f.) Spherical Geometry

a.) Motivation:

- $R = R(t)$, etc. $\Rightarrow$ effects of imploding system convergence, etc.

- Multiple Surfaces:

3 stages of R.T. instability:

a.) Acceleration phase:

- ablated material accelerated into shell $\Rightarrow$ R.T. instability of.

$R = 3\text{mm} \rightarrow 1.5\text{mm}$

b.) Coasting phase:

- shell coasts inward with no acceleration $\Rightarrow$

$R = 1.5\text{mm} \rightarrow 150\mu$ ballistic $\Rightarrow$ no acceleration $\Rightarrow$ no R.T.

c.) Deceleration phase:

- shell decelerates, compressing gaseous core (gas ignites) $\Rightarrow$ R.T. instability of (gast outward)
Impact on ICF design:
- naively, coating phase appears benign, as no R.T. instability
- but
  a) during acceleration early coating shell radius thin
  b) \[ \phi_1 = \exp \left[ -\frac{\ell}{R_1} (R_2 - R_1) \right] \]
  ⇒ outer surface R.T. seeds inner surface perturbation
  ⇒ inner surface perturbation grows due convergence prior to deceleration

Examples:

a) Pendulum

Adiabatic invariant action
\[ S = \int P \, dq = \int \mathcal{P} \, d\phi \quad ; \quad \text{here} \]
\[ H = \frac{\rho_0^2}{2mL^2} + \frac{1}{2} mgLQ^2 \Rightarrow \rho_0(0), N \text{ etc.} \]

Alternatively: \( S' = \text{Action} \)

\[ S' = (\frac{1}{2} mgLA^2)(\omega^{-1}) \]

\[ = \frac{1}{2} mgL^2A^2 \omega^{-1} \]

\[ = \frac{1}{2} m \omega L^2A^2 \]

\[ \Rightarrow S' \sim m L^{3/2}\sqrt{g} \ A^2 \]

\[ \Rightarrow L_0^{3/2}A_0^2 = L(t)^{3/2}A(t)^2 \]

\[ A(t)/A_0 = (L_0/L(t))^{3/4} \]

i.e. shortening string increases amplitude!

b.) Ocean Wave Impinging on Beach

\[ d = d(x) \]

Now, \( w = w(k, g, d(x)) \) for finite depth
An analogy with pendulum:

Action Density \( N = \frac{\varepsilon}{\omega} \)  

\( \varepsilon = \) wave energy density

Aside: QM:  \( E = (N+\frac{3}{2}) \frac{\hbar}{2} \)  

Semiclassical:  \( E = N \hbar \omega \)

Classical:  \( \hbar \rightarrow 1 \)  
\[ N \rightarrow \text{action} \]

Then:  \(- \) action density (= # waves) conserved along wave trajectories  

\[- \frac{\partial}{\partial t} N + D_x (\nabla_x N) = 0 \]

\[ \rightarrow \nabla_x (d(x)) \frac{\varepsilon(x)}{\omega(d(x))} = \text{const} \]

Key Point:
- during coasting phase, inner surface is \( RT \) stable but
Supports surface waves, seeded by outer surface RT perturbations.

As surface wave ~ harmonic oscillator, can expect growth as R shrinks during coasting.

\[ \omega = \sqrt{\frac{g}{L}} \]

\[ M = \rho V \]
\[ = \rho 4\pi R_i^2 \Delta R \text{ pert.} \]
\[ \Delta R \text{ pert} = \frac{R_i}{l} = \eta^{-1} \]
\[ M = \rho 4\pi R_i^3 / l \]

\[ S = \frac{1}{2} m \omega L^2 A^2 \]
\[ = \frac{1}{2} 4\pi \rho R_i^3 \left( -\frac{\omega}{R_i} \right) \frac{U}{\rho} \left( \frac{R_i}{R_0} \right)^{1/2} \eta^2 \]
\[ = \rho \frac{R_i^3 \omega \eta^2}{l} \]
\[ n^2 \sim \frac{S_0 l}{p R_i^3 w} \sim \frac{\text{const}}{R_i^{5/4}(R_i')^{1/2}} \]

\[ n \sim (R_i')^{-1/4} R_i^{-5/4} \]

\[ \Rightarrow \text{perturbation grows by (x10) during coasting phase!} \]

References:
1. K. O. Mikaelian; Phys. Rev. A 42 3400

b) Analysis

ii) Coasting Shell "Equilibrium"

- Mass conserved during implosion

\[ M = 4\pi \int_{R_i}^{R_i^2} \rho(R) R^2 dR \]

\[ = \frac{4\pi}{3} \bar{\rho} (R_i^3 - R_i^3) \]

\[ = \frac{4\pi}{3} \bar{\rho} R_0^3 \]

\[ (R_0 \equiv \text{radius of fully collapsed shell}) \]
- Shell incompressible: $V$ volume conserved

$$R^2 V = R^2 \dot{R} = R_i^2 \dot{R}_i$$

$$\dot{R} = \frac{R_i^2 \dot{R}_i}{R^2}$$

- For total energy: (H.E. of implosion)

$$W = \frac{1}{2} \int_{R_i}^{R_2} \rho(R) R^2 dR$$

$$= 2\pi \int_{R_i}^{R_2} R^2 \dot{R}_i^2 \frac{R^2}{R_i^4} dR$$

$$W = 2\pi \bar{\rho} \dot{R}_i^2 R_i^3 (1 - R_i/R_2)$$

- Time scale:

$$T_{\text{implosion}} = \left( \frac{M R_0^2}{2W} \right)^{1/2} \left\{ \text{radius} \right\}$$

- For $\dot{R}_i$ (i.e., dynamics of implosion)

$$W = 0 = 2\pi \bar{\rho} \int [2 \dot{R}_i R_i R_i^3 (1 - R_i/R_2)]$$
\[
+ \dot{R}_1 (3 \ddot{R}_1) R_2^2 (1 - R_1/R_2) \\
+ \dot{R}_1^2 R_3 \left( -\dot{R}_1 + \frac{R_1 \dot{R}_2}{R_2^2} \right)
\]

with \( R_2 \dot{R}_2 = R_1 \dot{R}_1 \)

\[ R_1 = -\frac{W}{4\pi \rho R_1^4} \left[ 3 + 2 \frac{R_1}{R_2} + \left( \frac{R_1}{R_2} \right)^2 \right] \]

observe for \( R_1 \ll R_2 \) (thick shell limit)

\[ R_1 \sim -\frac{c}{R_1^4} \]

\[ \dot{R}_1^2 \sim \frac{c}{R_1^3} \]

\( \Rightarrow \quad dR_1 R_1^{3/2} \sim dt \)

\[ R_1^{5/2} \sim (t_0 - t) \]

\( \Rightarrow \quad R_1 \sim (t_0 - t)^{2/5} \)

implosion radius evolution.

\( \text{ii.} \) Perturbations (RT/SH)

Take:
- rotational flow
- incompressible hydro \( (\ell_G/R_1 \gg \left| -\frac{\ddot{R}_1}{R_1} \right|) \)
Then, as usual, have:

(i) \( \nabla^2 \phi = 0 \); \( \nabla = \nabla \phi \)

(ii) At interface \( \Rightarrow \) Bernoulli Eqn.: \( \rho \left( \frac{\partial \phi}{\partial t} + \frac{(\nabla \phi)^2}{2} \right) + \rho = 0 \)

\[ \text{here, unperturbed } V \text{ from imploding shell} \]

\[ \frac{\partial \phi}{\partial t} + k \nabla^2 \phi + \tilde{\rho} = 0 \]  

(iii) Boundary conditions at interfaces:

\[ \frac{d\tilde{\rho}}{dt} = \tilde{\nabla} + \left( \frac{\partial V}{\partial \rho} \right) \tilde{R} \]

\( \tilde{R} \) in shell expanding, here

\[ \tilde{\rho} = \left( \frac{\partial \bar{R}}{\partial \rho} \right) \tilde{R} \]

\( \tilde{R} \) in shell expanding
but observe:

$$ p \dot{R} = -\frac{\partial p}{\partial R} $$

$$ R^2 \ddot{R} = R_j^2 \ddot{R}_j = R^2 \dot{V} $$

$$ \left( \frac{\partial V}{\partial R} \right)_j = \left[ \frac{R_j^2 \ddot{R}_j}{(R + \Delta R)^2} - \frac{R_j^2 \ddot{R}_j}{R^2} \right] / \Delta R $$

$$ = -2 \frac{R_j^2 \ddot{R}_j}{R^2} $$

$$ \Rightarrow \frac{d\vec{\eta}_j}{dt} = \vec{\dot{\eta}}_j - 2 \frac{\ddot{R}_j}{R_j} \vec{\dot{\eta}}_j $$  \hspace{1cm} (2)

Similarly:

$$ \vec{\dot{R}}_j = p \ddot{R}_j \vec{\dot{\eta}}_j $$  \hspace{1cm} (3)

Now, to crank:

$$ \nabla^2 \phi = 0 \Rightarrow $$

$$ \phi (R, t) = \sum_{l,m} \left[ R_1 \vec{V}_{l^m} (\vec{R}_l) (\frac{R_1}{R})^{l+1} + R_2 \vec{V}_{l^m} (\vec{R}_2) (\frac{R_2}{R})^l \right] * Y_{l,m} (\theta, \phi) $$
Then \( V = \nabla \phi \)

\[
V_{(r,t)} = \sum_{l,m} \left[ V_1 (R_1/R)^{l+2} \left[ -\vec{\nabla} \cdot (\ell+1) \gamma_{l,m} + R \partial \gamma_{l,m} 
+ V_2 (R/R_2)^{l-1} \left( \vec{\nabla} \cdot \gamma_{l,m} + R \partial \gamma_{l,m} \right) \right] \right]
\]

Substituting \( \phi, \partial \phi \) into Bernoulli Eqn (\( l_{th} \) mode)

\[
-\frac{\rho}{\rho_0} = \gamma_{l,m} \left\{ \left( \frac{R_1 \gamma_{l,m} + (\ell+2) \bar{R} \gamma_{l,m}}{R_1} \right) - (\ell+1) \frac{R_1 \gamma_{l,m}}{R_1} \right. 
\left. \times \left( \frac{R_1}{R} \right)^{\ell+1} + \left\{ \frac{R_2 \gamma_{l,m} - (\ell-1) \bar{R}_2 \gamma_{l,m}}{R_2} \right. \right. 
\right. 
\left. \left. + \ell \bar{R}_2 \gamma_{l,m} \left( \frac{R_2}{R} \right)^{3} \right\} \left( \frac{R_2}{R} \right)^{\ell} \right\}
\]

Further, taking \( \bar{V}_r \) into Eqn (2) to relate \( \bar{h}_1, \bar{V}_1, \bar{V}_2 \):

\[
\dot{\bar{h}}_1 + 2 (\bar{R}_1/R_1) \dot{\bar{V}}_1 = -(\ell+1) \bar{V}_1 + \ell A^{\ell+1} \bar{V}_2
\]

\[
\dot{\bar{h}}_2 + 2 (\bar{R}_2/R_2) \dot{\bar{V}}_2 = -(\ell+1) A^{\ell+2} \bar{V}_1 + \ell \bar{V}_2
\]

\( A = \frac{R_1}{R_2} \)
Similarly, plugging (3) into Bernoulli Eqn:

\[
\begin{align*}
(R_1 \dot{V}_1 + \left[ R_2 \dot{V}_2 + (l \ A^3 - l + 1) \dot{R}_2 V_2 \right]) A^{-l} &= -\dot{R}_2 \dot{V}_2 \\
(R_1 \dot{V}_1 + \left[ l + 2 - (l+1) \ A^3 \right] \dot{R}_1 V_1) A^{l+1} &= -R_2 \dot{V}_2
\end{align*}
\]

Now:

→ during coasting phase, consider thick shell \( R_1 / R_2 = A << 1 \) \( \Rightarrow l, A \rightarrow \infty \)

→ \( \dot{\dot{R}}_1 + 2 (R_1 / R_2) \dot{\dot{R}}_1 = -(l+1) \dot{V}_1 \)

→ \( R_1 \dot{V}_1 = -\dot{R}_2 \dot{V}_2 \)

→ \[
\frac{d}{dt} \left( \frac{1}{R_1} \frac{d}{dt} (R_1^2 \dot{R}_1) \right) = (l+1) R_1 \ddot{R}_1 \]

\( \ddot{\text{similarly}} \)
\[ \dot{R}_2 + 2 \left( \frac{\dot{R}_2}{R_2^2} \right) R_2 = \ell V_2 \]
\[ \dot{V}_2 = - \frac{\dot{R}_2}{R_2} \dot{R}_2 \]
\[ \frac{d}{dt} \left( \frac{1}{R_2} \frac{d}{dt} \left( R_2^2 \dot{R}_2 \right) \right) = -\ell R_2 \dot{R}_2 \dot{R}_2 \]

Two interfaces decouple. 

For inner interface (1):

- Coating shell \( R_1 \sim (t_0 - t)^{2/5} \)
- \( \dot{R}_1 \sim (t_0 - t)^{1/5} \) and \( \nabla \sim (t_0 - t)^{\alpha} \)

\[ \frac{d}{dt} \left( \frac{1}{R_1} \frac{d}{dt} \left( R_1^2 \dot{R}_1 \right) \right) = (\ell + 1) R_1 \dot{R}_1 \nabla \]

\[ \alpha = -\frac{1}{10} \pm \left( 25 - 24\ell \right)^{1/2} \]

\[ \dot{R}_1 \sim (t_0 - t)^{-1/10} \left[ \text{slow increase due to convergence} \right] \]

Note: Consequence: spherical convergence, not exponential growth.
Wave-like solution ($\omega R \tilde{R}$)

\[
\begin{align*}
\frac{d}{dt} \left( \frac{1}{R} \frac{d}{dt} (\mathbf{R}^2 \tilde{R}) \right) &= (l+1) \mathbf{R}^2 \tilde{R}, \\
\mathbf{R},(t) &= \mathbf{Z}(t) e^{i \int \omega(\tau^t) d\tau}
\end{align*}
\]

\[
-l_1 \mathbf{R}^2 \mathbf{Z} + i \left( 3\omega \mathbf{R}_1 \mathbf{Z} + 2\omega \mathbf{R} \mathbf{Z} + i \omega \mathbf{R}_1 \mathbf{Z} \right) + \mathbf{R}_1 \frac{d}{dt} \mathbf{Z} + 3 \mathbf{R} \frac{d}{dt} \mathbf{Z} + 2 \mathbf{R}_1 \frac{d}{dt} \mathbf{Z} = (l+1) \mathbf{R}_1^{l+1} \mathbf{Z}
\]

\textbf{lowest order:} \quad \omega^2 = -\frac{l''}{l+1} \quad \Rightarrow \text{eigen frequency}

\textbf{first order:} \quad 3\omega \mathbf{R}_1 \mathbf{Z} + 2\omega \mathbf{R} \mathbf{Z} + i \omega \mathbf{R}_1 \mathbf{Z} = 0

\Rightarrow \quad \mathbf{R}_1^3 \mathbf{Z}^2 \omega = \text{const.}

Recovery adiabatic in variant $||$

(SW Action, surface $\mathcal{O}$)

Implications follow $||$
\( \text{Note:} \)

d) Question significant to Nova upgrade advisability of long coasting phase

e) Generally, for spherical \( RT: \)

\[
\ddot{\eta} + 3 \frac{\dot{R}}{R} \dot{\eta} - n A(n) \frac{R}{\dot{R}^2} \eta = 0
\]

\( n \rightarrow \infty \)

\[ n A(n) = \sum_{n} (n-1) \rho_2 - (n+1)(n+2) \rho_2 \sqrt{\rho_2 + (n+1)} \]

need:

\[ n A(n) \dot{R} < 0 \]

\[ \frac{\partial}{\partial t} \left[ n A(n) R^5 \dot{R} \right] < 0 \]

(please et)

\( (\text{HW}) \)
% Nonlinear Rayleigh-Taylor Instability: Single Mode / Bubble

H. J. Hull, Review

A) Motivation and Heuristics

Recall:

(i) in linear phase, simple R.T. instability

\[ \gamma = \sqrt{g\alpha k} \]

(ii) in nonlinear phase expect algebraic growth (i.e. simple intuition) \[ \frac{\partial}{\partial t} \text{ bubble rise} \]

\[ l = \lambda^2 + t^2 \]

\[ (\alpha \ll 1) \]

Seek:\[ \rightarrow \text{ how recover algebraic growth?} \]

\[ \rightarrow \text{ how unify linear, nonlinear regimes?} \]

\[ \rightarrow \text{ understand flow structure in nonlinear regime} \]

Heuristically
\( N_L \); \( \partial \phi \partial A \sim \frac{\partial \phi}{\partial z} \)

\[ \Rightarrow k A \sim 1 \]

hence, for cylinder:

\[ A = 1 \]

\[ \text{c.e., air bubble rises in center} \]
\[ \text{H}_2\text{O spike falls at edge} \]

\[ \Rightarrow \text{"spike and bubble" picture R,} \]

\[ \Delta \text{ (interface variable)} \Rightarrow R \text{ (bubble radius)} \]

\( R \text{ (structure)} \)

then, linear theory \( \Rightarrow \)

\[ \gamma = \frac{1}{R} \frac{dR}{dt} = \sqrt{kg} \]

\[ \frac{dR}{dt} = v = \sqrt{kg} \frac{R}{R} \]

bubble rise velocity
\[ \phi \left( \frac{3}{2}, 0, I \right) = 0 \]

**B.C.'s:** Hand wall: \( \nu = 0 \) at \( y = 0 \)

\[ A = 1 \]
\[ \nu = 1 \]

**Unit:** \( R/\rho = 1 \)

Consider tube: the radius

\[ \frac{\partial \Gamma}{\partial r} \bigg|_{r=0} = 0 \]

\[ \text{Consider tube} \]

\[ \text{B)\text{A layer Calculation}} \]

Suggestions can understand nonlinear A.T. via

\[ \Delta r = \frac{\partial r}{\partial T} \]

- \( r = k \phi \Rightarrow \frac{\partial r}{\partial T} = k \frac{\partial \phi}{\partial T} \)

- \( k \phi = k \lambda \phi \Rightarrow \lambda = k \]

- \( \phi = R(1, e) \) for \( T > 1 \)

- \( \phi = R(1, e) \) for \( T > 1 \)

- \( \phi = R(1, e) \) for \( T > 1 \)

\[ \text{Nature of the dependence} \]
\[
\frac{dr}{dt} = v_r = -\partial_r \phi = + F(t) e^{-z} J_1(r)
\]
\[
\frac{dz}{dt} = v_z = -\partial_z \phi = + F(t) e^{-z} J_0(r)
\]

(Layer notation)

exploits potential flow structure, of problem — obviously not universal applicable

Then, for streamlines, can write:

\[
\frac{dz}{dr} = \frac{v_z}{v_r} = \frac{(dz/dt)/(dr/dt)}{J_0(r)/J_1(r)}
\]

but Bessel identity \( \Rightarrow \)

\[
J_1(r) = J_0(r) - \frac{J_1(r)}{r}
\]

\[
\frac{dz}{dr} = \frac{J_1(r) + J_0(r)}{J_1(r)}
\]

\[
= \frac{J_1}{J_1} + \frac{4}{r}
\]
\[ dZ = \frac{J_z}{J} \, dr + \frac{dv_z}{r} \]

\[ Z = \ln \left( J_z(r) \right) + \ln r \]

\[ e^Z = C \cdot r J_z(r) \]

→ parametrizes fluid streamlines, if interface slightly distorted.

No Bernoulli, yet.

**Note:**
- Generate stream surfaces via displacement along stream.
- Surfaces like:
  \[ \n \]
  → asymptotic to \( Z \) axis

\[ i.e. \ e^{-|Z|} \propto C \cdot r J_z(r) \]

\[ Z \to -\infty \quad r \to R \]

→ streamline structure is underpinning of spike + bubble intuition!

→ not really solution, but effect due to B.C.

i.e. generated

Now, to obtain equation of motion for interface...
Solution not yet satisfy Bernoulli end \( \Rightarrow \) interface distortion

but

message is that B.C.'s force spike structure

c.e. \[ 1 = c \cdot J_i(r) \cdot e^{-\frac{r}{l}} \]

\[ \text{small} \quad \text{large} \]

\[ \Rightarrow \text{spike + bubble picture consequence } \begin{cases} \text{B.C.'s symmetry} \\ \text{c.e. hexagonal symmetry (lateral convection)} \end{cases} \]

\[ \Rightarrow \text{spikes at vertices} \quad \text{bubble at center} \]

(top view)
\( \text{D integrated fluid equations of motion:} \)

\[
\frac{d\mathbf{r}}{dt} = + F(t) e^{-Z} J_1(r) \\
\frac{dz}{dt} = + F(t) e^{-Z} J_0(r)
\]

define: \( Z = e^Z \) \( \rightarrow \) vertical variable

\( V = r^2 \) \( \rightarrow \) radial

\( T(t) = \int_0^t dt F(t) + 1 \) \( \rightarrow \) time variable

\( k(V) = 2 J_1(r)/r \) \( \rightarrow \) shape

Now:

\[
\frac{dZ}{dt} = \frac{dz}{dt} e^Z = + F(t) J_0(r) \quad \text{(fluid eqns.)}
\]

\[
\frac{dZ}{dT} = J_0(\sqrt{V})
\]

and

\[
Z = V k(V)/ (dV/dT) \quad \rightarrow \text{phys.}
\]
Eliminating \( Z \):

\[
\frac{V}{V_0} = (T-1) \frac{h(V)}{Z_0 + 1} \quad \rightarrow \text{velocity}
\]

\[
\frac{Z}{Z_0} = \frac{h(V)}{h(V_0)} \left( \frac{(T-1) h(V)}{Z_0 + 1} \right)
\]

Now:

\[ \rightarrow \text{potential can't solve Bernoulli eqn. over full surface} \]

\[ \rightarrow \text{seek expansion valid near bubble vertex, i.e. weak distortion, see (266)} \]

\[ \rightarrow \text{un-perturbed surface flat } \left\{ \begin{array}{l}
V_0 = 0 \\
Z_0 = 1
\end{array} \right. \]

\[ \rightarrow \text{cylinder approximation} \]

\[ \text{n} \]

\[ \text{neglecting non-linearities in } V \quad (\text{no } \alpha) \]

\[ \frac{V}{V_0} = (T-1) \frac{h(V_0)}{Z_0 + 1} \quad \rightarrow \text{expansion} \]

\[ = (T-1) \frac{2 \gamma' \alpha}{\gamma - 1} \]

\[ \therefore \quad V = V_0 (T-1) \]
$\xrightarrow{\text{unperturbed}}$ $\xrightarrow{\text{weakly perturbed}}$ $\xrightarrow{\text{strongly perturbed}}$

$\approx$ 0 approx.

domain validity of treatment shrinks as bubble rises.
\[ n' \sim \frac{\Phi}{T} \]

Similarly:
\[ e^{n'} = (T-1) \left[ 1 - \frac{V}{(1-T^{-2})} \right] \]
\[ T \sim \ln(T) \quad \text{as} \quad T \to 1 \]

\[ V = V_0 (T-1) \]
\[ e^{\eta'} = (T-1) \left[ 1 - \frac{V}{(1-T^{-2})} \right] \]

Then:
\[ \phi = F(t) e^{-\frac{z}{2}} J_0(r) \]

\[ \frac{\partial \phi}{\partial t} - (\nabla \phi)^2 \eta = 0 \]

\[ \Rightarrow \left\{ T (T^2+1) \eta'' - T^2 - T^2 (T^2-1) = 0 \right\} \]

\[ \text{Let:} \quad F \to T' \]
\[ \Rightarrow \eta = \eta'(T, V) \quad \text{(above)} \]
\[ \Rightarrow \eta = \frac{d}{dt} \quad \text{(Bernoulli in real time)} \]

Check:

a) Linear Regime (small time)

\[ T = 1 + \tau , \quad \tau \ll 1 \]

\[ \Rightarrow (1+\tau)(1+\tau^2+1) \eta'' - \eta' - (1+\gamma)^2 \left( \eta + \gamma^2 \right) = 0 \]
1. \[ T'' - 2T' = 0 \]

i.e., \[ T = T(0) e^t \] \[ \Rightarrow \] exponential growth and linear theory.

In dimensional units:

\[ T = T(0) e^{\chi t} \]

\[ \chi = \sqrt{ \frac{R g}{\beta I} } \]

\[ \sim \sqrt{\frac{g}{\text{cylindrical geometry}}} \]

b.) Nonlinear Regime \[ T >> 1 \]

\[ T^3 T'' - T'T'' - T^4 = 0 \]

\[ T'' - T = 0 \]

\[ \Rightarrow \] \[ T = e^t \] \[ \Rightarrow \] \[ T' = \text{F}(+) = e^t \]

\[ V = \left. \frac{d \ln(T)}{dt} \right|_{T=0} \]

\[ = \frac{d}{dt} \frac{T}{T'} = \frac{1}{T} \frac{dT}{dt} = 1 \]

\[ \int e^{\gamma T} = (T-1) \]

\[ \gamma e^\gamma = T \] when \[ T >> 1 \]
Then \( V = \frac{4}{3} \), or in dimensional units:

\[
V = \left( \frac{9R}{\beta_1} \right)^{1/2}
\]

Agrees with free-fall intuition.

c.) Can find general vertex dynamics solution.

d.) \( T \sim 1 \) \( T \gg 1 \) limits establish \( k \eta \sim 1 \)
as criterion for entrance into nonlinear regime.

C.) Heuristics for Multiple Mode Systems

- Layer solution for single bubble/mode

- In reality, ICF target finnishes irregularities initialize many modes

Seek: multi-mode criterion for non-linearity

\( \Delta \) amplitude for exponential growth\( \Rightarrow \) cessation.

First observe: all modes can't grow to high spectral content diverges.

\[ \langle \beta^2 \rangle = \int dh h^2 \tilde{\eta}^2 \]
\[ = \int k dk \frac{1}{k^2} \]
\[ \rightarrow \infty \]

Some modes, especially long wavelength, remain in linear regime. (slow)

Natural, in multi-mode case, to suggest nonlinearity criterion \( \left( \frac{k}{k_{\text{peak}}} \right)^{\text{th}} \) where \( k_{\text{peak}} \) refers to peak wave number of spectrum.

\[ \left( \frac{k}{k_{\text{peak}}} \right)^2 \]

\[ \text{spectra} \rightarrow \text{peak + width} \]

long wavelength modes grow slowly, ablation, \( k \rightarrow \text{cut-off} \)
\[ k_{i=0}^{-2} = \frac{L^2}{(2\pi)^2} \int_0^\infty 2\pi k \, dk \, |\tilde{M}_{i=0}|^2 = \langle \tilde{M}^2 \rangle_{\text{MR}} \]

\[ = \frac{L^2}{2\pi} k_0 \Delta k \, |\tilde{M}_{k=0}|^2 \]

→ Establishes criterion:

\[ M_{k_0} \approx \frac{\sqrt{2\pi}}{L} \left( k_0 \Delta k \right)^{-1/2} \]

if \( \Delta k \approx k_0 \) (frequent state affairs)

\[ \sqrt{M_{k_0}} \approx \frac{\sqrt{2\pi}}{L k_0^{1/2}} \]

→ in multimode system, superposition (bubble competition) of many-mode interfacial displacements ⇒ \( k_0 \sqrt{M_{k_0}} \approx \frac{\lambda_0}{L} \), not \( \perp \) (single wave)

→ Transition to layering regime at lower amplitude.

→ Consistent with LNL simulations, experiments
c) Bubble Competition and Mix 4

- Goal is to:
  - Construct model of nonlinear Rayleigh–Taylor using Lezyer spike + bubble model
  - Generate model of growth of mixing layer

- Ingredients:
  - Single bubble model
  - Bubble-bubble interaction $\Rightarrow$ Competition

A.) Single Bubble

- Lezyer calculation $\Rightarrow$ bubble vertex rises
  \[ \frac{dV}{dt} = \left( \frac{QR}{\beta_i} \right)^{1/2} \]

1. Suggests that mixing layer grows as:
  \[ \frac{df}{dt} = \left( \frac{Q}{l} \right)^{1/2} \]
  \[ f \sim agt^2 \]
omits other effects which limit bubble/spike dynamics:

- compression

- drag, spike $kH$

1.0 Model:

\[
\begin{align*}
V(l, z, t) & \to \text{bubble rise velocity} \\
\ell(z, t) & \to \text{mixing layer width}
\end{align*}
\]

\[
\frac{\partial \ell}{\partial t} + V \frac{\partial \ell}{\partial z} = 0
\]

\[
p \frac{\partial V}{\partial t} = p g - p \rho_0 \frac{V^2}{\ell} \to \text{drag}
\]

gravitational force

Physics of Drag:

- view bubble rise as mixture of two inter-penetrating fluids:
\[ F_0 = C_0 \frac{V}{t} \frac{DV}{dy} \text{ (momenetum)} \]

\[ \text{time scale (l = "chunk" size for interpenetrating fluids)} \]

\[ \text{Drag slows bubble rise} \]

\[ \Rightarrow \frac{DV}{dt} = 0 \Rightarrow V = \left( \frac{1}{C_0} \right)^{1/2} (g \rho)^{1/2} \]

\[ l = C_0^{-1/2} g \rho^{1/2} \]

1. Drag coefficient contains physics of \( \propto 0.5 \) scaling

- alternatively, view drag term as manifestation of decay due to spike shear flow instability

\[ \sqrt{\text{repair} \rightarrow \text{full glue} \rightarrow \text{blunt tip}} \]

\[ \Rightarrow \text{sends spike} \]
Neg time scale for roll-up is flow shear rate

\[ \dot{\gamma} \sim |\nabla u| \]

dimensionally, \[ \dot{\gamma} \sim \frac{V}{x} \] \( \Rightarrow \) rate of drag

- drag coefficient essential to proper fit of explosion experiment data

B.) Bubble Competition

Observe:

- single bubble mixing model characterized by single length scale
  - but
   - multi-mode system \( \Rightarrow \) multi-bubbles

- need describe trend in evolution of bubble length scale

Simple example:
3 bubbles

Exterior larger

\[ h_3 < h_1, h_2 \]

\[ \rightarrow \text{but, in Rayzer regime:} \]

\[ V \sim (g \cdot \ell)^{1/2} \]

\[ \rightarrow \text{larger bubbles rise faster} \]

\[ \rightarrow \text{at that} \]

\[ \Rightarrow 1, 3 \text{ will tend to expand over } 2, \text{ thus "squeezing" it out} \]
Bubble, Bubble, Toil, and Trouble...

Last time:

- discussed D. Layzer's solution for single nonlinear bubble.
- why: observables in NL state
  \[ \gamma'(1\text{cm}) \sim 1/\text{sec.} \]

- what: singular mode (\( \chi \to R \)) growth
  \[ \left\{ \begin{array}{l}
  k_m \sim 1 \quad (m/R \sim 1) \quad \text{regime} \\
  t \to \infty \\
  \end{array} \right. \\
  \text{i.e.} \quad V = \#(gR)^{1/2} \\
  \to \text{connects exponential and algebraic growth regimes (NL saturation)} \]

- how: flat interface approximation at bubble tip
  \[ (\text{i.e., } t \to 0) \to \text{geometry} \]
  \[ \to \text{use of streamlines from } \nabla \phi = 0 \] and (assumed) self-similarity (valid at bubble tip).
Today:
(a) Lazy Man's Layerer
(b) Bubble Competition

I. Recall basic \( R - T \) equations:

- \( \nabla^2 \psi = 0 \)

- \( \frac{\partial \psi}{\partial t} + \nabla \psi \cdot \nabla = \frac{\nu}{\rho} \) or more generally just "interface moves with fluid"

- \( \frac{\partial \psi}{\partial t} + \nabla \psi \cdot \nabla \psi + g \eta = \text{const.} \)

For single mode (bubble):

\[ \psi(x, z, t) = a(t) \cos(kx) e^{-kz} \]

Upper limit \( t \to \infty \) since bubble is rising light fluid

Now, essence of layering bubble tip approximation is geometric assumed shape of bubble,

\[ Z_{e}(x, t) = Z_{b}(x, t) = Z_0 + Z_1 (x - x_0)^2 \]
c.e. parabolic shape approximation:

\[ x_i, z_0 \to \text{tip location} \]

\[ z_1 \to \text{radius of curvature of bubble} \]

\[ \text{c.e. } R_c = -1/2z_1 \]

Note: Linear: \[ \frac{\partial \varepsilon}{\partial z} = \frac{\partial \phi}{\partial z} \]

\[ \text{c.e. } \varepsilon \text{ forces } \eta \]

Layer: \[ \eta \leftrightarrow \text{geometry} \]

Then:

\[ \eta = z = z_0 + z_1 (x-x_i)^2 \]

\[ \frac{\partial \phi}{\partial z} + \nabla \cdot \nabla \phi = \frac{\partial \phi}{\partial \zeta} \]

\[ \Rightarrow \frac{d z_0}{d \tau} + (x-x_i)^2 \frac{d z_1}{d \tau} + (-2)z_1 (x-x_i) \frac{d x_i}{d \tau} \]

\[ + \sqrt{2} (x-x_i) = V_z \]
\[ V_z - \frac{\partial Z_0}{\partial t} - \frac{\partial Z_1}{\partial t} (X-X_1)^2 - 2 Z_1 (X-X_1) (V_X - \frac{\partial X_1}{\partial t}) = 0 \]

we have:

\[ \Phi = \alpha(t) \cos(kx) e^{-\frac{k^2 z_0}{2}} \quad ; \quad V = \nabla \Phi \]

\[ V_Z = \frac{\partial Z_0}{\partial t} = \frac{\partial Z_1}{\partial t} (X-X_1)^2 \]

\[ = -2 Z_1 (X-X_1) (V_X - \frac{\partial X_1}{\partial t}) \]

\[ \frac{\partial \Phi}{\partial t} + \frac{V_X^2 + V_Z^2}{2} + g z = \text{const.} \]

Plugging in, to 2nd order:

Interface case:

\[ O(\alpha t) j \]

\[ - k \alpha(t) \cos(kx) e^{-\frac{k^2 z_0}{2}} \left|_{x=0}^{x=\varepsilon} \right. - \frac{\partial Z_0}{\partial t} = 0 \]
\[
\frac{\partial}{\partial t} \left( a k^2 (z_1 + \frac{k}{2}) e^{-kz_0} \right) - \frac{\partial z_1}{\partial t} + 2z_1 a k^2 e^{-kz_0} = 0
\]

-and Bernoulli: (to O(2))

\[
\begin{align*}
&\frac{\partial}{\partial t} \left( k e^{-kz_0} (z_1 + \frac{k}{2}) \right) + a^2 k^3 \frac{z_1 e^{-2kz_0}}{\sqrt{2}} \\
&- q z_1 = 0
\end{align*}
\]

Recall: \( e^{-kz} \frac{a}{2} \left( z_0 + z_1 (x-x^2) \right) \) \( \rightarrow \) bubble tip \( (x \to 0) \) 

\( \Rightarrow \) equiv. Hayzen Solution.

Check: Linear Solution:

\[
\frac{k}{2} e^{-kz_0} \frac{\partial q}{\partial t} = q z_1
\]
\[
\frac{a k^3 e^{-k z_0}}{2} = \frac{d z_1}{dt}
\]

so, small perturbations: \( \Rightarrow \)

\[
\frac{k^2 e^{-k z_0}}{2} \frac{d^2 a}{dt^2} = \frac{g q k^3}{2}
\]

\[
\frac{d^2 a}{dt^2} = g k a \sqrt{.} \text{ etc.}
\]

Low, for late times:

\[
k e^{-k z_0} \left( z_1 + \frac{b}{a} \right) \frac{dx}{dt} + a^2 k^3 z_1 e^{-2k z_0} = g z_1
\]

made sorts: \( \text{balance} \)

\[
\begin{cases} 
V^2 \sim g M \sim g R \\
\end{cases}
\]

\[
\Rightarrow a^2 k^3 \sim g
\]

\[
(k a)^2 \sim g/k \Rightarrow V^2 \sim g R
\]

\[
\text{akin } V^2 \sim (g R)
\]

etc.
¬ What's important?

1. Birth/Death structure
2. Self-similarity (i.e. all $N$-agglomeration same)
3. Merger rules

1 - 3 $\Rightarrow$ inverse cascade.

More general form:

$$N(t) \frac{dg(\lambda, t)}{dt} = -2g(\lambda, t) \int_0^\infty \int g(\lambda', A, \omega(\lambda, \lambda')) d\lambda'$$

$$+ \int_0^\lambda g(\lambda - \lambda', t) g(\lambda', \omega(\lambda - \lambda', \lambda)) d\lambda'$$

Birth:

$$N = \int g(\lambda, t) d\lambda = \# \text{ bubbles at } t +$$

$$g(\lambda, t) d\lambda = \text{ size distribution function}$$

$$w(\lambda_j, \lambda_{i+1}, t) \rightarrow \text{ merger rate} \quad \lambda_j \rightarrow \lambda_i \quad \lambda_i \rightarrow \lambda_{i+1}$$