Basics of Turbulence I:
A Look at Homogeneous Systems

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Approach

• Highly Pedagogic

Turbulence

scale

Homogeneous Problems I
- Cascade
- Spectra
- Wave Interactions
...

space

Inhomogeneous Problems II
- Mixing length, profiles
- Pipe, wake flow
- ‘Turbulence spreading’
- Avalanches
...

• Focus on simplest problems
Outline

• Basic Ideas
• K41 and Beyond
• Turbulence in Flatland – 2D Fluid Turbulence
• First Look at MHD Turbulence
Model

- Unless otherwise noted:

\[
\rho \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} - \nu \nabla^2 \vec{v} \right) = -\nabla P + \vec{f}
\]

\[\nabla \cdot \vec{v} = 0\]

- Finite domain, closed, periodic
- \( Re = v \cdot \nabla v / \nu \nabla^2 v \sim VL/\nu \); \( Re \gg 1 \)

- Variants:
  - 2D, QG
  - Compressible flow
  - Pipe flow – inhomogeneity
  ....

Random forcing (usually large scale)
What is turbulence?

- Spatio-temporal “disorder”
- Broad range of space-time scales
- Power transfer / flux thru broad range of scales *
- Energy dissipation and irreversibility as $Re \to \infty$ *

And:

- Decay of large scales
- Irreversible mixing
- Intermittency / burstiness
What is difference between turbulence and noise/equilibrium fluctuations?

• Power transfer dominant

• Irreversibility for $\nu \rightarrow 0$

• Noisey thermal equilibrium: (ala’ Test Particle Model)

Emission <-> absorption balance, locally

Fluctuation-Dissipation Theorem applies

• Turbulence:

Flux ~ emission – absorption

Flux dominant for most scales
Why broad range scales?
What motivates cascade concept?

A) Planes, trains, automobiles…

DRAG

- Recall: $F_d \sim c_D \rho A V^2$
- $C_D = C_D(Re) \rightarrow$ drag coefficient
The Point:

- Energy dissipation is finite, and due to viscosity, yet does not depend explicitly on viscosity \( \Rightarrow \text{ANOMALY} \)
- ‘Irreversibility persists as symmetry breaking factors vanish’

i.e. \( \frac{dE}{dt} \sim F_d V \sim C_D \rho A V^3 \)

\[ \frac{dE}{dt} \sim \frac{V^3}{l_0} \equiv \epsilon \rightarrow \text{dissipation rate} \quad l_0 \rightarrow \text{macro length scale} \]

Where does the energy go?

Steady state \( \nu \langle (\nabla \vec{v})^2 \rangle = \langle \vec{f} \cdot \vec{v} \rangle = \epsilon \)
• So $\epsilon = \nu \langle (\nabla v)^2 \rangle \leftarrow$ independent of $\nu$

• $(\nabla v)_{rms} \sim \frac{1}{\nu^{1/2}} \Rightarrow$ suggests $\Rightarrow$ singular velocity gradients (small scale)

∴

• Flat $C_D$ in $Re \rightarrow$ turbulence must access small scales as $Re \rightarrow \infty$

• Obviously consistent with broad spectrum, via nonlinear coupling
B) … and balloons

- Study of ‘test particles’ in turbulence:
- Anecdotal:
  Titus Lucretius Caro: 99-55 BC
  “De rerum Nature” cf. section V, line 500
- Systematic:
  L.F. Richardson: - probed atmospheric turbulence by study of balloon separation
  Noted: $\langle \delta l^2 \rangle \sim t^3 \rightarrow$ super-diffusive
    - not $\sim t$, ala’ diffusion, noise
    - not exponential, ala’ smooth chaotic flow
Upshot:

\[ \delta V(l) = \left( \left( \hat{v}(\vec{r} + \hat{l}) - \hat{v}(\vec{r}) \right) \cdot \frac{\hat{l}}{|\hat{l}|} \right) \rightarrow \text{structure function} \rightarrow \text{velocity differential across scale} \]

Then: \( \delta V \sim l^\alpha \)

so, \( \frac{dl}{dt} \sim l^\alpha \rightarrow \text{growth of separation} \)

\[ \rightarrow \langle l^2 \rangle \sim t^{\frac{2}{1-\alpha}} \sim t^3 \]

\[ \rightarrow \alpha = \frac{1}{3} \]

so \( \delta V(l) \sim l^{1/3}, \langle \delta l^2 \rangle \sim t^3 \)

→ Points:

- large eddys have more energy, so rate of separation increases with scale

- Relative separation is excellent diagnostic of flow dynamics

cf: tetrads: Siggia and Shraiman
Roughness:

N.B. turbulence is spatially “rough”, i.e. \( \delta V(l) \sim \epsilon^{1/3} \ l^{1/3} \)

\[
\lim_{l \to 0} \frac{V(\vec{r} + \vec{l}) - V(\vec{r})}{l} = \lim_{l \to 0} \frac{\delta V(l)}{l} = \epsilon^{1/3} / l^{2/3}
\]

→ strain rate increases on smaller scales

- turbulence develops progressively rougher structure on smaller scales
• Where are we?
  
  – turbulence develop singular gradients to maintain $C_D$ indep. $Re$
  
  – turbulent flow structure exhibits
    
    • super-diffusive separation of test particles
    
    • power law scaling of $\delta V(l)$

→

• Cascade model – K41
K41 Model (Phenomenological)

- Cascade $\rightarrow$ hierarchical fragmentation

- Broad range of scales, no gaps

- Described by structure function
  
  - $\langle \delta v(l)^2 \rangle \leftrightarrow$ energy, of great interest
  
  - $\langle \delta V(l)^2 \rangle, \ldots \langle \delta V(l)^n \rangle, \ldots$

  Related to energy distribution $\leftrightarrow$ greatest interest

- higher moments more challenging
• Input:

• 2/3 law (empirical)

\[ S_2(l) \sim l^{2/3} \]

• 4/5 law (Rigorous) - TBD

\[ \langle \delta V(l)^3 \rangle = -\frac{4}{5} \epsilon l \]

\[ \frac{\delta V}{l^3} = \frac{-4}{5} \epsilon l \]

→ Ideas:

• **Flux** of energy in scale space from \( l_0 \) (input/integral scale) to \( l_d \) (dissipation) scale

  – set by \( \nu \)

• Energy flux is **same** at all scales between \( l_0, l_d \) \( \leftrightarrow \) self-similarity
And

- Energy dissipation – set as $\nu \to 0$ but not at $\nu = 0$
- * Asymmetry of breaking or stirring etc. lost in cascade: symmetry restoration
- N.B. intermittency <-> ‘memory’ of stirring, etc
- **Ingredients / Players**
  - Exciton $\rightarrow$ eddy (not a wave / eigenmode!)
  - $l$: scale parameter, eddy scale
  - $\delta V(l)$: velocity increment. Hereafter $V(l)$
- $V_o$: rms eddy fluctuation (large scale dominated)
- $\tau(l)$: eddy transfer / life-time / turn-over rate
- characteristic scale of transfer in cascade step

\[
\epsilon = \frac{V(l)^2}{\tau(l)}
\]

- Self-similarity → constant flow-thru rate $\epsilon = V(l)^2/\tau(l)$
- What is $\tau(l)$?? Consider…
The possibilities:

- Dimensionally, $\tau(l)$ is ‘lifetime’ of structure of scale $l$, time to distort out of existence

So

- $l' > l$
  - Larger scales advect eddy but don’t distort it
  - Physics can’t change under Galilean boost
cf: Rapid distortions, shearing

- $l' < l$
  - Irrelevant $\rightarrow$ insufficient energy

- $\tau(l) \sim l/V(l)$, set by $l' \sim l$
→ So

V(l)² \sim V(l)³ / l \rightarrow V(l) \sim (\epsilon l)^{1/3} \; ; \; 1/ \tau(l) \sim (\epsilon / l^2)^{1/3}

V(l)² \sim V_0² (l / l_0)^{2/3} \; (transfer rate increases as scale decreases)

And

E(k) \sim \epsilon^{2/3} k^{-5/3} \quad E = \int dkE(k)

→ Where does it end?
• Dissipation scale
  - cut-off at $1/\tau(l) \sim \nu/l^2$ i.e. $Re(l) \rightarrow 1$
  - $l_d \sim \nu^{3/4}/\epsilon^{1/4}$

• Degrees of freedom

\[ \#DOFs \sim \left(\frac{l_0}{l_d}\right)^3 \sim Re^{9/4} \]

For $l_o \sim 1km$, $l_d \sim 1mm$ (PBL)

$\Rightarrow N \sim 10^{18}$
Anything missing here?

- **Dynamics!**

  i.e.
  - How is the energy transferred?
  - How are small scales generated?
  - Where have the N.S. equations gone?
  ...

- **Enter vorticity!**

  \[ \omega = \nabla \times \mathbf{v} ; \quad \partial_t \mathbf{v} = \nabla \times \mathbf{v} \times \omega + \nu \nabla^2 \mathbf{v} \]

- \[ \Gamma = \int \phi \mathbf{v} \cdot d\mathbf{l} \sim \text{const. to } \nu \quad (\text{Kelvin's theorem}) \]

So

- Vortex tube stretching
  \[ \frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \nabla \omega = \omega \cdot \nabla \mathbf{v} + \nu \nabla^2 \omega \]
- Strain tensor \( \omega \cdot \mathbf{S} \)
• Stretching:

  ![Diagram](image)

  – Small scales generated ($\nabla \cdot \vec{v} = 0$)
  – Energy transferred to small scale

• Enstrophy $\Omega = \langle \omega^2 \rangle$

  \[
  \frac{d\omega^2}{dt} = \vec{\omega} \cdot (\vec{\omega} \cdot \nabla \vec{v}) + \cdots \sim \omega^3 + \cdots
  \]

  – Enstrophy increases in 3D N-S turbulence
  – Growth is strongly nonlinear

• Enstrophy production underpins forward energy cascade
• Where are we?

“Big whorls have little whorls that feed on their velocity. And little whorls have lesser whorls. An so on to viscosity.” – L.F. Richardson, 1920

After: “So naturalists observe a flea has smaller fleas that on him prey; And these have smaller yet to bite ’em, And so proceed ad infinitum. Thus every poet, in his kind, Is bit by him that comes behind.” – Jonathan Swift, “On Poetry, a Rhapsody”, 1793
The Theoretical Problem

• “We don’t want to think anything, man. We want to know.”
  – Marsellus Wallace, in “Pulp Fiction” (Quentin Tarantino)

• What do we know?
  – 4/5 Law (and not much else...)

\[
\langle V(l)^3 \rangle = -\frac{4}{5} \varepsilon l \to \text{asymptotic for finite } l, \nu \to 0
\]
\[
S_2 = \langle \delta V(l)^2 \rangle, \quad S_3 = \langle \delta V(l)^3 \rangle
\]

from:
\[
\frac{\partial S_2}{\partial t} = -\frac{1}{3l^4} \frac{\partial}{\partial l} (l^4 S_3) - \frac{4}{3} \varepsilon + \frac{2\nu}{l^4} \frac{\partial}{\partial l} \left( l^4 \frac{\partial S_2}{\partial l} \right)
\]

(Karman-Howarth)

• Stationarity, \( \nu \to 0 \)
\[ S_3(l) = -\frac{4}{5} \epsilon l \]

- Energy thru-put balance \( \langle \delta V(l)^3 \rangle / l \leftrightarrow \epsilon \)

- Notable:
  - Euler: \( \partial_t v + v \cdot \nabla v + \nabla P / \rho = 0; \) reversible; \( t \to -t, v \to -v \)
  - N-S: \( \partial_t v + v \cdot \nabla v + \nabla P / \rho = \nu \nabla^2 v; \) time reversal broken by viscosity
  - \( S_3(l): S_3(l) = -\frac{4}{5} \epsilon l; \) reversibility breaking maintained as \( \nu \to 0 \)

4/5 Law
- Asymptotically exact \( \nu \to 0, l \) finite
- Unique, rigorous result
• Extensions:

  MHD: Pouquet, Politano

  2D: Celari, et. al. (inverse cascade, only)

What of so called ‘entropy cascade’ in Vlasov turbulence?
• N.B.: A little history; philosophy:
  – ‘Anomaly’ in turbulence $\rightarrow$ Kolmogorov, 1941
  – Anomaly in QFT $\rightarrow$ J. Schwinger, 1951 (regularization for vacuum polarization)

• Speaking of QFT, what of renormalized perturbation theory?
  – Renormalization gives some success to low order moments, identifies relevant scales
  – Useful in complex problems (i.e. plasmas) and problems where $\tau_{int}$ is not obvious
  – Rather few fundamental insights have emerged from R.P.T
    Caveat Emptor
Turbulence in Flat Land

• 2D systems $\rightarrow$ 1 dimension constrained
  
i.e. Atmospheric $\leftrightarrow$ rotation $\Omega_0$
  Magnetized plasma $\leftrightarrow B_0$, $\Omega_c$
  Solar interior $\leftrightarrow$ stratification, $\omega_{B-V}$

• Simple 2D fluid:

$$\frac{d\omega}{dt} = \omega \cdot \nabla \vec{v} + \nu \nabla^2 \omega$$

$$\vec{v} = \nabla \phi \times \hat{z}$$
$$\omega = -\nabla^2 \phi$$

$\Rightarrow$

$$\partial_t \nabla^2 \phi + \nabla \phi \times \hat{z} \cdot \nabla \nabla^2 \phi = \nu \nabla^2 \nabla^2 \phi + \vec{s}$$

- $\omega$ constant along fluid trajectories, to $\nu$
- $\omega = \nabla^2 \phi$ akin conserved phase space density

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{q}{m} E \frac{\partial f}{\partial v} = C(f)$$

$V/L \ \Omega_{eff} < 1$

Low Rossby number
• The problem:

  – Enstrophy now conserved: $\vec{\omega} \cdot \nabla \vec{v} = 0$

  – Two inviscid invariants:
    
    • Enstrophy $\Omega = \langle (\nabla^2 \phi)^2 \rangle$
    
    • Energy $E = \langle (\nabla \phi)^2 \rangle$

  – Might ask: Where do these want to go, in scale?

  – Enstrophy:

  
  + turbulent flow ➔

  Isovorticity contour

  ➔ Stretched contour, $\langle (\nabla \omega)^2 \rangle \uparrow$

  ➔ Enstrophy to small scale
• Energy

– Expect \((\Delta k)^2\) increases

– What of centroid \(\vec{k}\)?

\[
(\Delta k)^2 = \frac{1}{E} \int dk (k - \vec{k})^2 E(k)
\]

\[
\vec{k} = \frac{1}{E} \int dk E(k)
\]

But

\[
(\Delta k)^2 = \frac{1}{E} \int dk (k^2 - 2k\vec{k} + \vec{k}^2) E(k) = \frac{1}{\Omega} \left(\Omega - \vec{k}^2\right)
\]

\[
\partial_t (\Delta k)^2 > 0 \Rightarrow \partial_t \vec{k} < 0 \quad \Omega \text{ conserved!}
\]

\Rightarrow \text{energy should head toward large scale}
• Dilemma:
  - Energy seeks large scale
  - Enstrophy seeks small scale
  - How accommodate self-similar transfer – i.e. cascade – of both?

⇒ Dual cascade (R.H. Kraichnan)
  - **Forward** self-similar transfer of enstrophy
    → toward small scale dissipation
  - **Inverse** transfer of energy
    → scale independent dissipation?

(Low $k$ sink)

Fig. 2.17. Schematic of energy spectrum for dual cascade.
• Spectra
  – Enstrophy range:
    \[ E(l) \to kE(k) \]
    \[ 1/\tau(l) \to k[kE(k)]^{1/2} \]
    \[ \to E(k) = \eta^{2/3} k^{-3} \]
  – Energy range: ala’ K41; \( E(k) = \epsilon^{2/3} k^{-5/3} \)

• Pair dispersion:
  – Energy range: ala’ Richardson
  – Enstrophy range: exponential divergence

• Scale independent dissipation critical to stationary state
→ Where do we stand now?

“Big whorls meet bigger whorls, And so it tends to go on. By merging they grow bigger yet, And bigger yet, and so on.”

- M. McIntyre, after L.F. Richardson
• Cautionary tale: coherent structures happen!

\[ \rho = -\nabla^2 \phi, \quad S = \frac{\partial^2 \phi}{\partial x \partial y} \rightarrow \text{local flow shear} \]

\[ \partial_t \nabla \rho = (s^2 - \rho^2)^{1/2}; \quad \text{criterion for “coherence”} \]

\[ \rightarrow \text{Gaussian curvature of stream function predicts stability} \]

• Depending upon forcing, dynamics be cascade or coherent structure formation, or both:

• Need a non-statistical criterion, i.e. Okubo-Weiss

\[ \rho = -\nabla^2 \phi, \quad S = \frac{\partial^2 \phi}{\partial x \partial y} \rightarrow \text{local flow shear} \]

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Decay experiment
\[ \rightarrow \text{Isolated coherent vortices appear in turbulent flow} \]

McWilliams, ‘84 et. seq.
Herring and McWilliams ‘85
• **MHD turbulence - A First Look**
  
  – HUGE subject – includes small scale and mean field dynamo problems (c.f. Hughes lectures)
  
  – Here, focus on Alfvenic turbulence i.e. (Kraichnan-Iroshnikov-Goldreich-Sridhar …) \( \rightarrow \) wave turbulence
    
    • Strong mean \( \vec{B}_0 \)
    
    • \( \delta B < B_0, \nabla \cdot \vec{v} = 0 \)
    
    • Shear-Alfven wave turbulence
  
  – Best described by reduced MHD: (Ohm’s Law, \( \nabla \cdot J = 0 \))

\[
\begin{align*}
\frac{\partial A_{\parallel}}{\partial t} + \nabla_{\perp} \times \hat{z} \cdot \nabla_{\perp} A_{\parallel} &= B_0 \partial_z \phi + \eta \nabla^2 A_{\parallel} \\
\frac{\partial}{\partial t} \nabla^2 \phi + \nabla_{\perp} \phi \times \hat{z} \cdot \nabla_{\perp} \nabla^2 \phi^2 &= B_0 \partial_z \nabla^2 A_{\parallel} + \nabla_{\perp} A_{\parallel} \times \hat{z} \cdot \nabla_{\perp} \nabla^2 A_{\parallel} + \nu \nabla^2 \nabla^2 \phi + \tilde{S}
\end{align*}
\]
• Observations:
  – All nonlinear scattering is perpendicular
  – Contrast N-S, eddys with $\omega = 0$

Now: Alfven waves: $\omega^2 = k_\parallel^2 V_A^2$

– If uni-directional wave population:
  i.e. $A = f(z - V_A t) + g(z + V_A t)$
  then $f$ is exact solution of MHD

⇒ Need counter-propagating populations to manifest nonlinear interaction

– See also resonance conditions
  \[ \omega_1 + \omega_2 = \omega_3 \]
  \[ k_{\parallel 1} + k_{\parallel 2} = k_{\parallel 3} \]
• For Alfven wave cascade:

\[ \epsilon = T(k \rightarrow k + \Delta k) E(k) \rightarrow E(k)/\tau(k) \]

transition rate

• Recall Fermi Golden Rule:

\[ T_{i,j} \sim \frac{2\pi}{\hbar} |\langle i|H_{int}|j \rangle|^2 \delta(E_j - E_i - \hbar \omega) \]

\[ \Rightarrow T \sim \frac{V(l_d)^2}{l^2} \tau_{int}(l_\perp) \]

\[ V(l_\perp)^2 \rightarrow \text{scatter energy} \]

\[ 1/l^2 \rightarrow (cc)^2 \]

• \( \tau_{int}(l) = 1/(\Delta k_\parallel)V_A \)

\[ \Rightarrow \text{Alfvenic transit time (} \Delta k_\parallel \sim k_\parallel) \]

Packet passage

\[ \rightarrow \]
Enter the Kubo number

\[
\frac{l_{\parallel ac}}{\Delta \perp} \frac{\delta B}{B_0} \sim \left( \frac{V_A \delta B / B}{l_{\perp}} \right) |\Delta k_{\parallel} V_A|
\]

- Basically: \( B \cdot \nabla \rightarrow B_0 \partial_z + \tilde{B} \cdot \nabla_{\perp} \)
  - Linear: \( B_0 \partial_z \)
  - Nonlinear: \( \tilde{B} \cdot \nabla_{\perp} \)

\( l_{\parallel ac} \), \( \Delta \perp \), \( B_0 \), \( V_A \), \( \Delta k_{\parallel} \)

i.e. \( K < 1 \rightarrow \) weak scattering, diffusion process

\( K > 1 \rightarrow \) strong scattering, \( \sim \) de-magnetization \( \sim \) percolation

\( K = 1 \rightarrow \) (critical) balance
Why Kubo?

• But… “It ain’t over till its over”

- Eastern (division) philosopher

• As $l_\perp$ drops, $V(l_\perp)/l_\perp \to (\Delta k_\parallel)V_A$

$\Rightarrow \quad \tau_\perp \to \tau_\parallel \quad Ku \to 1$

• Critically balanced cascade, $Ku \sim 1$

\[ i.e. \quad \frac{V(l_\perp)}{l_\perp} \sim V_A \frac{\delta B(l_\perp)}{B_0} \sim (\Delta k_\parallel)V_A, \quad \text{unavoidable at small scale} \]

- Statement that transfer sets $K \approx 1$

- Attributed to G.-S. ‘95 but:

  “the natural state of EM turbulence is $K \sim 1$”

  - Kadomtsev and Pogutse ‘78
• If now \( \frac{1}{\tau_{int}(l_\perp)} \sim \frac{V(l_\perp)}{l_\perp} \)

- Recover K41 scaling in MHDT, \( F(k_\perp) \sim \epsilon^2 k_\perp^{\frac{5}{3}} \)
- "Great Power Law in the Sky"

• Eddy structure:

\[
k_\parallel V_A \sim \frac{V(l_\perp)}{l_\perp} \Rightarrow k_\parallel \sim k_\perp^3 \epsilon^\frac{1}{3} / V_A \quad \Rightarrow \text{anisotropy increases as } l_\perp \downarrow
\]

• Many variants, extensions, comments, "we did it too's"…
Fate of Energy?

- End point is dissipation

- What is dissipative structure?
  - Dimension < 3 $\Rightarrow$ fractal and multi-fractal intermittency models
  - Structure:
    - Vortex sheet
    - Current sheet
    $\Rightarrow$ Stability $\Rightarrow$ micro-tearing, etc.

- Energy leak to kinetic scales?
  - Electron vs ion heating
  - Particle acceleration ($2^{nd}$ order Fermi)
Conclusion

• This lecture is not even the “end of the beginning”

• A few major omissions:
  – pipe flow turbulence – Prandtl law of the wall
  – spatial structures, mixing, spreading
  – general theory of wave turbulence - Qiu, P.D.
  – MHDT + small scale dynamo - Hughes
  – kinetic/Vlasov turbulence – Sarazin, Qiu, Dif-Pradalier
  – Langmuir collapse … - Kosuga