow \text{Constrained Dynamics} \]

- Recall Taylor - Araidman Theorem

\[ \text{in rotating fluid, } (\omega + 2 \Omega) \frac{1}{\rho} \]

\[ \Rightarrow \Omega >> \text{other rates} \]

\[ 2 \Omega \partial_z u = 0 \]

\[ \Rightarrow \sim 2D \text{ dynamics} \]
Immediately realize that
~ 2D dynamics

\[ \text{Ro} = \sqrt{\frac{\nu}{\lambda}} \]

Rossby

\[ \sim \sqrt{\frac{\nu}{\lambda}} \rightarrow \text{(other terms)} \]

In vortex vs 2D x 2D

Contrast Re

\[ \text{favors slow, large scale motion in (thin) rotating system} \]

\text{i.e., atmosphere, ocean, etc.}

Ways to 2D case:

- rotation \hspace{1cm} \text{Ro} = \sqrt{\frac{\nu}{\lambda}}

- stable stratification

- # \sim \sqrt{\frac{\nu}{\lambda}} N

\[ N^2 = g \frac{\partial \rho}{\partial z} \]

- strong magnetic field

\[ \text{Ro} \rightarrow \sqrt{\frac{\nu}{\lambda}} \Omega \text{cylindrical frequency} \]

\text{Mima model}
- Low $\lambda_0$ dynamics

Given $\lambda_0 < 1$, have fundamentals relation between pressure and vorticity. \( \Omega \) includes centrifugal.

\[
\frac{d\lambda}{dt} = -\nabla \left( \frac{P^*}{\rho} \right) - 2\Omega \times \mathbf{v}
\]

$\lambda_0 < 1 \Rightarrow$ Geostrophic balance

\[
\mathbf{v} = -\nabla \left( \frac{P^*}{\rho} \right) - 2\Omega \times \mathbf{v}
\]

\[
\Rightarrow \quad \lambda^* = \frac{2 \times \nabla \left( \frac{P^*}{\rho} \right)}{\Omega^2}
\]

Pressure - $\phi$ - Stream Function:

Clockwise: High Pressure

Counter-Clockwise: Low Pressure

Fluid rotation about High pressure cells.
- **B-plane Model**

- Quick derivation

  - B plane tangent to spherical shell atmosphere
  - Strong stable stratification on $L$

So describe/approximate dynamics in 2D plane tangent to sphere i.e. B-plane

Now, consider displacement of fluid/vortex element:

- $\omega + 2\Omega$ frozen ($\Omega$)
- $C = \int \sigma g \cdot (\omega + 2\Omega)$

Circulation conserved.
Point: displacing fluid element implies change in

\[ \oint \text{d}l \cdot \mathbf{n} = \oint \vec{v} \cdot \text{d}S = \cos \Theta \mathbf{n} \cdot \mathbf{p} \]

- there must be a change in fluid vorticity to conserve circulation
- since planetary vorticity piece of circulation changed by displacement

\[ A = \text{area of vortex} \]

\[ \frac{d\omega}{dt} = 0 \]

\[ \frac{d}{dt} (A \omega + A^2 \frac{L}{R} \sin \Theta) = 0 \]

projection factor

\[ \frac{d\omega}{dt} = 2 \omega \cos \Theta \frac{d\Theta}{dt} \]

\[ = -\frac{2 \omega}{R} \cos \Theta \frac{d}{dt} (R \Theta) \]

\[ = -\Theta \omega \]

\[ \Theta = \frac{2 \Omega - \cos \Theta}{R} \rightarrow \Theta \text{ gradient of Coriolis force} \]
Of course \( \frac{d}{dt} (R \Theta) = \frac{d}{dt} y = v_y \)

\[ \frac{d \omega}{dt} = -c \nu \omega \]

+ add dissipation

\[ \frac{d}{dt} \omega + u \cdot \nabla \omega + \mu \omega = \nu \nabla^2 \omega + f \]

\[ \frac{d}{dt} V = \frac{d}{dt} + u \cdot \nabla \]

\[ V = -\frac{\partial p}{\partial z} \to \phi \times \phi \]

\[ \omega = \frac{\partial^2 \phi}{2 \pi} \]

\[ R \to \infty \quad \beta \to 0 \]

\[ \frac{d}{dt} \nabla^2 \phi + \nabla \times \nabla \cdot \nabla^2 \phi + \mu \nabla^2 \phi = -\nu \nabla^2 \phi + f \]

2D fluid equation.

\( \beta \)-plane equation is next simplest \( \Rightarrow \) supports waves, eddies, and flows.
Observe:

Can re-write 2D inviscid equation as

\[ \partial_t \omega_2 + \sum \omega_2, H^3 = 0 \]

\[ H = \phi \]

Conservative Hamiltonian evolution

Similar to Liouville or Vlasov equation:

\[ \partial_t f + \{ f, H \} = 0 \]

\[ H = \frac{p^2}{2m} + \frac{1}{2} m \phi + \text{Poisson's equation} \]

\[ \partial_t f + v \partial_x f + \frac{2}{m} E \partial_u f = 0 \]

i.e.

\[ \omega_2 \rightarrow f \Rightarrow \text{conserved (phase space) density} \]

Which brings us to:

Potential Vorticity

Observe can write equations in conservative form, i.e.
\[
\frac{d}{dt} \nu = 0 \quad \text{(same for \( PV \))}
\]

\[
\frac{d}{dt} (\omega + \Omega y) = 0 \quad \text{(\( \Omega \)-plane)}
\]

\[\text{fluid} \rightarrow \text{planetary vorticity} \]
\[\text{vorticity} \quad \text{(L.D. \& expansion)}\]

\[\omega + \Omega y = \text{simple example of potential vorticity (PV)}\]

- generalized vorticity akin to phase space density

\[\text{GFD = the Study of Fluids with PV''}\]
\[= "\text{The Fluid Dynamics of PV"}\]

More generally on PV:

- recall for rotating fluid:

\[
\frac{d}{dt} \left( \frac{\omega + 2\Omega}{\rho} \right) = \frac{(\omega + 2\Omega) \cdot \nabla \rho}{\rho}
\]

akin:

\[
\frac{d}{dt} \frac{\rho}{\rho} = \rho \cdot \nabla \cdot V
\]

some eqn \( \leftrightarrow \frac{\omega + 2\Omega}{\rho} \) frozen (M)
Now consider conserved scalar field $\psi$

\[ \frac{d}{dt} \psi = 0 \]

\[ \frac{d}{dt} (\psi - \psi_2) = 0 \]

\[ \Delta \psi = \nabla \psi \cdot \partial \]

\[ \frac{d}{dt} \left( \frac{\nabla \psi \cdot \partial}{c} \right) = 0 \]

and $\rho c^2 \rightarrow \frac{\omega + 2\Omega}{c^2}$

so if $\rho \psi$ satisfies $\frac{\omega + 2\Omega}{c^2}$ must satisfy

\[ \frac{d}{dt} \left( \frac{\omega + 2\Omega}{c} \cdot \nabla \psi \right) = 0 \]

along trajectories

\[ q = \frac{\omega + 2\Omega \cdot \nabla \psi}{c^2} \]

$PV$ (general) any $\partial \psi$
\[ \psi = 0 \]
\[ \tilde{H} = V + \frac{1}{2} \nabla \cdot \mathbf{u} \]
\[ \nabla \psi = \mathbf{u} \]

- PV conservation \iff particle re-labeling symmetry (i.e., particles can be re-labeled without changing thermodynamic state)

N.B. IF consider finite thickness shell

\[ Z = V \psi + \frac{f a^2}{2} \alpha \left( \frac{\rho}{N^2} \frac{\partial \alpha}{\partial \rho} \right) \]

\[ f_0 = 2 \omega \sin \theta \quad \text{rotation} \]
\[ N^2 = g \rho \quad \text{buoyancy} \]

Relevance of finite thickness? Scale 1 \rightarrow 1/L^2 \quad \text{vs} \quad \frac{f a^2}{N^2 H^2} \quad \text{layer thickness} \]
\[ \frac{f}{L} \quad \text{(deformation radius)} \]
\[ \frac{1}{L^2} \sim \frac{1}{L^3} \]

\[
\begin{cases} 
\sim 100 \text{ km ocean} \\
\sim 1000 \text{ km atmosphere}
\end{cases}
\]

\[ L \ll \lambda_d \Rightarrow B\text{-plane.} \]

\[ \text{relative vorticity and deformation effects contribute equally} \]
2D Turbulence

Issues: conservation of energy, enstrophy
- trends in constrained spectral evolution
- self-similarity ranges, inverse cascade
- Fate of energy

Issues:

2D turbulence is the generic problem of GFD

\[ \frac{\partial}{\partial t} \phi + \mathbf{u} \cdot \nabla \phi - \nu \Delta \phi - \frac{\partial \mathbf{u}^3}{\partial \mathbf{x}^3} \phi + \frac{\partial \mathbf{u}^2}{\partial \mathbf{x}^2} \phi \]

\[ \Rightarrow \phi \]

Any scale
drag = scale invariant damping
\Rightarrow control large scale

2 inviscid invariants:

\[ \langle (\nabla \phi)^2 \rangle \Rightarrow \text{energy} \]

\[ \langle (\nabla^2 \phi)^2 \rangle \Rightarrow \text{enstrophy} \]
N.B.:
- in 3D enstrophy produced:
  \[ \frac{d}{dt} \langle \omega^2 \rangle = \langle \omega, (\omega \cdot \nabla) \rangle \]
  \[ \langle \Delta_k \omega \rangle \sim k^{\frac{5}{3}} - k \]
- 2D, \( \omega \cdot \nabla \omega \rightarrow 0 \)
  \( \Rightarrow \) all powers \( \int d^2 x \, \omega^n \) conserved
  \[ \int d^2 x \, \omega \rightarrow \langle \omega^2 \rangle \] conserved on finite box
  \( \therefore \) story incompatible with \( \text{KHI} \)

Problem of 2D Fluid:
- given forcing at any scale \( L \) is \( \delta x \)
  \( L \geq \delta_k > \delta r \)
  \( \Rightarrow \) how does dual conservation of \( E, S \) constrain transfer?
  \( \Rightarrow \) \( \delta \text{eff} \) similarity ranges?
Theoretical "clues":

- consider 3 modes interacting (3 to conserve quadratic quantity)

\[
\begin{align*}
\text{LHS} & : & \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} & \begin{pmatrix} k_1^2 & k_2^2 & k_3^2 \end{pmatrix} \\
\text{RHS} & : & \begin{pmatrix} k_1^2 < k_2^2 < k_3^2 \end{pmatrix} & \begin{pmatrix} k_2^2 = k_3^2 \end{pmatrix}
\end{align*}
\]

Conservation:

\[
\begin{align*}
E_2 &= E_1 + E_3 \\
\Omega_2 &= \Omega_1 + \Omega_3
\end{align*}
\]

but \( \Delta \mathcal{E}(h) = h^2 \mathcal{E}(h) \)

\[
\begin{align*}
\begin{cases}
E_2 &= E_1 + E_3 \\
k_1^2 E_2 &= k_1^2 E_1 + k_3^2 E_3
\end{cases}
\end{align*}
\]

\[
\therefore E_1 = \left( \frac{k_3^2 - k_1^2}{k_3^2 - k_1^2} \right) E_2 \Rightarrow E_1 \sim E_2
\]

\[
E_3 = \left( \frac{k_2^2 - k_1^2}{k_3^2 - k_1^2} \right) E_2 \Rightarrow E_3 \sim \frac{k_2^2}{k_3^2} E_2
\]
Theorem 1:

Given a and b, prove that:

\[ a + b = b + a \]

Proof:

Let \( a + b = c \) and \( b + a = d \) for some \( c, d \in \mathbb{R} \).

By the commutative property of addition, we have:

\[ c = d \]

Therefore, \( a + b = b + a \).
as \( \Omega(k) = k^2 E(k) \)

\[ E_1 \sim E_2 \quad \Rightarrow \quad \text{energy transferred to large scale mode!} \]

\[ \delta_1 \sim \delta_2 \quad \Rightarrow \quad \text{eustrophy transferred to small scale mode!} \]

\( \sim \) suggests: energy accumulates at large scale; eustrophy accumulates at small scale.

\( \Rightarrow 2 \) self-similar transfer ranges on 2D turbulence.

N.B. Analogy: Asymmetric Top.

Conservation:
\[ \sum L_i^2 = L^2 \]

\[ E = \sum \frac{L_i^2}{2I_i} \]

\[ L^2 = L_1^2 + L_2^2 + L_3^2 \]

\[ E = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3} \]

\( I = \frac{1}{k^2}, \) etc.

and energizing intermediate axis \( \Rightarrow \) decay to \( \frac{1}{2} \delta \) etc.
Consider a spectral 'slyg' of turbulence, initialized.

How will $\overline{P}$ evolve, given $\partial \langle \overline{\omega k^2} \rangle > 0$?

I.e., assume spectrum spreads.

N.B. Does $\langle \overline{\omega k^2} \rangle$ exist?

\[
\langle \overline{\omega k^2} \rangle = \int dk \left( k - \overline{k} \right)^2 E(k) / \int dk E(k)
\]

\[
= \int dk \left( k^2 - 2k \overline{k} + \overline{k}^2 \right) E(k) / \int dk E(k)
\]

\[
= \int dk \left( k^2 E(k) - 2k \overline{k} E(k) + \overline{k}^2 E(k) \right) / \int dk E(k)
\]

\[
= \left( \overline{k}^2 E_0 - 2 \overline{k} E_0 + \overline{k}^2 E_0 \right) / E_0
\]

\[
= \Omega_0 / E_0 - 2 \overline{k}^2
\]

\[
= (\Omega_0 / E_0) - \overline{k}^2
\]
Here: \( \int dk \, E(k) = E_0 \to \text{const} \)
\( \int dk \, k^2 E(k) = \Omega \to \text{const} \)
\( \int dk \, k E(k) = \tilde{h} E_0 \to \text{defines centroid} \)

\( a \neq \, \frac{2}{2} \left( \frac{\Lambda k^2}{\langle a k^2 \rangle} \right) > 0 \Rightarrow a \neq \frac{2}{2} \frac{\tilde{h}}{E(k)} \)

- Spectrum broadens but also shifts toward large scales
- Energy content shuffled/coupled to larger scale
  - suggestive of inverse energy cascade
  - similar story for enstrophy \( \Rightarrow \) forward cascade!

\[ \therefore \text{Enter the Dual Cascade!} \]
Dual cascade (Kraichnan '67)

From forcing, system supports self-similarity relation:

- forward energy cascade (k_f < k_c)
  - no forward energy flux
  - no energy dissipation by viscosity (Re → ∞)

- inverse energy cascade
  - no inverse energy flux
  - no viscous power dissipation
  - damping by drag, etc.
  - not stationary

Cascade ≡ range self-similar transfer energy self-similarity

\[ \frac{d}{dt} <u^2> \sim \left( \frac{U_f}{k_f} \right)^3 \]

\[ \epsilon = \frac{d}{dt} <v^2> \sim \frac{U_f^3}{k_f} \rightarrow \text{ergy rate not dissipation} \]
of course $\nu^2 E \sim M$.

- Forward & Enstrophy

$\frac{\Omega(k)}{\Gamma(k)} \sim M$

$\frac{\nu(k)}{\Gamma(k)} \sim M^{-1}$

$\nu(k) \sim M$

$\omega(k) \sim M^2$

$\omega^2(k) \sim M^4$

but $\omega^2(k) \sim k^2 \Omega(k)$

$\mathcal{E}(k) = \frac{\Omega(k)}{\Gamma(k)} \sim M^{2/3}$

energy spectrum in enstrophy range

- no forward energy flux in enstrophy range

- observe $\frac{1}{T^2} \sim M^{-4/3} \to$ const, hence

$\frac{1}{T^2} \sim \frac{c^{1/3}}{p^{2/3}} \to$ frozen for smaller
⇒ tip-off that since all scales transfer at some rate, non-local transfer of enstrophy can occur
⇒ Corrections! [Logarithmic due to straining]

- Inverse Energy

\[ e = \frac{v(\ell)^2}{\ell} \] as before but \( e \equiv \text{energy straining rate} \)

⇒

\[ E(\ell) = e^{2/3} \ell^{-5/3} \] inverse energy cascade range

⇒ akin 3D, but upscale

⇒ \( \ell(\ell) \sim \ell(\ell) \) cascade slows as larger scales approached.

⇒ not stationary state

\[ \frac{1}{\ell^2} \]

\[ \frac{1}{\ell} \]

⇒ eventually encounters drag boundary, etc.
- Straining (non-local) effect on scale?
  - Structure at large scales.

- No inverse energy flux in energy range.

- No forward energy flux, \( P_d \) by viscosity \( \to 0 \).
  - Dissipated power.

- So, where does the energy go?
  - Friction \( M \)
  - Boundary effects
  - Straining on small scales

- \(< \delta(r^3) > = +3/2 \leq I \)
  - Analogue of 4/5 for inverse energy range
  - For \( l < l_c \)
  - \( \varepsilon \) here is stirring rate.

- (Static) vorticity contours in inverse energy range exhibit statistics of percolation cluster.

- No "conventional" intermittency in inverse cascade. Deviations from Gaussian occurs...
- what do particles particle dispersion?

Revisiting Richardson:

\[ l_{1/2} \rightarrow \text{energy range} \]

\[ \frac{dl_{1/2}}{dt} = \nu (l_{1/2}) = \varepsilon \frac{l_{1/2}}{l} \]

\[ l_{1/2} \sim \varepsilon t^{1/3} \quad \text{as before} \]

\[ l_{1/2} \rightarrow \text{enstrophy} \]

\[ \frac{dl}{dt} = \nu (\varepsilon l) = \left[ \nu \varepsilon \lambda \right] l = \eta \frac{l_{1/3}}{l} \]

\[ \Rightarrow \text{enstrophy grows exponentially in enstrophy range} \]

N.B.: Dual cascade used to justify selective decay - minimum enstrophy

\[ \frac{\partial^2}{\partial z^2} \left( \sigma_B \right)^2 + \lambda \frac{\partial^2}{\partial x^2} \left( \sigma_B \right)^2 \]

\[ \frac{\partial^2}{\partial z^2} \sigma_B = 0 \]
\( f(x) = \frac{1}{x^2} \)
\[ \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{\beta} \frac{\partial V_y}{\partial y} + \frac{f}{\beta} \frac{\partial^2 \phi}{\partial x^2} \]

Ignoring: \( V, M, f \)

\[ \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial x^2} = -\beta V_y \]

\[ \Rightarrow \text{waves} \]

\[ w_z = -\frac{\beta k_x}{k_x^2}, \quad \frac{\partial V_y}{\partial y} = -\frac{2\beta k_x k_y}{(k_x^2)^2} \]

\[ \Rightarrow \text{Rossby waves} \]

\[ \Rightarrow \text{flows} \]

how does large scale order emerge?

\[ k_x \to 0^2 \]

\[ k_y \text{ finite} \]

\[ w_z \to 0 \]

\[ zonal \text{ mode} \]

jets, belts, jetstream

Numerous questions:

2. How do waves, flows, and inverse cascade interact?
3. Scale of zonal flows?
4. Implications for atmospheric phenomenology

On zonal flows:

- ZFs ubiquitous
- Flows produced by momentum transport
- Simplest perspective: wave propagation

Reynolds stress
(Linear) wave propagation
in account for ZF formation
beach

Recall: 

Excitation (storms etc.)
Radiation on latitude

\[ S = \sqrt{\mathbf{v} \cdot \mathbf{v}} = 2 \lambda_x \lambda_y \mathbf{E} \mathbf{E} \]

- Outgoing waves \( \mathbf{u} \cdot \mathbf{k} \mathbf{x} \mathbf{y} \)
  \[ S \sim (+) \]
- \( \mathbf{S} \sim -\mathbf{y} \)
  \[ \mathbf{u} \cdot \mathbf{k} \mathbf{x} \mathbf{y} \quad \Theta \]

but

\[ \langle \mathbf{v}_y \mathbf{v}_x \rangle = \sum \frac{h_x k_y}{4} \mathbf{E} \mathbf{E} \mathbf{D}^2 \]

- \( \Theta \rightarrow \Pi x, y \quad < 0 \)
- \( \Theta \rightarrow \Pi y, x \quad > 0 \)

Point:

Outgoing wave energy density flux generated in incoming momentum flux.
Remarkable property!

A beautiful example of:

... the central result that a rapidly rotating flow, when started in a localized region, will converge angular momentum into this region (stemming from spin-up)

Flows $\rightarrow$ energy transfer

$\rightarrow$ wave mechanism requires separation of excitation and distribution (beach)

Requires:
- waves
- vorticity/momentum transport or speed
- irreversibility $\rightarrow$ outgoing waves
- symmetry breaking, $D$ has directivity
- set of forcing/demography
Useful to investigate wave theorems for flow production.

Key observation:
(inhomogeneous PV mixing)
\[
\left< \nabla \cdot \mathbf{V} \right>_z = \text{const.}
\]
\[
\left< \nabla \cdot \mathbf{V} \right>_z = \left< \nabla \cdot \left( \begin{array}{c} \phi \\ \mathbf{x} \end{array} \right) \right>
\]
\[
= \left< \nabla \cdot (\mathbf{a} \cdot \nabla \phi) \right>
\]
\[
= \left< \nabla \cdot \left( \begin{array}{c} \mathbf{x} \\ \mathbf{x} \end{array} \right) \right>
\]

Recall essence of PV conservation force by planetary flow transport exchange.

\[
\mathbf{F} = \mathbf{F}(\nabla \times (\mathbf{a} \cdot \nabla \phi))
\]

Symmetry

\[
\left< \nabla \cdot \mathbf{V} \right>_z = -\left< \nabla \cdot (\mathbf{a} \cdot \nabla \phi) \right>
\]
\[
= -\partial_y \left< \nabla \cdot \left( \begin{array}{c} \phi \\ \mathbf{x} \end{array} \right) \right> + \left< \nabla \cdot (\mathbf{x} \cdot \nabla \phi) \right>
\]

Taylor Identity
\[
\left< \nabla \cdot \mathbf{V} \right>_z = \left< \nabla \cdot \left( \begin{array}{c} \mathbf{x} \\ \mathbf{x} \end{array} \right) \right>
\]

(Comment 3D - EP. = corollary hereafter.)

Reynolds Force drives flow.

Look at potential enstrophy balance.
Zonally averaged Latitudinal flow

\[ \text{PV flux} = \text{zonal} \times \text{averaged} \]

Latitudinal Reynolds Force \( \rightarrow \) drives Flow.

As Reynolds stress controls flow:

\[ \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + (2 \eta \Delta \mathbf{u}) \]

Geostrophic balance

\[ \rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \nabla \times (\nabla \times \mathbf{u}) \]

then PV evolution

\[ \text{Entropy, Enstrophy} \]

\[ \text{control flow}. \]

What are essential to ZF generation:

- PV mixing/transport in space
- translation symmetry in direction of the flow.
Now consider P.E balance:

\[ \frac{d}{dt} \| v \|^2 - \nu \| \nabla \| v \|^2 = 0 \]

\[ \frac{d}{dt} \left( \frac{1}{2} \| \nabla \| v \|^2 \right) + \nu \left( \frac{1}{2} \nabla \cdot \nabla \| v \|^2 \right) = -\nabla \cdot \left( \nabla \| v \|^2 \right) \]

\[ = -\left[ \frac{d}{dy} \left( \nabla \cdot \| v \|^2 \right) \right] \]

\[ = \left[ \frac{d}{dy} \left( \frac{d}{dy} \| v \|^2 \right) + \frac{d}{dx} \left( \frac{d}{dx} \| v \|^2 \right) \right] \]

\[ = -\left[ \frac{d}{dy} \left( \nabla \cdot \| v \|^2 \right) \right] = -\left( \frac{d}{dy} \left( \frac{d}{dx} \| v \|^2 \right) \right) \]

\[ = -\left( \frac{d}{dy} \left( \frac{d}{dx} \| v \|^2 \right) \right) = -\left( \frac{d}{dy} \left( \nabla \cdot \| v \|^2 \right) \right) \]
\[
\begin{align*}
\left(\frac{\partial^2}{\partial y'^2}\right) \varphi &= (\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial t^2}) \varphi \\
\text{WAD} \quad \text{WAD} \\
2 \left\{ \frac{\partial^2}{\partial t^2} \right\} \varphi + 2 \frac{\partial^2}{\partial y'^2} \varphi &= -\frac{\partial \varphi}{\partial t} \\
\text{Wave}\quad \text{Activity}\quad \text{Density} \\
\text{WAD} \quad \text{WAD} \\
\text{pseudo-momentum} \\
2 \frac{\partial^2}{\partial t^2} \varphi - 2 \frac{\partial}{\partial y} \left( \frac{\partial \varphi}{\partial y} \right) &= -\frac{\partial}{\partial t} \left\{ \left( \frac{\partial \varphi}{\partial x} \right)^2 \right\} \\
\text{absent} \\
\text{drag} \\
\text{damping} \\
\text{mixing (Cabot order)} \\
\text{Flow behavior to wave momentary density} \\
\text{Cheney - Osher) Thm.} \\
\text{non-acceleration thm.} \\
\text{ZF's wave momentum density.} \\
\text{Cannot accelerate (or maintain as drag)} \\
\text{zonal flow without changing convection) wave intensity.}
\end{align*}
\]
Note:\[ -\frac{\mathbf{\hat{k}} \cdot \langle \mathbf{\hat{z}}^2 \rangle}{2 \mathbf{\hat{k}} \cdot \mathbf{d} \langle \mathbf{\hat{z}} \rangle / \mathbf{d} \mathbf{y}} \]

\[ I = \partial \Phi + \mathbf{\mathbf{\hat{E}}} \cdot \mathbf{\mathbf{\hat{n}}} \]
absent mean field

\[ \frac{d \langle \mathbf{\hat{z}} \rangle}{d \mathbf{y}} = \mathbf{\mathbf{\hat{E}}} \]

\[ \langle \mathbf{\mathbf{\hat{z}}} \rangle = \frac{\mathbf{\mathbf{\hat{k}}}^2 \mathbf{\mathbf{\hat{z}}}}{2} \]

\[ -\frac{\mathbf{\mathbf{\hat{k}}} \cdot \mathbf{\mathbf{\hat{k}}} \mathbf{\mathbf{\hat{z}}} \mathbf{\mathbf{\hat{z}}}}{2 \mathbf{\mathbf{\hat{k}}} \cdot \mathbf{\mathbf{\hat{k}}} \mathbf{\mathbf{\hat{z}}} / d \mathbf{y}} = \frac{\mathbf{\mathbf{\hat{k}}} \mathbf{\mathbf{\hat{E}}}}{-\mathbf{\mathbf{\hat{k}}} \cdot \mathbf{\mathbf{\hat{D}}} \mathbf{\mathbf{\hat{D}}}} = \frac{\mathbf{\mathbf{\hat{k}}} \mathbf{\mathbf{\hat{E}}}}{\mathbf{\mathbf{\hat{W}}}} \]
Action Density

\[ = \mathbf{\mathbf{\hat{k}}} \cdot \mathbf{\mathbf{\hat{N}}} \mathbf{\mathbf{\hat{N}}} \]

\[ = \mathbf{\mathbf{\hat{p}}} \mathbf{\mathbf{\hat{w}}} \text{ wave momentum density} \]

\[ \mathbf{\mathbf{\hat{d}}} \left\{ \langle \mathbf{\mathbf{\hat{u}}} \rangle - \mathbf{\mathbf{\hat{p}}} \mathbf{\mathbf{\hat{w}}} \right\} = -\mathbf{\mathbf{\hat{m}}} \langle \mathbf{\mathbf{\hat{u}}} \rangle \]

\[ - \frac{\mathbf{\mathbf{\hat{z}}} \mathbf{\mathbf{\hat{y}}} / \mathbf{d} \mathbf{y}}{d \mathbf{y}} + \ldots \]
\[ u \sim \frac{v(0)}{e^{\frac{m}{2} - \frac{e^2}{2}}} \]
\[ e^{\frac{m}{2}} \sim \frac{m}{8e^2} \sim \frac{e^2}{m} \]
\[ e \sim e^{1/2} \cdot e^{\frac{3}{2}} \leq \ell \]