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Chapter 1

Preliminaries

1.1 Introduction

The quantum Hall effect (QHE) refers to a set of phenomena and associated phases of matter found in two-dimensional electron gases subjected to a large perpendicular magnetic field\(^1\). The phenomena are typically divided into two classes, the integer quantum Hall effect (IQHE) and the fractional quantum Hall effect (FQHE), depending on the Landau level filling fraction, given by \( \nu = n \frac{hc}{eB} \), where \( n \) is the two-dimensional electron density and \( B \) the magnetic field strength. The combination \( \phi_0 = \frac{hc}{e} = 4.137 \times 10^{-7} \text{G} \cdot \text{cm}^2 \) is the Dirac flux quantum\(^2\), hence

\[
\nu = 4.14 \cdot n[10^{11} \text{cm}^{-2}] / B[\text{T}] .
\]

Thus, in a field of \( B = 4.14 \text{T} \), the Landau level (LL) filling fraction \( \nu = 1 \) occurs for an electron density \( n = 10^{11} \text{cm}^{-2} \).

The IQHE was discovered by von Klitzing in 1980 in routine magnetotransport studies of silicon MOSFETs\(^3\). The FQHE was discovered by Tsui, Störmer, and Gossard in 1982 \(^4\), in GaAs–Al\(_x\)Ga\(_{1-x}\)As heterojunctions. The experimental setup is depicted in Fig. 1.1, and some spectacular data shown in Fig. 1.2. An electrical current \( I \) is established along the \( \hat{x} \) direction, and the longitudinal and transverse voltage drops \( V_L \) and \( V_H \) are measured, from which one obtains, in the linear response regime, the resistances \( R_L = V_L / I \) and \( R_H = V_H / I \). In the IQHE, one observes that \( R_H \) remains constant along plateaus as the filling fraction \( \nu \) is varied (either by varying the electron density \( n \), typically with a gate, or by varying the magnetic field \( B \)). The plateau values are given by \( R_H = \frac{h}{pe^2} \), where \( p \in \mathbb{Z} \) is an integer, for \( \nu = p \). In the FQHE,

\(^1\)The effect has been seen in hole gases as well.
\(^2\)This is often more conveniently expressed as \( \phi_0 = 4.137 \times 10^5 \text{T} \cdot \text{Å}^2 \), where \( 1 \text{T} = 10^4 \text{G} \).
one observes plateaus at rational fractions $\nu = p/q$, typically with $q$ odd, where $R_H = qh/pe^2$. The quantity $R_Q = h/e^2 = 25.812.8 \, \Omega$ is known as the quantum of resistance.

### 1.1.1 Resistance, conductance, resistivity, conductivity

In the linear response regime, one has $V_\alpha = R_{\alpha\beta} I_\beta$, where $R$ is the resistance tensor. It's matrix inverse, $G = R^{-1}$, is known as the conductance, with $I_\alpha = G_{\alpha\beta} V_\beta$. The units of each element $R_{\alpha\beta}$ of the resistance tensor are Ohms ($\Omega$), hence the units of $G_{\alpha\beta}$ are $\Omega^{-1}$.

Resistance and conductance are not materials parameters (i.e. intensive quantities); you can’t look up the resistance of copper in a table, for example. If, *ceteris paribus*, you double the length of a copper wire, its resistance doubles. What doesn’t change is the metal’s resistivity, $\rho$, which *is* a materials parameter. The corresponding linear response relation is between current density and electric field, viz. $E_\alpha = \rho_{\alpha\beta} j_\beta$. The inverse of the resistivity tensor is the conductivity tensor $\sigma = \rho^{-1}$, for which $j_\alpha = \sigma_{\alpha\beta} E_\beta$.

For an isotropic $d$-dimensional cube of side length $L$, in zero magnetic field, if the current along one of the cubic axes is $I$ then the current density is $j = I/L^{d-1}$. Similarly, if the voltage drop along this axis is $V$, the electric field is $E = V/L$. Thus $R = V/I = \rho L^{2-d}$, and we see that resistance and resistivity in general have different units. Similarly $G = \sigma L^{d-2}$. In two dimensions, resistance and resistivity have the same dimensions, but nevertheless resistance is a geometric quantity. Consider a $L_x \times L_y$ rectangular sample with conductivity tensor

$$
\sigma = \begin{pmatrix}
\sigma_{xx} & \sigma_{xy} \\
\sigma_{yx} & \sigma_{yy}
\end{pmatrix}
$$

---

5. The even denominator quantum Hall effect is very interesting and distinct from the odd denominator effect.

6. Assuming, that is, that the length $L$ is longer than the inelastic scattering (or phase breaking) length, $\ell_\phi$. For $L < \ell_\phi$, quantum interference effects become important and Ohm’s law is no longer valid.

7. The resistivity will in general depend on the temperature, and on the density and type of impurities present, as well as on the material itself.
1.1. INTRODUCTION

Figure 1.2: Low temperature ($T \approx 150 \text{mK}$) longitudinal resistivity $\rho_{xx}$ and Hall resistivity $\rho_{xy}$ as a function of applied magnetic field in a two-dimensional electron gas system (GaAs/AlGaAs heterostructure), from R. Willett et al., Phys. Rev. Lett. 59, 1776 (1987). Each dip in $\rho_{xx}$ and concomitant plateau in $\rho_{xy}$ corresponds to a distinct phase of matter.

with $j = \sigma E$. In general, linear response transport is described by the set of equations $J_i = L_{ik}F_k$, where the $\{J_i\}$ are generalized currents and the $\{F_k\}$ generalized forces. Onsager reciprocity\(^8\) then requires

$$L_{ik}(B) = \eta_i \eta_k L_{ki}(-B),$$

(1.3)

with no sum on $i$ or $k$, where $\eta_i = \pm 1$ according to whether $J_i$ is symmetric or antisymmetric under time reversal, i.e. $J_i^T = \eta_i J_i$. Thus, $\sigma_{yx}(B) = \sigma_{xy}(-B)$ since both $j_x$ and $j_y$ are odd under time reversal. But $B \rightarrow -B$ reverses the orientation of the $(\hat{x}, \hat{y}, \hat{B})$ triad, hence $\sigma_{xy}(-B) = -\sigma_{xy}(B)$, and we have that the off-diagonal elements of the conductivity tensor are antisymmetric: $\sigma_{yx}(B) = -\sigma_{xy}(B)$. Now let's write the current densities as $j_x = I_x/L_y$ and $j_y = I_y/L_x$, and the fields as $E_x = V_x/L_x$ and $E_y = V_y/L_y$. We then have

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} \begin{pmatrix} L_y^{-1} & 0 \\ 0 & L_x^{-1} \end{pmatrix} \begin{pmatrix} I_x \\ I_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} L_x^{-1} & 0 \\ 0 & L_y^{-1} \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix},$$

(1.4)

\(^8\)See L. D. Landau and E. M. Lifshitz, Statistical Physics, part I, §120.
from which we read off the relation between conductance and conductivity tensors,
\[
\begin{pmatrix}
G_{xx} & G_{xy} \\
G_{yx} & G_{yy}
\end{pmatrix} = \begin{pmatrix}
L_y & 0 \\
0 & L_x
\end{pmatrix} \begin{pmatrix}
\sigma_{xx} & \sigma_{xy} \\
-\sigma_{xy} & \sigma_{yy}
\end{pmatrix} \begin{pmatrix}
L_x^{-1} & 0 \\
0 & L_y^{-1}
\end{pmatrix} = \begin{pmatrix}
L_y \sigma_{xx} & L_y \sigma_{xy} \\
L_x \sigma_{xy} & L_x \sigma_{yy}
\end{pmatrix}. \tag{1.5}
\]

Similarly, the relation between resistance and resistivity tensors is
\[
\begin{pmatrix}
R_{xx} & R_{xy} \\
R_{yx} & R_{yy}
\end{pmatrix} = \begin{pmatrix}
L_y \rho_{xx} & L_y \rho_{xy} \\
L_x \rho_{xy} & L_x \rho_{yy}
\end{pmatrix}. \tag{1.6}
\]

Finally,
\[
\rho = \begin{pmatrix}
\rho_{xx} & \rho_{xy} \\
-\rho_{xy} & \rho_{yy}
\end{pmatrix} = \begin{pmatrix}
\sigma_{xx} & \sigma_{xy} \\
-\sigma_{xy} & \sigma_{yy}
\end{pmatrix}^{-1} = \frac{1}{\sigma_{xx}^2 + \sigma_{xy}^2} \begin{pmatrix}
\sigma_{yy} & -\sigma_{xy} \\
\sigma_{xy} & \sigma_{xx}
\end{pmatrix} = \sigma^{-1}. \tag{1.7}
\]

Along the QH plateaus, as \( T \to 0 \), the longitudinal resistivity vanishes as \( \rho_{xx}(T) \propto e^{-\Delta/k_BT} \), where \( \Delta \) is the energy gap for transport. Thus, at \( T = 0 \) the resistivity and conductivity tensors are purely off-diagonal, with \( \rho_{\alpha\beta} = \rho_{xy} \epsilon_{\alpha\beta} \) and \( \sigma_{\alpha\beta} = \sigma_{xy} \epsilon_{\alpha\beta} \), with \( \rho_{xy} = 1/\sigma_{xy} \).

### 1.1.2 Semiclassical magnetotransport theory

Combining Newton’s second law with the Lorentz force law for a particle of charge \(-e\) and mass \(m\), we have
\[
\frac{dp}{dt} = -eE - \frac{e}{c} \frac{p}{m} \times B - \frac{p}{\tau}, \tag{1.8}
\]
where the last term is a frictional force which in metals and semiconductors typically comes from electron-impurity scattering\(^9\), with \( \tau \) the transport scattering time\(^10\). We take \( B = B\hat{z} \), and write the current density as \( j = -nep/m \). Defining the cyclotron frequency \( \omega_c = eB/mc \), and setting \( \dot{p} = 0 \) in steady state, we obtain
\[
\frac{ne^2\tau}{m} E + \omega_c \tau j \times \hat{z} + j = 0 \tag{1.9}
\]
the solution of which is \( j = \sigma E \), where the conductivity tensor is
\[
\sigma = \frac{ne^2\tau/m}{1 + \omega_c^2\tau^2} \begin{pmatrix}
1 & -\omega_c\tau \\
\omega_c\tau & 1
\end{pmatrix}. \tag{1.10}
\]

Taking the inverse, we have \( E = \rho j \), with resistivity tensor
\[
\rho = \sigma^{-1} = \frac{m}{ne^2\tau} \begin{pmatrix}
1 & \omega_c\tau \\
-\omega_c\tau & 1
\end{pmatrix}. \tag{1.11}
\]

\(^9\)Electron-phonon scattering, electron-electron scattering, and boundary scattering are also present.

\(^{10}\)There is an important difference between the single particle scattering time \( \tau_{sp} \) and the transport scattering time \( \tau_{tr} \). See, e.g., §1.5 of my Physics 211B lecture notes for details.
What is \( n \)? Naïvely one might think it is the total electron density, but of course this is wrong. As we know from elementary solid state physics, filled Bloch bands are inert and carry no net current. A somewhat more realistic linearized Boltzmann equation approach, assuming an isotropic parabolic conduction band with electron carriers, yields the same result, with \( n = \int d\varepsilon \, g_c(\varepsilon) \, f^0(\varepsilon - \mu) \) the conduction electron density, with \( g_c(\varepsilon) \) the conduction band density of states and \( f^0(\varepsilon - \mu) \) the Fermi function, and \( m \) replaced by the effective mass \( m^* \) of the conduction band. All the fully occupied bands below the conduction band contribute nothing to the current. Note that \( \rho_{xx} = \rho_{yy} = m^*/ne^2\tau \) because the system is isotropic. For the anisotropic parabolic band, where the effective mass tensor \( m^* \) has eigenvalues \( m_{x,y}^* \), then along its principal axes one of course has \( \rho_{xx} = m_x^*/ne^2 \) and \( \rho_{yy} = m_y^*/ne^2 \), with \( \rho_{xy} = B/nec \) as in the isotropic case.

One interesting feature of the semiclassical Boltzmann result is that the diagonal terms of the resistivity tensor are independent of magnetic field. Thus, \( \partial \rho_{xx}/\partial B = 0 \), and the magnetoresistance \( \Delta \rho_{xx}(B) \equiv \rho_{xx}(B) - \rho_{xx}(0) \) vanishes. This is in general not the case if one has multiple bands contributing to the transport current (say conduction electrons as well as valence holes), or in the case where the Fermi surface has open orbits which span the Brillouin zone. Thus, as a function of \( B \), the semiclassical result says that \( \rho_{xx}(B) \) is constant and \( \rho_{xy}(B) \) is perfectly linear. This it completely different from the results shown in Fig. 1.2, except in the very low field regime.

### 1.1.3 Mobility, cyclotron frequency, and electron-electron interactions

The mobility \( \mu \) is defined by the combination \( \mu = e\tau/m^* \). Thus, in zero field, the steady state electron velocity is \( v = \mu E \), so mobility has units of \([\mu] = \text{cm}^2/\text{V} \cdot \text{s}\). In MOSFETs, mobilities are seldom more than a few tens of thousands in these units. But in MBE-grown GaAs heterostructures, mobilities as high as \( 10^7 \text{ cm}^2/\text{V} \cdot \text{s} \) have been achieved. In GaAs, where the conduction band is isotropic and has effective mass \( m^* = 0.067 m_0 \), one finds

\[
\tau = 3.8 \times 10^{-17} \text{s} \cdot \mu \left[ \text{cm}^2/\text{V} \cdot \text{s} \right] .
\]

Thus, for \( \mu = 10^6 \text{ cm}^2/\text{V} \cdot \text{s} \), one obtains \( \tau \simeq 38 \text{ ps} \).

The cyclotron frequency is given by the combination \( \omega_c = eB/m^*c \). With

\[
\phi_0 = \frac{hc}{e} = 4.14 \times 10^{-7} \text{G} \cdot \text{cm}^2 , \quad h = 6.63 \times 10^{-27} \text{erg} \cdot \text{s} = 4.14 \times 10^{-15} \text{eV} \cdot \text{s} , \quad k_B = 8.62 \times 10^{-5} \text{eV/K} ,
\]

Thus, for GaAs conduction electrons, one obtains

\[
\omega_c = 2.63 \times 10^{12} \text{Hz} \cdot B[T] , \quad \omega_c \tau = 10^{-4} \mu \left[ \text{cm}^2/\text{V} \cdot \text{s} \right] B[T] , \quad h\omega_c = 1.73 \text{ meV} \cdot B[T] = 20 \text{ K} \cdot B[T] .
\]

At fields \( B \sim 10 \text{ T} \) and in samples of mobility \( \mu \sim 10^6 \text{ cm}^2/\text{V} \cdot \text{s} \), we have \( \omega_c \tau \sim 1000 \gg 1 \).
As we shall see, quantization introduces a new length scale, \( \ell = (\hbar c/eB)^{1/2} \), called the magnetic length. This depends only on physical constants and the magnetic field strength. One finds \[ \ell = (\phi_0/2\pi B)^{1/2} = 257 \text{Å}/\sqrt{B[T]} \tag{1.15} \]

From this length scale, we construct the energy scale \( e^2/\ell \) for electron-electron interactions. For GaAs, where \( \epsilon = 13 \), we have \[ \frac{e^2}{\ell} = 4.31 \text{meV} \cdot \sqrt{B[T]} = 50.0 \frac{\text{K}}{k_B} \cdot \sqrt{B[T]} \tag{1.16} \]

### 1.1.4 \( \vec{E} \times \vec{B} \) Drift and Separation of Time Scales

For a classical particle of charge \( e \) moving in the (x,y) plane and subjected to a magnetic field \( B = B\hat{z} \), the equations of motion are given by the Lorentz force law, \[ m\ddot{r} = -\nabla V - \frac{e}{c} B \dot{r} \times \hat{z} \tag{1.17} \]

We now write \( r(t) = \mathcal{R}(t) + \xi(t) \). We presume that the guiding center motion \( \mathcal{R}(t) \) executes large excursions, slowly drifting along equipotentials of \( V(r) \), while the cyclotron motion \( \xi(t) \) executes fast small excursions with characteristic time scale \( 2\pi/\omega_c \). This assumption will be borne out in the following analysis.

The zeroth order theory is simply given by \[ \ddot{\mathcal{R}} = -\frac{e}{c} B \dot{\mathcal{R}} \times \hat{z} \] \[ \ddot{\xi} = \omega_c \dot{\mathcal{R}} \times \xi \tag{1.18} \]

Thus, the guiding center executes a slow drift in the direction of \( \nabla V \times \hat{z} \), while the cyclotron coordinate executes counterclockwise circular motion as viewed from above.

Proceeding with the expansion in powers of the cyclotron motion, we have \[ m\ddot{\mathcal{R}}_\alpha + m\ddot{\xi}_\alpha = -\partial_\alpha V(\mathcal{R}) - \xi_\beta \partial_\alpha \partial_\beta V(\mathcal{R}) - \frac{1}{2} \xi_\beta \xi_\gamma \partial_\alpha \partial_\beta \partial_\gamma V(\mathcal{R}) + \ldots \]

\[ -\frac{eB}{c} \epsilon_{\alpha\beta} \dot{\mathcal{R}}_\beta - \frac{eB}{c} \epsilon_{\alpha\beta} \dot{\xi}_\beta \tag{1.19} \]

Here we have used the relation, for any vector \( u \), \[ \epsilon_{\alpha\beta} u_\beta = (u_y, -u_x) = (u \times \hat{z})_\alpha \tag{1.20} \]

We assume \( \dot{\mathcal{R}}_\alpha = 0 \) on average, leading to the slow equation, \[ -\frac{eB}{c} \epsilon_{\alpha\beta} \dot{\mathcal{R}}_\beta = -\partial_\alpha V(\mathcal{R}) - \frac{1}{2} \langle \xi_\beta \xi_\gamma \rangle \partial_\alpha \partial_\beta \partial_\gamma V(\mathcal{R}) - \ldots \tag{1.21} \]
where \( \langle \xi_\alpha \xi_\beta \rangle \) is averaged over the fast motion, and the fast equation,

\[
m \ddot{\xi}_\alpha = -\xi_\beta \partial_\alpha \partial_\beta V(\mathcal{R}) - \frac{eB}{c} \epsilon_{\alpha\beta\gamma} \dot{\xi}_\gamma + \ldots .
\]  

(1.22)

On the fast scale of the \( \xi(t) \) motion, the guiding center \( \mathcal{R}(t) \) is assumed constant. Fourier transforming the fast motion, we write

\[
\xi(t) = \text{Re} \xi^0 e^{-i\omega t},
\]

with

\[
\begin{pmatrix}
-m\omega^2 + V_{xx} & im\omega_c + V_{xy} \\
-im\omega_c + V_{xy} & -m\omega^2 + V_{yy}
\end{pmatrix}
\begin{pmatrix}
\xi^0_x \\
\xi^0_y
\end{pmatrix} = 0
\]

(1.23)

where \( V_{\alpha\beta} \equiv \partial_\alpha \partial_\beta V(\mathcal{R}) \). Solving for \( \omega \), we take the fast root of the resulting quadratic equation and obtain

\[
\omega_+^2 = \frac{1}{2} \left( \omega_c^2 + \frac{V_{xx} + V_{yy}}{m} \right) + \frac{1}{2} \omega^2 \sqrt{1 + \frac{2(V_{xx} + V_{yy})}{m\omega_c^2} + \frac{(V_{xx} - V_{yy})^2}{m^2\omega_c^4} + \frac{V_{xy}^2}{m^2\omega_c^4}}
\]

(1.24)

Thus the local cyclotron frequency is given by \( \omega_c(\mathcal{R}) = \omega_c + \frac{\nabla^2 V(\mathcal{R})}{2m\omega_c} \) to lowest nontrivial order.

We will need the corresponding eigenvector for the high frequency root. Writing \( \xi^0 \equiv (u\xi_0, v\xi_0) \), with \(|u|^2 + |v|^2 = 1\), we have

\[
u = -\frac{V_{xx} - m\omega_+^2}{\sqrt{(V_{xx} - m\omega_+^2)^2 + |V_{xy} + im\omega_c\omega_+|^2}}.
\]

(1.25)

Averaging over the cyclotron motion, we find

\[
\langle \xi_\alpha \xi_\beta \rangle = \frac{1}{2} \xi^2_0 \begin{pmatrix}
|u|^2 & \text{Re}(u\bar{v}) \\
\text{Re}(u\bar{v}) & |v|^2
\end{pmatrix}
\]

(1.26)
Since $\omega_c \approx \omega_c$, we obtain $u \approx \frac{1}{\sqrt{2}}$ and $v \approx \frac{1}{\sqrt{2}}$. Thus the guiding center motion is given by the equation

$$\frac{eB}{c} \epsilon_{\alpha\beta} \dot{R}_\beta = -\partial_\alpha V - \frac{1}{4} \xi_0^2 \partial_\alpha \nabla^2 V \equiv -\partial_\alpha V_{\text{eff}}(\mathcal{R}) \quad (1.27)$$

where the effective guiding center potential is

$$V_{\text{eff}}(\mathcal{R}) = V(\mathcal{R}) + \frac{1}{4} \xi_0^2 \nabla^2 V(\mathcal{R}) \quad (1.28)$$

This makes good physical sense: as the electron moves slowly along the equipotential, it samples, through its small and fast cyclotron excursions, the local environment, inducing a gradient squared correction to the local value of $V(\mathcal{R})$.

For a classical electron moving in a circular orbit of radius $r$, setting the centrifugal force $F_c = \frac{mv^2}{r}$ equal to the Lorentz force $evB/c$ yields the relation $v = \frac{eB}{mc}$. The kinetic energy is then $T = \frac{1}{2}mv^2 = \frac{e^2 B^2 r^2}{2mc^2}$. If we now quantize semiclassically, demanding $\pi r^2 \cdot B = (n + \frac{1}{2})\phi_0$, then $r_n^2 = (2n + 1)\ell^2$ where $\ell = (hc/eB)^{1/2}$ is the magnetic length. The kinetic energy is then $T = (n + \frac{1}{2})\hbar \omega_c$. Thus $\xi_0 = r_n$ in our above derivation of the effective potential, with $n$ the Landau level index.

The potential $V(r)$ is due to extrinsic disorder, arising typically from the irregular placement of the dopant atoms in a heterostructure, or interface irregularities in a MOSFET. In heterostructures, the dopant ions are typically several hundred Ångstroms removed from the 2DEG layer, and $V(r)$ is smooth on this length scale. MOSFETS are typically much dirtier, with correspondingly lower mobilities, hence $V(r)$ there is disordered on shorter length scales. Indeed disorder is essential to the quantum Hall effect, since in a pristine system we can always perform a Lorentz boost to a frame where $B = 0$ and deduce $\sigma_{xy} = -\frac{ne\xi_0}{B}$. (This argument is quite a bit more subtle if there are other features breaking translational symmetry, such as leads and surfaces.)

### 1.2 MOSFETs and Heterojunctions

Where do two-dimensional electron gases (2DEGs) come from? As noted above, the IQHE was first discovered in silicon MOSFETs while the FQHE was first discovered in GaAs heterostructures. Details of the modeling and important semiconductor physics in these systems is discussed in the 1982 review by Ando, Fowler, and Stern\(^\text{11}\). Today, we have new two-dimensional systems which exhibit the QHE, such as graphene. Graphene is particularly interesting because it is a ‘Dirac material’ in which the electronic band structure features Dirac points, which are conical intersections of conduction and valence bands described by a two-dimensional Dirac Hamiltonian. More on this later.

In a metal, internal electric fields are efficiently screened and excess charge migrates rapidly to the surface, with charge density fluctuations attenuated exponentially as one enters the bulk. The Thomas-Fermi screening length, \( \lambda_{TF} = \left( \frac{4\pi e^2 g(\varepsilon_F)}{\varepsilon_F} \right)^{-1/2} \), is short (a few Ångstroms) due to the large density of states at the Fermi level. In semiconductors, the Fermi level lies somewhere in the gap between valence and conduction bands, and the density of states at \( \varepsilon_F \) is quite low. Screening is due to thermally excited charge carriers, and since the carrier density is small in comparison to that in metals, the screening length is many lattice spacings.

Consider now a junction between a semiconductor and a metal, with an intervening insulating layer. This is called MIS, or metal-insulator-semiconductor, junction. If the metal is unbiased relative to the semiconductor, their chemical potentials will align. The situation for a \( p \)-type semiconductor-metal junction is depicted in the left panel of Fig. 1.4. Next consider the case in which the metal is biased negatively with respect to the semiconductor, \( i.e. \) the metal is placed at a negative voltage \( -V \). There is then an electric field \( E = -\nabla \phi \) pointing out of the semiconductor. Electric fields point in the direction positive charges want to move, hence in this case valence holes are attracted to the interface, creating an accumulation layer of holes, as depicted in the right panel of Fig. 1.4. On the metallic side, electrons migrate to the interface for the same reason. No charges move across the insulating barrier. Thus, a dipole layer is created across the barrier, with the dipole moment pointing into the semiconductor. This creates an internal potential whose net difference \( \phi_{metal} - \phi_{semiconductor}(-\infty) = V \) exactly cancels the applied bias. This condition in fact is what determines the width of the accumulation layer.

What happens when the metal is biased positively? In this case, the electric field points into the semiconductor, and valence holes are repelled from the semiconductor surface, which is then negatively charged. This, in turn, repels electrons from the nearby metallic surface. The

![Figure 1.4: Junction between a \( p \)-type semiconductor and a metal. Left: Zero bias. Right: Metal biased negative with respect to semiconductor, creating an accumulation layer of holes and a net dipole moment at the interface.](image-url)
CHAPTER 1. PRELIMINARIES

Figure 1.5: Junction between a \textit{p}-type semiconductor and a metal. Left: Metal biased positive with respect to semiconductor, creating a space charge depletion layer. Right: Strong positive bias creates an inversion layer of \textit{n}-type carriers in the \textit{p}-type material.

result is a space charge \textit{depletion layer} in the semiconductor, which is devoid of charge carriers (\textit{i.e.} valence holes). This situation is sketched in the left panel of Fig. 1.5.

Finally, what happens if the bias voltage on the metal exceeds the band gap? In this case, the field is so strong that not only are valence holes expelled from the surface, but conduction electrons are present, as shown in the right panel of Fig. 1.5. The presence of \textit{n}-type carriers in a \textit{p}-type semiconductor is known as \textit{n-inversion}.

Remember this:

- \textit{Accumulation}: presence of additional \textit{n}-carriers in an \textit{n}-type material, or additional \textit{p}-carriers in a \textit{p}-type material.
- \textit{Depletion}: absence of \textit{n}-carriers in an \textit{n}-type material, or \textit{p}-carriers in a \textit{p}-type material.
- \textit{Inversion}: presence of \textit{n}-carriers in a \textit{p}-type material, or \textit{p}-carriers in an \textit{n}-type material.

Inversion occurs when the presence of a depletion layer does not suffice to align the chemical potentials of the two sides of the junction.

The MOSFET

A MOSFET (\textbf{M}etal-\textbf{O}xide-\textbf{S}emiconductor-\textbf{F}ield-\textbf{E}ffect-\textbf{T}ransistor) consists of two back-to-back \textit{p}-\textit{n} junctions, and, transverse to this, a gate-bulk-oxide capacitor. The situation is depicted in Fig. 1.6. If there is no gate voltage ($V_g = 0$), then current will not flow at any bias voltage $V$.
because necessarily one of the $p$-$n$ junction will be reverse-biased. The situation changes drastically if the gate is held at a high positive potential $V_g$, for then an $n$-type accumulation layer forms at the bulk-gate interface, thereby connecting source and drain directly and resulting in a gate-controlled current flow. Although not shown in the figure, generally both source and drain are biased positively with respect to the bulk in order to avoid current leakage.
1.2.1 Heterojunctions

Potential uses of a junction formed from two distinct semiconductors were envisioned as early as 1951 by Shockley. Such devices, known as heterojunctions, have revolutionized the electronics industry and experimental solid state physics, the latter due to the advent of epitaxial technology which permits growth patterning to nearly atomic precision. Whereas the best inversion layer mobilities in Si MOSFETs are $\mu \approx 2 \times 10^4 \text{cm}^2/\text{V} \cdot \text{s}$, values as high as $10^7 \text{cm}^2/\text{V} \cdot \text{s}$ are possible in MBE-fabricated GaAs–Al$_x$Ga$_{1-x}$As heterostructures. There are three reasons for this:

(i) MBE (molecular beam epitaxy), as mentioned above, can produce layers which are smooth on an atomic scale. This permits exquisite control of layer thicknesses and doping profiles.

(ii) Use of ternary compounds such as Al$_x$Ga$_{1-x}$As makes for an excellent match in lattice constant across the heterojunction interface, i.e. on the order of or better than 1%. By contrast, the Si–SiO$_2$ interface is very poor, since SiO$_2$ is a glass.

(iii) By doping the Al$_x$Ga$_{1-x}$As layer far from the interface, Coulomb scattering between inversion layer electrons and dopant ions is suppressed.

Let’s consider the chemical potential alignment problem in the case of an $n$-$n$ heterojunction, sketched in Fig. 1.8. In the GaAs–Al$_x$Ga$_{1-x}$As heterojunction, GaAs has the smaller of the two band gaps. Initially there is a mismatch, as depicted in the left panel of the figure. By forming a depletion layer on the side with the larger band gap (Al$_x$Ga$_{1-x}$As), and an accumulation layer on the side with the smaller gap (GaAs), an internal potential $\phi(x)$ is established which aligns the chemical potentials.

Fig. 1.9 shows the phenomena of accumulation and inversion in different possible heterojunctions. There are four possibilities: (a) $n$-$n$, (b) $p$-$p$, (c) $n$-$p$ with the $n$-type material having the larger gap, and (d) $n$-$p$ with the $p$-type material having the larger gap.

Figure 1.8: Accumulation layer formation in an $n$-$n$ heterojunction.
1.2. MOSFETS AND HETEROJUNCTIONS

1.2.2 QM of electron motion normal to 2DEG planes

Consider the case of an $n$-accumulation or $n$-inversion layer as depicted in Fig. 1.9. Let the direction perpendicular to the 2DEG be $\hat{z}$, and let the 2DEG lie on the $z > 0$ side of the interface. Assuming that $\hat{z}$ is a principal axis for the effective mass tensor (with eigenvalue $m_z$), and the magnetic field is along $\pm \hat{z}$, the single electron Hamiltonian is separable into degrees of freedom in the $(x, y)$ plane and those in the $\hat{z}$ direction, i.e. $H = H_\perp + H_\parallel$. The eigenvalues and eigenfunctions for $H_\perp$, which governs the planar degrees of freedom, were discussed in §1.3. Here we consider $H_\parallel$, which we model as

$$H_\parallel = -\frac{\hbar^2}{2m_z} \frac{\partial^2}{\partial z^2} + V(z) , \quad (1.29)$$

with

$$V(z) = -e\phi(z) \approx \begin{cases} 2\pi e\sigma & \text{if } z \geq 0 \\ \infty & \text{if } z < 0 \end{cases} \quad (1.30)$$

Here $\sigma$ is the 2D charge density of the space charge layer and $\epsilon$ the dielectric constant for $z > 0$. Thus, we have a triangular potential.
Next, define the length scale
\[ \lambda \equiv \left( \frac{\sigma m_{\text{e}}}{4\pi \varepsilon_{0} m_{\text{e}}^{2}} \right)^{1/3}, \]  
(1.31)
the energy scale \( \varepsilon_{\parallel} \equiv \hbar^{2}/2m_{\parallel}\lambda^{2} \), and the dimensionless length \( s \equiv z/\lambda \). Then
\[ H_{\parallel} = \varepsilon_{\parallel} \left( -\frac{\partial^{2}}{\partial s^{2}} + s \right) \]  
(1.32)
with wavefunctions subject to the boundary condition \( \varphi(0) = 0 \). The solutions are Airy functions. Recall the Airy differential equation,
\[ Ai''(z) - z Ai(z) = 0 \]  
(1.33)
Thus, the eigenfunctions of \( H_{\parallel} \) are given by \( \varphi_{n}(z) = Ai(z + \zeta_{n}) \), where \( Ai(\zeta_{n}) = 0 \). The first few zeros of \( Ai(z) \) are given by
\[ \zeta_{1} = -2.3381, \quad \zeta_{2} = -4.0879, \quad \zeta_{3} = -5.5206, \quad \zeta_{4} = -6.7867, \quad \zeta_{5} = 7.9441. \]  
(1.34)
The energy eigenvalue corresponding to the eigenfunction \( \varphi_{n}(z) \) is \( \varepsilon_{n} = -\zeta_{n} \varepsilon_{\parallel} \).

1.3 Quantization of Transverse Motion

1.3.1 Cyclotron and guiding center operators

Initially we shall assume spinless (i.e. spin-polarized) electrons. Later on we will include effects of the Zeeman term and explore exchange interactions within a Landau level. The single
1.3. QUANTIZATION OF TRANSVERSE MOTION

The particle Hamiltonian is then

\[ H = \frac{1}{2m} \left( p + \frac{e}{c} A \right)^2 + V(r) , \tag{1.35} \]

where \( V(r) \) is the potential. On a toroidal base space, \( V(r) \) is a doubly periodic function with spatial periods \( L_{1,2} \), and \( V(r + L_a) = V(r) \) for \( a = 1, 2 \). We assume \( B = -B \hat{z} \) is constant\(^{12} \). The cyclotron and guiding center momenta are defined to be

\[
\pi = p + \frac{e}{c} A \\
\kappa = p + \frac{e}{c} A - \frac{e}{c} B \times r . \tag{1.36} \]

In component notation, we have \( \kappa_\mu = p_\mu + \frac{e}{c} A_\mu - \frac{e}{c} B \times r \). The commutators are

\[
\begin{align*}
[\pi_\mu, \pi_\nu] & = \frac{e\hbar}{ic} \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) = \frac{i\hbar^2}{\ell^2} \epsilon_{\mu\nu} \\
[\kappa_\mu, \kappa_\nu] & = \frac{e\hbar}{ic} \left( \partial_\mu A_\nu - \partial_\nu A_\mu + 2B \epsilon_{\mu\nu} \right) = -\frac{i\hbar^2}{\ell^2} \epsilon_{\mu\nu} \\
[\pi_\mu, \kappa_\nu] & = \frac{e\hbar}{ic} \left( \partial_\mu A_\nu - \partial_\nu A_\mu + B \epsilon_{\mu\nu} \right) = 0 , \tag{1.37}
\end{align*}
\]

where \( \ell = \sqrt{\hbar/c} B \) is the magnetic length. Now we write

\[ A = \frac{1}{2} By \hat{x} - \frac{1}{2} Bx \hat{y} - \frac{\hbar c}{e} \nabla \chi , \tag{1.38} \]

where \( \chi(r) = \chi(r + L_a) \) is an arbitrary gauge function which is periodic on the torus.

Now define the complexified operators

\[
\begin{align*}
\pi & = \pi_x + i\pi_y = \frac{\hbar}{i} \left( \partial_x + i\partial_y \right) + \frac{eB}{2c} \left( y - ix \right) - \hbar \left( \partial_x + i\partial_y \right) \chi \\
& = \frac{2\hbar}{i} \left( \bar{\partial} + \frac{z}{4\ell^2} - i \bar{\partial} \chi \right) = e^{ix} e^{-z^2/4\ell^2} \left( -2i\hbar \bar{\partial} \right) e^{z^2/4\ell^2} e^{-ix} \tag{1.39} \end{align*}
\]

and

\[
\begin{align*}
\kappa & = \kappa_x + i\kappa_y = \frac{\hbar}{i} \left( \partial_x + i\partial_y \right) - \frac{eB}{2c} \left( y - ix \right) - \hbar \left( \partial_x + i\partial_y \right) \chi \\
& = \frac{2\hbar}{i} \left( \bar{\partial} - \frac{z}{4\ell^2} - i \bar{\partial} \chi \right) = e^{ix} e^{z^2/4\ell^2} \left( -2i\hbar \bar{\partial} \right) e^{-z^2/4\ell^2} e^{-ix} , \tag{1.40} \end{align*}
\]

where \( \ell = \sqrt{\hbar/c} B \) is the magnetic length. We have used \( z = x + iy \), \( \bar{z} = x - iy \), in which case

\[
\begin{align*}
\partial & = \frac{\partial}{\partial z} = \frac{1}{2} ( \partial_x - i \partial_y ) \\
\bar{\partial} & = \frac{\partial}{\partial \bar{z}} = \frac{1}{2} ( \partial_x + i \partial_y ) . \tag{1.41}
\end{align*}
\]

\(^{12}\)By orienting \( B \) along \( -\hat{z} \), the non-Gaussian part of the lowest Landau level wavefunctions will be holomorphic in \( z = x + iy \), rather than in \( \bar{z} = x - iy \).
CHAPTER 1. PRELIMINARIES

Note that
\[ \partial^\dagger = -\bar{\partial} \] (1.42)
under Hermitian conjugation. Thus, we can write
\[
\pi = \frac{2\hbar}{i} \left( \bar{\partial} + \frac{z}{4\ell^2} - i \bar{\partial} \chi \right) = -2i\hbar e^{ix} e^{-z\bar{z}/4\ell^2} \bar{\partial} e^{z\bar{z}/4\ell^2} e^{-ix}
\] (1.43)
and
\[
\kappa = \frac{2\hbar}{i} \left( \partial - \frac{\bar{z}}{4\ell^2} - i \partial \chi \right) = -2i\hbar e^{ix} e^{z\bar{z}/4\ell^2} \partial e^{-z\bar{z}/4\ell^2} e^{-ix}
\] (1.44)
The commutators are
\[ [\pi, \pi^\dagger] = [\kappa^\dagger, \kappa] = \frac{2\hbar^2}{\ell^2} , \] (1.45)
with \([\pi, \kappa] = [\pi^\dagger, \kappa] = 0\). Thus, we can define cyclotron and guiding center ladder operators
\[ a = \frac{i\ell \pi}{\sqrt{2} \hbar} = \sqrt{2\ell} e^{ix} e^{-z\bar{z}/4\ell^2} \bar{\partial} e^{z\bar{z}/4\ell^2} e^{-ix} \] (1.46)
and
\[ a^\dagger = -\frac{i\ell \pi^\dagger}{\sqrt{2} \hbar} = -\sqrt{2\ell} e^{ix} e^{z\bar{z}/4\ell^2} \bar{\partial} e^{-z\bar{z}/4\ell^2} e^{-ix} \] (1.47)
The kinetic term in the Hamiltonian is then
\[ \frac{\pi^2}{2m} = \frac{\pi^\dagger \pi}{2m} + \frac{\hbar^2}{2m\ell^2} = \hbar \omega_c (a^\dagger a + \frac{1}{2}) , \] (1.48)
where \(\omega_c = eB/mc\) is the cyclotron frequency. Note that
\[ z = \frac{i\ell^2}{\hbar} (\pi - \kappa) \] (1.49)
We may also define the complexified guiding center and cyclotron coordinates, \(R\) and \(\xi\), as follows:
\[ R = \frac{i\ell^2}{\hbar} \kappa = \sqrt{2\ell} b^\dagger , \quad \xi = \frac{i\ell^2}{\hbar} \pi = \sqrt{2\ell} a \] (1.50)
which satisfy \([R, R^\dagger] = -2\ell^2, [\xi, \xi^\dagger] = 2\ell^2\), and \(R + \xi = z\).

Exercise: Show that the angular momentum operator satisfies
\[ L_z = e^{ix}(xp_y - yp_x) e^{-ix} = \hbar (b^\dagger b - a^\dagger a) \] (1.51)
1.3. QUANTIZATION OF TRANSVERSE MOTION

1.3.2 The lowest Landau level

The eigenvalue of \( a^\dagger a \) is an integer which corresponds to the Landau level index. For states in the lowest Landau level, we have

\[
a \psi(r) = 0 \implies \psi(r) = e^{i\chi(r)} e^{-r^2/4\ell^2} f(z),
\]

(1.52)

where \( z = x + iy \). At this point, \( f(z) \) is any analytic function. As we shall soon see, periodicity on the torus further constrains the form of \( f(z) \).

In zero magnetic field, the density of states (per unit area, per unit energy) is constant:

\[
g(\varepsilon, B = 0) \, d\varepsilon = \frac{d^2k}{(2\pi)^2} = \frac{k \, dk}{2\pi} \implies g(\varepsilon) = \frac{m}{2\pi\hbar^2}
\]

(1.53)

since \( \varepsilon = \hbar^2 k^2/2m \). When \( B \) is finite, the spectrum collapses into discrete Landau levels with energies \( \varepsilon_n = (n + \frac{1}{2})\hbar\omega_c \). The density of states is

\[
g(\varepsilon, B) = \frac{B}{\phi_0} \sum_{n=0}^{\infty} \delta(\varepsilon - (n + \frac{1}{2})\hbar\omega_c) \quad .
\]

(1.54)

The number of Landau levels below energy \( E \) is \( E/\hbar\omega_c \), rounded to the nearest integer. To check the coefficient \( B/\phi_0 \) in the above expression note that the total number of states per unit area below energy \( E \) is then

\[
\frac{B}{\phi_0} \cdot \frac{E}{\hbar\omega_c} = \frac{mE}{2\pi\hbar^2}
\]

(1.55)

which agrees with the \( B = 0 \) result. Below, we shall count the number of states precisely using a toroidal geometry.

Figure 1.11: Density of states in \( d = 2 \) for \( B = 0 \) (red) and \( B > 0 \) (purple).
We define the wavefunction \( \psi_0(r) \) to satisfy \( a \psi_0 = b \psi_0 = 0 \). Imposing normalization,

\[
\psi_0(r) = (2\pi \ell^2)^{-1/2} e^{i\chi(r)} e^{-z\bar{z}/4\ell^2} .
\]  

(1.56)

A complete and orthonormal set of wavefunctions is given by the collection

\[
\psi_{m,n}(r) = \frac{(a^\dagger)^n}{\sqrt{n!}} \frac{(b^\dagger)^m}{\sqrt{m!}} \psi_0(r) = \frac{1}{\sqrt{2\pi \ell^2 m! n!}} \left( \frac{z}{\sqrt{2\ell}} \right)^m e^{-z^2/4\ell^2} \left( e^{x-m} \frac{dx}{dx^m} e^{-x} \right)_{x=z\bar{z}/2\ell^2} .
\]

(1.57)

Note that, per Eqn. 1.51, this is also an eigenbasis of angular momentum, viz.

\[
L_z | m, n \rangle = \hbar (b^\dagger b - a^\dagger a) | m, n \rangle = \hbar (m - n) | m - n \rangle .
\]

(1.58)

Completeness entails the relation

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} | m, n \rangle \langle m, n | = 1 .
\]

(1.59)

i.e. \( \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \psi_{m,n}(r) \psi_{m,n}^*(r') = \delta(r - r') \). Note, however, that if we sum over only the states in the lowest Landau level, then

\[
\sum_{m=0}^{\infty} \psi_{m,0}(r) \psi_{m,0}^*(r') = \frac{e^{i\chi(r)} e^{-i\chi(r')}}{2\pi \ell^2} \sum_{m=0}^{\infty} \left( \frac{z}{\sqrt{2\ell}} \right)^m e^{-z^2/4\ell^2} e^{-z'^2/4\ell^2}
\]

\[
= \frac{e^{i\chi(r)} e^{-i\chi(r')}}{2\pi \ell^2} \exp \left( - \frac{z z'}{4\ell^2} - \frac{z' z}{4\ell^2} + \frac{z^2}{2\ell^2} \right)
\]

\[
= \frac{e^{i\chi(r)} e^{-i\chi(r')}}{2\pi \ell^2} \exp \left( - \frac{|z - z'|^2}{4\ell^2} + i \frac{\Im z z'}{2\ell^2} \right)
\]

\[
= \frac{e^{i\chi(r)} e^{-i\chi(r')}}{2\pi \ell^2} \exp \left( - \frac{(r - r')^2}{4\ell^2} - i \frac{r \times r' \cdot \hat{z}}{2\ell^2} \right)
\]

(1.60)

rather than \( \delta(r - r') \). This tells us that the shortest distance scale on which we can localize an electron in the lowest Landau level is the magnetic length \( \ell \).

In the lowest Landau level (LLL), we may write

\[
\psi_m(r) = \frac{e^{i\chi(r)}}{\sqrt{2\pi \ell^2 m!}} \left( \frac{z}{\sqrt{2\ell}} \right)^m e^{-z\bar{z}/4\ell^2} .
\]

(1.61)

This is generally known as the angular momentum basis.
1.3. QUANTIZATION OF TRANSVERSE MOTION

1.3.3 Magnetic translation operators

The magnetic translation operators are defined as

\[ t(d) = \exp(i\bm{\kappa} \cdot \bm{d}/\hbar) = \exp[(ib - \ddbar{b})/\sqrt{2}\ell] \]  

(1.62)

Note that [t(d), \pi] = 0, so the magnetic translations commute with the kinetic energy. Acting on any function of the coordinates, we have

\[ t(d) \psi(r) t^\dagger(d) = \psi(r + d) \]  

(1.63)

which is why \( t(d) \) is a translation operator. If we make use of the results of eqn. 1.95 below, we can show that while \( t_0(d) \psi(r) = \psi(r + d) \), where \( t_0(d) = e^{i\mu \cdot d/\hbar} = e^{d \cdot \nabla} \) is the translation operator for \( B = 0 \), then

\[ t(d) \psi(r) = e^{i\chi(r)} e^{-i\chi(r+d)} e^{-i\ddbar{d} \cdot \hat{z}/2\ell^2} \psi(r + d) \]  

(1.64)

Due to the magnetic field, two arbitrary magnetic translations do not necessarily commute. Rather,

\[ t(d_1) t(d_2) = e^{i\ddbar{d_1} \cdot \ddbar{d_2}/2\ell^2} t(d_1 + d_2) = e^{i\ddbar{d_1} \times \ddbar{d_2}/\ell^2} t(d_2) t(d_1) \]  

(1.65)

Thus, [t(d_1), t(d_2)] = 0 if and only if \( \hat{z} \cdot d_1 \times d_2 = 2\pi \ell^2 q \), where \( q \) is an integer.

1.3.4 Coherent state wavefunctions

Having tired of carrying the stupid gauge function \( \chi(r) \) with us for so long, we will now drop it, which means working in the symmetric gauge, with \( \chi(r) = 0 \). Consider again the MTO,

\[ t(R) = \exp(i\bm{\kappa} \cdot R/\hbar) = e^{(Rb - \ddbar{R}b)/\sqrt{2}\ell} \]  

(1.66)

where we have used the Baker-Campbell-Hausdorff equality

\[ e^{A+B} = e^{A} e^{B} e^{-\frac{1}{2}[A,B]} \]  

(1.67)

which is true when both \( A \) and \( B \) commute with their commutator \([A, B]\). The coherent state wavefunction is defined to be

\[ \big| R \big> = t^\dagger(R) \big| 0, 0 \big> = e^{-Rb \cdot \ddbar{b}/\sqrt{2}\ell} e^{Rb/\sqrt{2}\ell} \big| 0, 0 \big> \]  

(1.68)

It is left as an exercise to the reader to verify the following formulae:

\[ \langle R | t(R) \big| 0, 0 \big> = \frac{1}{\sqrt{2\pi \ell^2}} e^{-ir \cdot R \hat{z}/2\ell^2} e^{-(r-R)^2/4\ell^2} \]  

\[ \langle R | R' \rangle = \exp \left( -\frac{(R - R')^2}{4\ell^2} - i\frac{R \times R' \cdot \hat{z}}{2\ell^2} \right) \]  

(1.69)

\[ \int \frac{d^2R}{2\pi \ell^2} | R \rangle \langle R | = \sum_{m=0}^{\infty} | m, 0 \rangle \langle m, 0 | \equiv \Pi_0 \]  

Thus, the coherent state wavefunction $\varphi_R(r)$ is Gaussianly localized about $r = R$, but contains phase information as well. The coherent states admit a resolution of unity within the lowest Landau level, or indeed for any Landau level if we define $| R, n \rangle \equiv t^\dagger (R) | 0, n \rangle$.

### 1.3.5 Landau strip basis

Had we instead chosen the gauge $A = -B x \hat{y}$ (again corresponding to $B = -B \hat{z}$), then

$$H = \frac{p_x^2}{2m} + \frac{(p_y - eB c x)^2}{2m}.$$  

(1.70)

There is now translational invariance along $\hat{y}$, hence the wavefunctions may be written as $\psi(x, y) = e^{ik_y y} \phi(x)$, with $\phi(x)$ an eigenfunction of

$$H(k_y) = \frac{p_x^2}{2m} + \frac{(\hbar k_y - eB c x)^2}{2m} = \frac{p_x^2}{2m} + \frac{1}{2} m \omega_c^2 (x - \ell^2 k_y)^2 .$$  

(1.71)

This is the one-dimensional harmonic oscillator, with eigenfunctions $\phi_n(x - \ell^2 k_y)$, where

$$\phi_n(x) = \frac{(m \omega_c / \pi \hbar)^{1/4}}{\sqrt{2^n n!}} H_n \left( \sqrt{\frac{m \omega_c}{\hbar}} x \right) e^{-m \omega_c x^2 / 2\hbar} ,$$  

(1.72)

and corresponding eigenvalues $\varepsilon_n = (n + \frac{1}{2}) \hbar \omega_c$. The full basis set of wavefunctions as a function of $(x, y)$ is labeled by a discrete Landau level index $n$ and a continuous index $k_y$, viz.

$$\psi_{n,k_y}(x, y) = \phi_n(x - \ell^2 k_y) e^{ik_y y} .$$  

(1.73)

On a cylinder $\mathbb{R} \times S^1$ where $y \in [0, L]$, periodic boundary conditions requires $k_y$ to be quantized due to the relation $e^{ik_y L} = 1$.

### 1.3.6 Toroidal base space

On the torus, we require that the area $\hat{z} \cdot \mathbf{L}_1 \times \mathbf{L}_2 = 2\pi \ell^2 N$ be quantized in units of $2\pi \ell^2$, where $N$ is the number of flux quanta, if we are to have $[ t(\mathbf{L}_1) , t(\mathbf{L}_2) ] = 0$. Then since $V(r + \mathbf{L}_a) = V(r)$, we have that $\{ H , t(\mathbf{L}_1) , t(\mathbf{L}_2) \}$ is a complete set of commuting observables. Since the $t(\mathbf{L}_{1,2})$ are unitary, the eigenstates $| \psi_\alpha \rangle$ of $H$ may be chosen to satisfy

$$t(\mathbf{L}_a) | \psi_\alpha \rangle = e^{i \theta_a} | \psi_\alpha \rangle$$  

(1.74)

for all $\alpha \in \{1, \ldots, N\}$. The dimension of the Hilbert space is $N$, as we shall see.
We will consider potentials $V(r)$ which are periodic on the torus. This is equivalent to a periodic potential, and any reciprocal lattice vector $G$ as

$$ G = n_1 b_1 + n_2 b_2 , $$

where

$$ b_1 = \frac{2\pi}{\Omega} L_2 \times \hat{z} , \quad b_2 = \frac{2\pi}{\Omega} \hat{z} \times L_1 , $$

with integer $n_{1,2}$, where $\Omega = \hat{z} \cdot L_1 \times L_2 = 2\pi \ell^2 N$ is the area of the torus. With the above definitions of the primitive reciprocal lattice vectors $b_{1,2}$, we have $b_a \cdot L_{a'} = 2\pi \delta_{aa'}$. Thus,

$$ \ell^2 \hat{z} \times G = \frac{2\pi\ell^2}{\Omega} (n_1 L_2 - n_2 L_1) . $$

Thus,

$$ (\ell^2 \hat{z} \times G) \times L_a = -2\pi \ell^2 n_a \hat{z} , $$

and therefore $[ t(\ell^2 \hat{z} \times G) , t(L_a) ] = 0$ for $a = 1, 2$.

For any vector $Q$, we define its complexification as $Q \equiv Q_x + iQ_y$. The complexified elementary reciprocal lattice vectors are then

$$ b_1 = b_{1,x} + ib_{1,y} = -\frac{2\pi i}{\Omega} L_2 , \quad b_2 = b_{2,x} + ib_{2,y} = \frac{2\pi i}{\Omega} L_1 . $$

The modular parameter $\tau = \tau_1 + i\tau_2$ is defined as the complex ratio

$$ \tau \equiv \frac{L_2}{L_1} = \frac{L_{2,x} + iL_{2,y}}{L_{1,x} + iL_{1,y}} . $$

For a general reciprocal lattice vector $G = n_1 b_1 + n_2 b_2$, then, we have

$$ G = G_x + iG_y = \frac{2\pi i}{\Omega} L_1 (n_2 - n_1 \tau) . $$

The unit cell area is then

$$ \Omega = 2\pi \ell^2 N = \text{Im} (\bar{L}_1 L_2) = |L_1|^2 \tau_2 . $$

Then we have

$$ \frac{1}{4} G^2 \ell^2 = \frac{1}{4} |G|^2 \ell^2 = \frac{\pi}{2N\tau_2} \left( (n_1 \tau_1 - n_2)^2 + n_1^2 \tau_2^2 \right) . $$
1.3.7 Lowest Landau level Hamiltonian

The potential may be written in terms of its Fourier components, \( \text{viz.} \)

\[
V(r) = \sum_G V_G e^{iG \cdot r} = \sum_G V_G e^{i(G \vec{z} + \vec{G} \vec{z})/2}
\]

\[
= \sum_G V_G e^{i\ell(G \pi - \vec{G} \pi)/2\hbar} e^{i\ell(\vec{G} \kappa - \vec{G} \kappa^*)/2\hbar}
\]

\[
= \sum_G V_G e^{-G^2 \ell^2 / 4} e^{i\ell^2 G \pi / 2\hbar} e^{-i\ell^2 G \pi / 2\hbar} t(\ell^2 \hat{z} \times \vec{G}) .
\]

(1.84)

If we project onto the lowest Landau level, we obtain

\[
\tilde{V} = P_0 V(r) P_0 = \sum_G V_G e^{-G^2 \ell^2 / 4} t(\ell^2 \hat{z} \times \vec{G}) .
\]

(1.85)

Define the unitary operators

\[
t_1 \equiv t(\vec{L}_1 / N) , \quad t_2 \equiv t(\vec{L}_2 / N) .
\]

(1.86)

Then it is easy to show

\[
t_1 t_2 = e^{2\pi i / N} t_2 t_1 .
\]

(1.87)

Furthermore, we have

\[
t(\ell^2 \hat{z} \times \vec{G}) = t\left(\frac{n_1 \vec{L}_2}{N} - \frac{n_2 \vec{L}_1}{N}\right)
\]

\[
= e^{-i\pi n_1 n_2 / N} t_2^{n_1} t_1^{-n_2} .
\]

(1.88)

We can define an \( N \)-element basis \( \{|k\rangle\} \) which satisfies the following:

\[
t_1 |k\rangle = e^{i\theta_1 / N} |k - 1\rangle
\]

\[
t_2 |k\rangle = e^{i\theta_2 / N} e^{2\pi i k / N} |k\rangle ,
\]

(1.89)

with \( |k + N\rangle \equiv |k\rangle \). Note that \( t_1 t_2 |k\rangle = e^{2\pi i / N} t_2 t_1 |k\rangle \) for all \( k \), and furthermore that \( t(\vec{L}_a) |k\rangle = t_a^N |k\rangle = e^{i\theta_a} |k\rangle \) for all allowed \( a \) and \( k \). Thus,

\[
\langle k | t(\ell^2 \hat{z} \times \vec{G}) | k'\rangle = e^{-i\pi n_1 n_2 / N} e^{i n_1 \theta_2 / N} e^{2\pi i n_1 / N} e^{-i n_2 \theta_1 / N} \langle k | k' + n_2\rangle
\]

\[
= e^{-i\pi n_1 n_2 / N} e^{i(n_1 \theta_2 - n_2 \theta_1) / N} e^{2\pi i n_1 / N} \delta_{k,k'+n_2} ,
\]

where

\[
\delta_{k,l} = \delta_{k,l \text{ mod } N} .
\]

(1.90)
Thus, the Hamiltonian for our system is
\[ H_{kk'} = \sum_{n_1, n_2} V_{n_1, n_2} e^{-\pi [(n_1 r_1 - n_2)^2 + n_1^2 r_2^2]/2N r_2} e^{-i\pi n_1 n_2/N} e^{i(n_1 \theta_2 - n_2 \theta_1)/N} e^{2\pi i n_1 k/N} \delta_{k, k'+n_2}. \] (1.92)

Checking that the Hamiltonian is Hermitian, we have
\[ H_{kk'}^* = \sum_{n_1, n_2} V_{n_1, n_2}^* e^{-\pi [(n_1 r_1 - n_2)^2 + n_1^2 r_2^2]/2N r_2} e^{-i\pi n_1 n_2/N} e^{i(n_1 \theta_2 - n_2 \theta_1)/N} e^{2\pi i n_1 (k'-n_2)/N} \bar{\delta}_{k', k+n_2} \]
\[ = \sum_{n_1, n_2} V_{n_1, n_2} e^{-\pi [(n_1 r_1 - n_2)^2 + n_1^2 r_2^2]/2N r_2} e^{-i\pi n_1 n_2/N} e^{i(n_1 \theta_2 - n_2 \theta_1)/N} e^{2\pi i n_1 (k'-n_2)/N} \bar{\delta}_{k', k+n_2} \]
\[ = \sum_{n_1, n_2} V_{n_1, n_2} e^{-\pi [(n_1 r_1 - n_2)^2 + n_1^2 r_2^2]/2N r_2} e^{-i\pi n_1 n_2/N} e^{i(n_1 \theta_2 - n_2 \theta_1)/N} e^{2\pi i n_1 k'/N} \bar{\delta}_{k', k+n_2} = H_{k'k}, \]

where we have used \( V_{n_1, n_2}^* = V_{-n_1, -n_2} \) and then replaced \( n_{1,2} \) with \( -n_{1,2} \) in the second line.

As an example, consider a case where \( V(r) = V_{1,1} e^{i(b_1+b_2) \cdot r} + V_{1,1}^* e^{-i(b_1+b_2) \cdot r} \) and \( N = 3 \). We then find
\[ H = \begin{pmatrix} 0 & -V_{1,1}^* e^{i\alpha} & V_{1,1} e^{i\pi/3} e^{-i\alpha} \\ -V_{1,1} e^{-i\alpha} & 0 & V_{1,1}^* e^{i\pi/3} e^{i\alpha} \\ V_{1,1}^* e^{-i\pi/3} e^{-i\alpha} & V_{1,1} e^{i\pi/3} e^{i\alpha} \end{pmatrix} e^{-\pi [(r_1-1)^2 + r_2^2]/6 r_2}, \] (1.94)

where \( \alpha \equiv \frac{1}{3}(\theta_1 - \theta_2) \). For example, to compute \( H_{kk'} \) with \( k = 1 \) and \( k' = 2 \), we need \( n_2 = -1 \) to satisfy the Kronecker delta in eqn. 1.92, and therefore \( n_1 = -1 \) as well, corresponding to \( V_{-1,-1} = V_{1,1} \). Working out the phase of the matrix element, we then have \( e^{-i\pi/3} e^{i(\theta_1 - \theta_2)/3} e^{-2\pi i/3} = -e^{i\alpha} \). For \( k = 1 \) and \( k' = 3 \), we need \( n_2 = 1 \) in order to satisfy \( k = k' + 1 \mod N \). Thus \( n_1 = 1 \) as well, corresponding to \( V_{1,1}^* \) and the phase of the matrix element is \( e^{-i\pi/3} e^{i(\theta_1 - \theta_2)/3} e^{2\pi i/3} = e^{i\pi/3} e^{-i\alpha} \).

### 1.3.8 Basis wavefunctions on a torus

Periodic boundary conditions on the torus requires \( t(L_{a}) \mid \psi \rangle = e^{i\theta_a} \mid \psi \rangle \) for all states \( \mid \psi \rangle \). Let’s examine what this requires for the analytic part \( f(z) \). We have
\[ t(L) = e^{i\alpha L/h} = e^{i(\kappa L + n L)/2h} \]
\[ = e^{-LL/4 \ell^2} e^{i\kappa L/2h} e^{i\kappa L/2h} \]
\[ = e^{-LL/4 \ell^2} e^{ix} e^{z^2/4 \ell^2} e^{-z^2/2 \ell^2} e^{L \partial} e^{z^2/4 \ell^2} e^{-i\chi}. \] (1.95)

Thus, with \( \psi(r) = e^{ix} e^{-z^2/4 \ell^2} f(z) \), we have
\[ t(L) \psi(r) = e^{-LL/4 \ell^2} e^{ix} e^{z^2/4 \ell^2} e^{L \partial} e^{-z^2/2 \ell^2} e^{L \partial} f(z) \]
\[ = e^{ix} e^{-LL/4 \ell^2} e^{-z^2/4 \ell^2} e^{-zL/2 \ell^2} f(z + L). \] (1.96)
Thus, we must have
\[ f(z + L_0) = e^{i\theta a} e^{L_aL_a/4\ell^2} e^{zL_a/2\ell^2} f(z) , \]
valid for \( a = 1, 2 \). Note that integrating the logarithmic derivative of \( f(z) \) around the torus yields
\[
\oint_{\Omega} \frac{dz}{2\pi i} \frac{f'(z)}{f(z)} = \int_{0}^{L_1} \frac{dz}{2\pi i} \ln \left( \frac{f(z)}{f(z + L_2)} \right) + \int_{0}^{L_2} \frac{dz}{2\pi i} \ln \left( \frac{f(z + L_1)}{f(z)} \right)
\]
\[ = - \int_{0}^{L_1} \frac{dz}{2\pi i} \frac{L_2}{2\ell^2} + \int_{0}^{L_2} \frac{dz}{2\pi i} \frac{L_1}{2\ell^2} = \frac{L_1L_2 - L_1\bar{L}_2}{2\pi i} \cdot \frac{1}{2\ell^2} = \frac{\Omega}{2\pi \ell^2} = N , \]
which establishes that \( f(z) \) has precisely \( N \) zeros on the torus.

Now consider the Jacobi theta function,
\[ \vartheta_1(w \mid \sigma) = -i \sum_{n=-\infty}^{\infty} (-1)^n e^{i\pi\sigma(n + \frac{1}{2})^2} e^{(2n+1)iw} , \]
where \( \text{Im}(\sigma) > 0 \). This function is quasiperiodic over the fundamental domain for \( w \), which is a parallelogram with sides \( 1 \) and \( \sigma \), satisfying
\[ \vartheta_1(w + \pi \mid \sigma) = -\vartheta_1(w \mid \sigma) \]
\[ \vartheta_1(w + \pi \sigma \mid \sigma) = -e^{-2iw} e^{-i\pi\sigma} \vartheta_1(w \mid \sigma) . \]
From the above relations, integrating the logarithmic derivative of \( \vartheta_1(w \mid \sigma) \) establishes that the function has one zero in the fundamental domain, located at \( w = 0 \). Iterating the second of the above relations, one has
\[ \vartheta_1(w + j\pi \sigma \mid \sigma) = (-1)^j e^{-2iw} e^{-ij^2\pi\sigma} \vartheta_1(w \mid \sigma) , \]
for any integer \( j \). In addition, using the Poisson summation formula,
\[ \sum_{n=-\infty}^{\infty} \delta(x - n) = \sum_{m=-\infty}^{\infty} e^{2\pi imx} , \]
one derives the identity
\[ \vartheta_1(w \mid \sigma) = \sqrt{\frac{i}{\sigma}} e^{-i\pi(1+\sigma)/4} e^{-iw^2/\pi\sigma} \vartheta \left( w - \frac{1}{\sigma} \right) . \]
Now consider the function
\[ f(z) = e^{L_1z^2/4\ell^2} e^{\lambda z/L_1} \vartheta_1 \left( \frac{\pi z}{L_1} - \pi \zeta \bigg| \frac{L_2}{N L_1} \right) . \]
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Then

\[ f(z + L_1) = -e^{L_1/L_1/4\ell^2} e^{L_1 z/2\ell^2} e^{i\lambda} f(z) \]

\[ f(z + L_2) = (-1)^N e^{L_2/L_2/4\ell^2} e^{L_2 z/2\ell^2} e^{i\lambda \tau} e^{2\pi N\zeta} f(z) , \]

where \( \tau \equiv L_2/L_1 \). Invoking the periodicity requirements, we obtain

\[ e^{i\lambda} = -e^{i\theta_1} \]

\[ e^{i\lambda \tau} e^{2\pi N\zeta} = (-1)^N e^{i\theta_2} . \]

Thus, we have

\[ \lambda = \theta_1 + (2k_1 + 1)\pi \]

and

\[ \zeta = \frac{\theta_2 + \pi N + 2\pi k_2}{2\pi N} - \frac{\theta_1 + (2k_1 + 1)\pi}{2\pi N} \cdot \tau , \]

where \( k_1 \) and \( k_2 \) are integers. Since \( k_1 \to k_1 + 1 \) increases the argument of the \( \vartheta \)-function by a multiple of \( \sigma \equiv \tau/N \), one can invoke the quasiperiodicity relation, whence one finds that this results in a multiplication of \( f(z) \) by a constant. Therefore, we are free to select a fixed value for \( k_1 \). We choose \( k_1 \equiv -1 \). Then

\[ \lambda = \theta_1 - \pi \]

\[ \zeta = \frac{\theta_2 + \pi N + 2\pi k_2}{2\pi N} + \frac{(\pi - \theta_1) \tau}{2\pi N} . \]

So our basis functions are

\[ \psi_k(r) = C e^{ix} e^{i\pi k/N} e^{-z^2/4\ell^2} e^{L_1 z^2/4\ell^2 L_1} e^{i(\theta_1 - \pi)z/L_1} \vartheta_1 \left( \frac{\pi z}{L_1} - \pi \zeta_k \left| \frac{L_2}{N} L_1 \right. \right) , \]

where

\[ \zeta_k = \frac{\theta_2 + \pi N + 2\pi k}{2\pi N} + \frac{(\pi - \theta_1) \tau}{2\pi N} . \]

and \( C \) is a constant independent of \( k \). Then after some work one can show that indeed

\[ t_1 \psi_k(r) = e^{i\theta_1/N} \psi_{k-1}(r) \]

\[ t_2 \psi_k(r) = e^{i\theta_2/N} e^{2\pi ik/N} \psi_k(r) . \]

Finally, define

\[ w \equiv \frac{z}{L_1} \equiv u + \tau v \]
where \((u, v) \in [0, 1] \times [0, 1]\). Then with \(\phi_k(w) \equiv \psi_k(z = wL_1, \bar{z} = \bar{w}\bar{L}_1)\),

\[
\phi_k(w) = Ce^{i\pi k/N} e^{i\pi N(w-\bar{w})/2\tau} e^{i(\theta_1-\pi)w} \vartheta_1 \left(\pi u - \frac{k\pi}{N} - \frac{\theta_2 + \pi N}{2N} + \pi v + \frac{(\theta_1 - \pi)\tau}{2N} \right),
\]

which is holomorphic in \(\tau\). The normalization condition is

\[
1 = \Omega \int_0^1 du \int_0^1 dv |\phi_k(u, v)|^2 = |C|^2 \left(\frac{2\pi \ell^2 N^3}{\tau_2}\right)^{1/2} \exp\left(\frac{(\theta_1 - \pi)^2 \tau_2}{2\pi N}\right). \tag{1.115}
\]

### 1.4 Appendix: Primer on Coherent States

The quantum-classical correspondence is elucidated with the use of coherent states. Recall that the one-dimensional harmonic oscillator Hamiltonian may be written

\[
\hat{H}_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2 = \hbar \omega_0 \left(a^\dagger a + \frac{1}{2}\right), \tag{1.116}
\]

where \(a\) and \(a^\dagger\) are ladder operators satisfying \([a, a^\dagger] = 1\), which can be taken to be

\[
a = \ell \frac{\partial}{\partial q} + \frac{q}{2\ell}, \quad a^\dagger = -\ell \frac{\partial}{\partial q} + \frac{q}{2\ell}, \tag{1.117}
\]

with \(\ell = \sqrt{\hbar/2m\omega_0}\). Note that

\[
q = \ell (a + a^\dagger), \quad p = \frac{\hbar}{2i\ell} (a - a^\dagger). \tag{1.118}
\]

The ground state satisfies \(a \psi_0(q) = 0\), which yields

\[
\psi_0(q) = (2\pi \ell^2)^{-1/4} e^{-q^2/4\ell^2}. \tag{1.119}
\]

The normalized coherent state \(|z\rangle\) is defined as

\[
|z\rangle = e^{-\frac{1}{2}|z|^2} e^{za^\dagger} |0\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle. \tag{1.120}
\]
The overlap of coherent states is given by
\[ \langle z_1 \mid z_2 \rangle = e^{-|z_1|^2/2} e^{-|z_2|^2/2} e^{z_1 \bar{z}_2} = e^{-|z_1 - z_2|^2/2} e^{i \text{Im}(\bar{z}_1 z_2)} , \] (1.121)

hence different coherent states are not orthogonal. Despite this nonorthogonality, the coherent states allow a simple resolution of the identity,
\[ 1 = \int \frac{d^2z}{2\pi i} \langle z \mid z \rangle ; \quad \frac{d^2z}{2\pi i} \equiv \frac{d \text{Re} z \, d \text{Im} z}{\pi} , \] (1.122)

which is straightforward to establish.

To gain some physical intuition about the coherent states, define
\[ z \equiv \frac{Q}{2\ell} + i \frac{P}{\hbar} \] (1.123)
and write \[ |z\rangle \equiv |Q, P\rangle . \] One finds (exercise!)
\[ \psi_{Q, P}(q) = \langle q \mid z \rangle = (2\pi \ell^2)^{-1/4} e^{-iPQ/2\hbar} e^{iPq/\hbar} e^{-(q-Q)^2/4\ell^2} , \] (1.124)

hence the coherent state \[ \psi_{Q, P}(q) \] is a wavepacket Gaussianly localized about \( q = Q \), but oscillating with average momentum \( P \).

For example, we can compute
\[ \langle Q, P \mid q \mid Q, P \rangle = \langle z \mid \ell (a + a^\dagger) \mid z \rangle = 2\ell \text{Re} z = Q \] (1.125)
\[ \langle Q, P \mid p \mid Q, P \rangle = \langle z \mid \frac{\hbar}{2i\ell} (a - a^\dagger) \mid z \rangle = \frac{\hbar}{\ell} \text{Im} z = P \] (1.126)
as well as
\[ \langle Q, P \mid q^2 \mid Q, P \rangle = \langle z \mid \ell^2 (a + a^\dagger)^2 \mid z \rangle = Q^2 + \ell^2 \] (1.127)
\[ \langle Q, P \mid p^2 \mid Q, P \rangle = -\langle z \mid \frac{\hbar^2}{4\ell^2} (a - a^\dagger)^2 \mid z \rangle = P^2 + \frac{\hbar^2}{4\ell^2} . \] (1.128)

Thus, the root mean square fluctuations in the coherent state \[ |Q, P\rangle \] are
\[ \Delta q = \ell = \sqrt{\frac{\hbar}{2m\omega_0}} , \quad \Delta p = \frac{\hbar}{2\ell} = \sqrt{\frac{m\hbar\omega_0}{2}} , \] (1.129)
and \( \Delta q \cdot \Delta p = \frac{1}{2} \hbar \). Thus we learn that the coherent state \[ \psi_{Q, P}(q) \] is localized in phase space, i.e. in both position and momentum. If we have a general operator \( \hat{A}(q, p) \), we can then write
\[ \langle Q, P \mid \hat{A}(q, p) \mid Q, P \rangle = A(Q, P) + \mathcal{O}(\hbar) , \] (1.130)
where $A(Q,P)$ is formed from $\hat{A}(q,p)$ by replacing $q \to Q$ and $p \to P$.

Since
\[
\frac{d^2z}{2\pi i} = \frac{d\text{Re} z \, d\text{Im} z}{\pi} = \frac{dQ \, dP}{2\pi \hbar},
\]
we can write the trace using coherent states as
\[
\text{Tr} \hat{A} = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} dQ \int_{-\infty}^{\infty} dP \langle Q,P | \hat{A} | Q,P \rangle.
\]

We now can understand the origin of the factor $2\pi \hbar$ in the denominator of each $(q_i, p_i)$ integral over classical phase space in the metric $d\mu = \prod_i dq_i \, dp_i / 2\pi \hbar$.

Note that $\omega_0$ is arbitrary in our discussion. By increasing $\omega_0$, the states become more localized in $q$ and more plane wave like in $p$. However, so long as $\omega_0$ is finite, the width of the coherent state in each direction is proportional to $\hbar^{-1/2}$, and thus vanishes in the classical limit.

The resolution of the identity in Eqn. ?? prompts the following question. Suppose we consider an infinite discrete lattice $\{ | z_{m,n} \rangle \}$ of coherent states, with $z_{m,n} = (m + in)\sqrt{\pi}$, called a von Neumann lattice of coherent states. The dimensionless phase space area $A$ per unit cell of this lattice is $\pi$, which is the denominator in the integral resolution of the identity in Eqn. ???. One might expect, then, that while this basis is not orthogonal, since
\[
\langle z_{m,n} | z_{m',n'} \rangle = e^{-\pi (m-m')^2/2} e^{-\pi (n-n')^2/2} e^{i\pi (mn-m'n')},
\]
that the Gaussian overlaps could be undone, i.e. that the overlap matrix could be inverted, and a proper complete and orthonormal set of localized LLL wavefunctions could be constructed. Alas, this is impossible! As shown by Perelomov$^{13}$, there is a single linear dependence relation among the von Neumann lattice of coherent states, when $A = \pi$. For $S < \pi$, the lattice is over-complete by a finite amount per unit area. Similarly, when $A > \pi$, the lattice is undercomplete by a finite amount per unit area. But when $A = \pi$, it is undercomplete by precisely one state. This state is necessary to include in order for the filled Landau level to carry a Chern number $C = 1$. Thus, the von Neumann lattice cannot be used as a basis to describe the quantum Hall effect.