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A simple dynamical model of intermittent fully developed turbulence

By URIEL FRISCH, PIERRE-LOUIS SULEM
CNRS, Observatoire de Nice, France
AND MARK NELKIN
School of Applied and Engineering Physics, Cornell University, Ithaca, New York 14853

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We present a phenomenological model of intermittency called the β-model and related to the Novikov–Stewart (1964) model. The key assumption is that in scales $\sim l_0 2^{-n}$ only a fraction β^n of the total space has an appreciable excitation. The model, the idea of which owes much to Kraichnan (1972, 1974), is dynamical in the sense that we work entirely with inertial-range quantities such as velocity amplitudes, eddy turnover times and energy transfer. This gives more physical insight than the traditional approach based on probabilistic models of the dissipation.

The β-model leads in an elementary way to the concept of the self-similarity dimension D, a special case of Mandelbrot's (1974, 1976) 'fractal dimension'. For three-dimensional turbulence, the correction B to the $\frac{5}{3}$ exponent of the energy spectrum is equal to $\frac{1}{3}(3-D)$ and is related to the exponent μ of the dissipation correlation function by $B = \frac{1}{3}\mu$ (0·17 for the currently accepted value). This is a borderline case of the Mandelbrot inequality $B \leqslant \frac{1}{3}\mu$. It is shown in the appendix that this inequality may be derived from the Navier–Stokes equation under the strong, but plausible, assumption that the inertial-range scaling laws for second- and fourth-order moments have the same viscous cut-off.

The predictions of the β-model for the spectrum and for higher-order statistics are in agreement with recent conjectures based on analogies with critical phenomena (Nelkin 1975) but generally diasgree with the 1962 Kolmogorov lognormal model. However, the sixth-order structure function $\langle \delta v^6(l)\rangle$ and the dissipation correlation function $\langle \epsilon(\mathbf{r})\,\epsilon(\mathbf{r}+\mathbf{l})\rangle$ are related by

$$\langle \delta v^6(l)\rangle / l^2 \sim \langle \epsilon(\mathbf{r})\,\epsilon(\mathbf{r}+\mathbf{l})\rangle$$

in both models. We conjecture that this relation is model independent.

Finally, some possible directions for further numerical and experimental work on intermittency are indicated.

1. Introduction

In trying to understand some of the recent work on intermittency, particularly that of Mandelbrot, we have constructed a simple dynamical model which embodies many of the observed features. These include spottiness of the small-scale structure (Batchelor & Townsend 1949; Kuo & Corrsin 1971), higher-order structure functions which do not follow the Kolmogorov (1941) scaling (Van Atta & Park 1972) and a dissipation correlation function which follows a power law (Gibson, Stegen & McConnell 1970).

A number of probabilistic models have already been developed by the Soviet School (see Monin & Yaglom 1975, vol. 2, chap. 25 for a review). Mandelbrot (1972, 1974, 1976) has shown that such models give rise to the very interesting geometric concept of a fractal dimension D, which is a measure of the extent to which the regions in which dissipation is concentrated fill space. He has shown that the exponent in the dissipation correlation function, usually denoted by μ, is equal to $3-D$ and that the correction, which he denotes by B, to the $\frac{5}{3}$ exponent of the energy spectrum satisfies

$$B \leqslant \tfrac{1}{3}\mu. \tag{1.1}$$

He has also shown that the Kolmogorov (1962) lognormality assumption, which leads to $B = \frac{1}{9}\mu$, is only one of many possibilities. In all the above models the key quantity is the *dissipation*, the effect of which is restricted to very small scales. It would be preferable to work with dynamical quantities directly related to inertial-range energy transfer (Kraichnan 1974; Nelkin 1974; Frisch, Lesieur & Sulem 1976; Nakano 1976).

The β-model introduced in this paper is essentially a dynamical version of the Novikov–Stewart (1964) model, and relates naturally to the concept of a fractal dimension. An important feature of the β-model is that we do not have to assume the Kolmogorov (1941) law initially and then derive its modified version by somehow mysteriously taking fluctuations into account. Instead we introduce a simple dynamical argument, mostly borrowed from Kraichnan (1972), from which we derive both the Kolmogorov 1941 theory (§ 2) and the scaling laws for intermittent turbulence (§§ 3 and 4), depending on whether or not the small scales are space filling. The β-model leads immediately to the concept of a self-similarity dimension D, a special case of Mandelbrot's fractal dimension. Furthermore, all the exponents defined by scaling laws can be simply expressed in terms of the single parameter D (or μ). In § 5 we show that the β-model can be extended to other space dimensions, including the inverse energy cascade in two dimensions and the one-dimensional Burgers equation. In § 6 we discuss the relation of our model to previous work on intermittency, particularly that of Oboukhov (1962), Kolmogorov (1962), Novikov & Stewart (1964), Yaglom (1966) and Mandelbrot (1974, 1976). In § 7 we consider the relations among measurable exponents predicted by the β-model. We discuss the possibility of experimental choice among models, and make a conjecture about the sixth-order structure function which is probably model independent.

2. Kolmogorov (1941) revisited

By the Kolmogorov (1941, hereafter K41) theory, we mean the general class of arguments developed by Kolmogorov, Oboukhov, Onsager and others which has led, in particular, to the $\frac{5}{3}$ law (see Batchelor 1953; Monin & Yaglom 1975 for reviews). The $\frac{5}{3}$ law may be derived from dimensional analysis, but more insight is gained from a simple dynamical argument borrowed from Kraichnan (1972, p. 213).† We define the energy spectrum $E(k)$ as the kinetic energy per unit mass and unit wavenumber k.

† *See also the closely related argument of Onsager (1949, p. 284).*

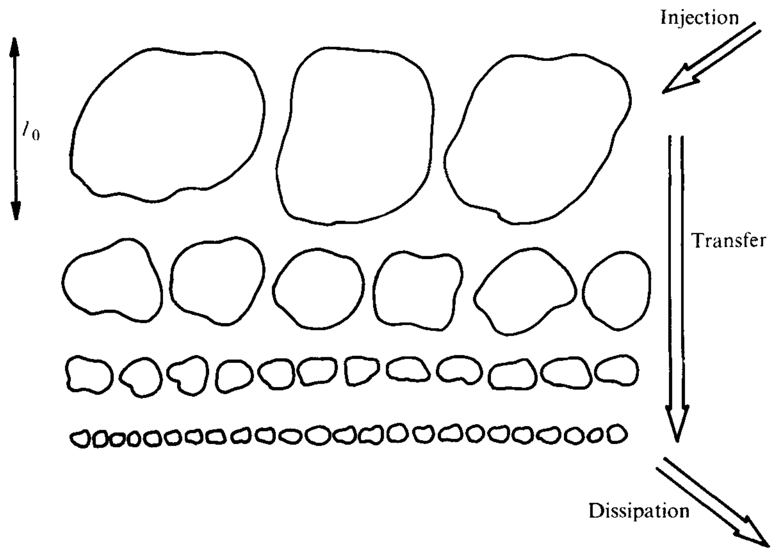

FIGURE 1. The energy cascade according to the 1941 Kolmogorov theory. Notice that at each step the eddies are space filling.

It is a convenient simplification, with no significant loss of generality, to consider a discrete sequence of scales or 'eddies'

$$l_n = l_0 2^{-n}, \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

and a discrete sequence of wavenumbers $k_n = l_n^{-1}$. The kinetic energy per unit mass in scales $\sim l_n$ is defined as

$$E_n = \int_{k_n}^{k_{n+1}} E(k)\, dk. \tag{2.2}$$

Let us assume that we have statistically stationary turbulence where energy is introduced into the fluid at scales $\sim l_0$ and is then transferred successively to scales $\sim l_1$, $\sim l_2, \ldots$ until some scale l_d is reached where dissipation is able to compete with non-linear transfer (figure 1). If we now make the essential assumption that *eddies of any generation are space filling*, as indicated in figure 1, we may define a characteristic velocity v_n of nth-generation eddies (n-eddies for short) by

$$E_n \sim v_n^2. \tag{2.3}$$

In (2.3) and subsequently factors of the order of unity will be systematically dropped except when such factors would accumulate multiplicatively in successive cascade steps.

Note that v_n is *not* the velocity with which n-eddies move with respect to the reference frame of the mean flow, this being mostly due to advection by the largest eddies. In a local cascade v_n is rather a typical velocity difference $v_n = \delta v(l_n)$ across a distance $\sim l_n$, the latter being the only dynamically significant quantity. We now define the eddy turnover time

$$t_n \sim l_n/v_n. \tag{2.4}$$

The quantity t_n^{-1} may be considered as the typical shear in scales $\sim l_n$, and therefore defines the characteristic rate at which excitation at scales $\sim l_n$ is fed into scales $\sim l_{n+1}$. There are, however, at least two important exceptions to this statement. First, we may define a viscous dissipation time

$$t_n^{\text{diss}} \sim l_n^2/\nu, \tag{2.5}$$

where ν is the kinematic viscosity of the fluid. In the dissipation range

$$t_n^{\text{diss}} \ll t_n, \tag{2.6}$$

transfer is no longer able to compete with dissipation, and most of the excitation in scales $\sim l_n$ is lost to viscosity. Second, if

$$t_n \gg t_0 = l_0/v_0 \tag{2.7}$$

then the shear acting on scales $\sim l_n$ comes mostly from scales $\sim l_0$, and t_0 should be used instead of t_n as a dynamical time, so that the cascade is not local.

In the inertial range of three-dimensional turbulence, where inequalities (2.6) and (2.7) are reversed (as may be checked *a posteriori*), we make the fundamental assumption that in a time of the order of t_n a sizable fraction of the energy in scales $\sim l_n$ is transferred to scales $\sim l_{n+1}$. The rate of transfer of energy per unit mass from n-eddies to $(n+1)$-eddies is then given by

$$\epsilon_n \sim E_n/t_n \sim v_n^3/l_n. \tag{2.8}$$

Since we assume a stationary process in which energy is introduced at scales $\sim l_0$ and removed at scales $\sim l_d$, conservation of energy requires that

$$\epsilon_n \equiv \bar{\epsilon}, \quad l_d \leqslant l_n \leqslant l_0. \tag{2.9}$$

Notice that $\bar{\epsilon}$ can be thought of as a rate of energy injection, a rate of energy transfer or a rate of energy dissipation. From the point of view of inertial-range dynamics, the second of these three definitions is the most relevant. Using (2.8) and (2.9) we solve for v_n and E_n:

$$v_n \sim (\bar{\epsilon} l_n)^{\frac{1}{3}}, \quad E_n \sim (\bar{\epsilon} l_n)^{\frac{2}{3}}. \tag{2.10}$$

This is Kolmogorov's result for the structure function, which after Fourier transformation yields the K41 spectrum

$$E(k) \sim \bar{\epsilon}^{\frac{2}{3}} k^{-\frac{5}{3}}. \tag{2.11}$$

The eddy turnover time of (2.4) is given by

$$t_n \sim \bar{\epsilon}^{-\frac{1}{3}} l_n^{\frac{2}{3}}. \tag{2.12}$$

Equating (2.12) to the viscous diffusion time (2.5) determines the Kolmogorov microscale

$$l_d \sim (\nu^3/\bar{\epsilon})^{\frac{1}{4}}. \tag{2.13}$$

Equation (2.13) gives the length scale at which the cascade is terminated by viscous dissipation.

3. Intermittency: the β model

Since the first experiments of Batchelor & Townsend (1949) strong evidence has been obtained that the small-scale structures of turbulence become less and less space filling as the scale size decreases (Kuo & Corrsin 1971; see also Monin & Yaglom 1975 for a review). Dynamically this spottiness of the small scales can be made plausible by a simple vortex-stretching argument somewhat similar to the argument of Corrsin & Kistler (1954) for the phenomenon of boundary sharpness. Consider the point M within a large-scale structure which at the initial time has the largest vorticity amplitude $|\omega|$. This point is also likely to have a large velocity gradient $|\nabla v| \sim |\omega|$. The straining action of the velocity gradient on the vorticity may then be described by a crude form of the vorticity equation:

$$D|\omega|/Dt \sim |\omega|^2. \tag{3.1}$$

Hence it is expected that the vorticity downstream of M will rise to very large values (possibly infinite at zero viscosity) in a time of the order of the large-eddy turnover time $t_0 \sim |\omega|^{-1}$.

Even if the initial vorticity has a very flat spatial distribution, the nonlinearity of (3.1) will lead to a very narrowly peaked spatial distribution at times $\approx t_0$. So we see that small-scale structure may be generated in a very localized fashion. This argument can be made fully rigorous for the Burgers equation, but not for the Navier–Stokes equation (Léorat 1975). For the Navier–Stokes equation there is the important complication that the velocity gradient at a point $\mathbf{x}$ is not related in any simple way to the vorticity at $\mathbf{x}$; instead it is given by a Poisson integral with a fairly substantial local contribution but also some coupling to nearby points. This could smooth out the vorticity peak, but the smallest-scale structures will still have some tendency not to occur uniformly. Note that the intermittency will of necessity also be temporal because of the sweeping of small structures by large ones, but that there probably exists in addition an intrinsic temporal intermittency (Kraichnan 1974; Siggia 1977, 1978).

Assuming that the small eddies do indeed become less and less space filling, let us now define the β-model. At each step of the cascade process any n-eddy of size $l_n = l_0 2^{-n}$ produces on the average N $(n+1)$-eddies. If the largest eddies are space filling, after n generations only a fraction

$$\beta_n = \beta^n \quad (\beta = N/2^3 \leqslant 1) \tag{3.2}$$

of the space will be occupied by active fluid (see figure 2). Furthermore, we assume that $(n+1)$-eddies are positionally correlated with n-eddies by embedding or attachment (for the sake of pictorial clarity this feature is not included in figure 2).

It is straightforward to work out how the β-model modifies K41. Let v_n now denote a typical velocity difference over a distance $\sim l_n$ *in an active region.* The kinetic energy per unit mass on scales $\sim l_n$ is then given by

$$E_n \sim \beta_n v_n^2. \tag{3.3}$$

The characteristic dynamical time for transfer of energy from active n-eddies to smaller scales is still given by the turnover time $t_n = l_n/v_n$ as in K41: the generation

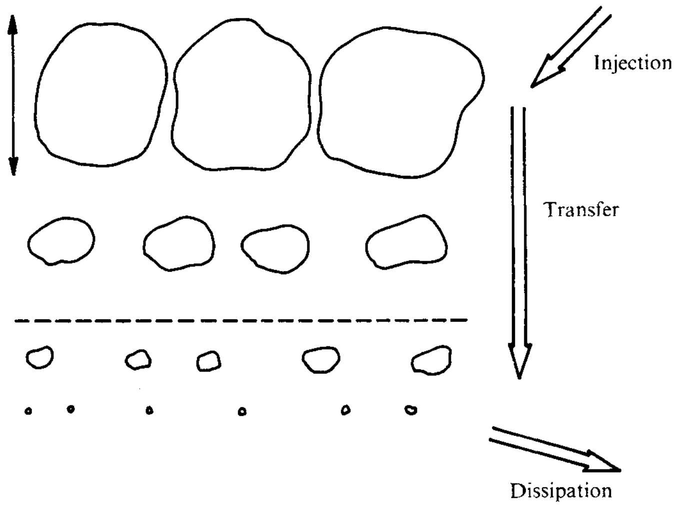

FIGURE 2. The energy cascade for intermittent turbulence: the eddies become less and less space filling. The reader is warned that this picture is very schematic: the eddies are in fact embedded within each other and the eventual product of the cascade, where dissipation takes place, should be thought of as a highly convoluted sheet.

of an $(n+1)$-eddy arises from the internal dynamics of the n-eddy in which it is embedded. We can express the rate of energy transfer from n-eddies to $(n+1)$-eddies exactly as in K41, and as in K41 this quantity must be independent of n in the inertial range:

$$\epsilon_n \sim E_n/t_n \sim \beta_n v_n^3/l_n \sim \bar{\epsilon}. \tag{3.4}$$

Combining (3.2)–(3.4), we obtain

$$v_n \sim \bar{\epsilon}^{\frac{1}{3}} l_n^{\frac{1}{3}} (l_n/l_0)^{-\frac{1}{3}(3-D)}, \tag{3.5}$$

$$t_n \sim \bar{\epsilon}^{-\frac{1}{3}} l_n^{\frac{2}{3}} (l_n/l_0)^{\frac{1}{3}(3-D)}, \tag{3.6}$$

$$E_n \sim \bar{\epsilon}^{\frac{2}{3}} l_n^{\frac{2}{3}} (l_n/l_0)^{\frac{1}{3}(3-D)} \tag{3.7}$$

and

$$E(k) \sim \bar{\epsilon}^{\frac{2}{3}} k^{-\frac{5}{3}} (kl_0)^{-\frac{1}{3}(3-D)}. \tag{3.8}$$

In (3.5)–(3.8) all the intermittency corrections have been expressed in terms of the *self-similarity dimension* D, which is a special case of Mandelbrot's (1975) *fractal dimension* and is related to the number of offspring N by

$$N = 2^D. \tag{3.9}$$

That D can suitably be called a dimension is made clear by figure 3, which shows three very familiar objects: a unit interval, a square and a cube, which have dimensions D equal to 1, 2 and 3, respectively. If we reduce the linear dimensions of these objects by a factor of 2, as in the cascade process, the number of offspring needed to reconstruct the original object is 2^D. For more complicated self-similar objects, a natural interpolation is $N = 2^D$, where D need no longer take only integer values: some rather exotic examples can be found in Mandelbrot (1975). Equation (3.8), which relates the correction to the $\frac{5}{3}$ exponent of the K41 theory and the fractal dimension, was first derived by Mandelbrot (1976) using the Novikov–Stewart (1964) model. In

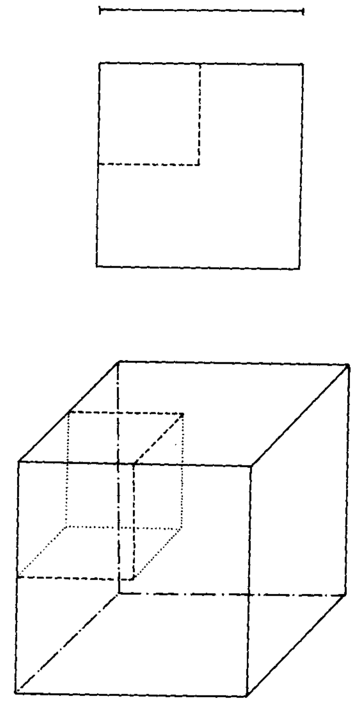

FIGURE 3. When the linear dimensions of a D-dimensional object are reduced by a factor λ (here 2), λ^D pieces are needed to reconstruct the original. More exotic examples with non-integral D, such as probably occur in turbulence, may be found in Mandelbrot (1975).

this context, D is also the Hausdorff dimension of the dissipative structures in the limit of zero viscosity. $D = 2$ would correspond to sheet-like structures, but in view of the experimental value of the exponent for the dissipation correlation function (see § 4) a more likely value is $D \approx 2{\cdot}5$, which corresponds to highly convoluted sheets.

Remark (3.1). The mean energy transfer (or dissipation) rate can be evaluated in terms of the scale l_0 of energy-carrying eddies and the r.m.s. turbulent velocity v_0 by putting $n = 0$ in the inertial-range expression (3.5). This gives the classical Kolmogorov result

$$\bar{\epsilon} \sim v_0^3/l_0, \tag{3.10}$$

which is not affected by the value of the intermittency parameter D. In contrast, the *dissipation scale* l_d is affected. Indeed, by equating the turnover time (3.6) to the viscous diffusion time l_n^2/ν, we obtain the dissipation scale

$$l_d \sim l_0 R^{-3/(1+D)}, \tag{3.11}$$

where we have introduced the Reynolds number

$$R = l_0 v_0/\nu \sim \bar{\epsilon}^{\frac{1}{3}} l_0^{\frac{4}{3}} \nu^{-1}. \tag{3.12}$$

Notice that l_d differs from the Kolmogorov microscale $(\bar{\epsilon}/\nu^3)^{-\frac{1}{4}}$ whenever $D \neq 3$.

More generally, we expect that the kind of intermittency considered in this paper will not influence processes depending essentially on the large-scale dynamics (e.g. transport processes). But there are questions, including some of practical interest, where the small-scale dynamics are relevant, such as chemical reactions in turbulent flows; it is then not safe to ignore intermittency (Herring 1973, private communication).

Remark (3.2). There is no need, *a priori*, to identify D with a (non-negative) dimension. Indeed, from (3.2) and (3.9) we have $D = \log N/\log 2$; since N is an average number of offspring, which can be less than unity, D can assume arbitrary negative values. There is however a dynamical reason to impose $D > -1$: otherwise it may be checked from (3.11) that the cascade will never be terminated by viscosity when the viscosity is too small. Furthermore, Sulem & Frisch (1975) have shown that an $\frac{8}{3}$ upper bound for the inertial-range spectral exponent can be rigorously derived from the Navier–Stokes equation (at least for turbulence of finite total energy). From (3.8) this imposes $D \geqslant 0$. Finally, Mandelbrot (1974) and Sulem & Frisch (1975) give heuristic arguments to show that $D > 2$. Notice also that, for $D < 2$, (3.5) implies that v_n *increases* with n (i.e. with decreasing scale); this appears most unlikely for ordinary fluid turbulence.

4. Higher-order statistics

The scaling laws and exponents of the K41 theory have strong experimental support at the level of the energy spectrum (see Monin & Yaglom 1975, chap. 23 for a review). There may be small corrections to the exponents, but much larger effects due to intermittency can be seen in higher-order statistics (see Monin & Yaglom 1975, chap. 25). We denote by $\delta v(l)$ any fluctuating component of the difference between the velocities at two points $\mathbf{r}$ and $\mathbf{r}'$ separated by a distance l. For homogeneous isotropic turbulence the statistics of $\delta v(l)$ can depend only on l (Van Atta & Park 1972). According to K41 the dimensionless structure functions

$$a_p(l) = \langle|\delta v(l)|^p\rangle / \langle|\delta v(l)|^2\rangle^{\frac{1}{2}p} \tag{4.1}$$

should not depend on l in the inertial range. Experimentally they are found to vary as negative powers of l. Using the β-model these results can be readily understood, at least in their qualitative features. We assume the following scale-similarity principle:

The complete statistics of velocity differences within active regions of different sizes become identical under an appropriate scaling of velocities.

Since such scaling factors are already fixed by the energy cascade (3.5), we can immediately determine the l dependence of the structure functions. Let us denote by $\langle|\delta v(l_n)|^p\rangle_{\text{cond}}$ the structure functions evaluated under the condition that the velocity difference is measured across an active n-eddy. Since such eddies fill only a fraction β_n of the space, we have

$$\begin{aligned}\langle|\delta v(l_n)|^p\rangle &\sim \beta_n \langle|\delta v(l_n)|^p\rangle_{\text{cond}} \\ &\sim \beta_n v_n^p,\end{aligned} \tag{4.2}$$

where the second line follows from the scale-similarity principle. From (3.2), (3.5) and (3.9) we obtain

$$\langle|\delta v(l_n)|^p\rangle \sim \bar{\epsilon}^{\frac{1}{3}p} l_n^{\frac{1}{3}p} (l_n/l_0)^{\zeta_p}, \tag{4.3}$$

where

$$\zeta_p = \tfrac{1}{3}(3-D)(3-p). \tag{4.4}$$

[A similar linear relationship between ζ_p and p was obtained by Novikov & Stewart (1964); see § 6.] The dimensionless structure functions are given by

$$a_p(l_n) \sim (l_n/l_0)^{\xi_p}, \tag{4.5}$$

with

$$\xi_p = \zeta_p - \tfrac{1}{2}p\zeta_2 = \tfrac{1}{2}(3-D)(2-p). \tag{4.6}$$

In contrast to the linear dependence on p in (4.4), the Kolmogorov (1962) lognormal theory predicts a quadratic dependence, namely

$$\zeta_p = \tfrac{1}{18}\mu p(3-p), \tag{4.7}$$

where μ is the exponent in the dissipation correlation function (see below).

The deviation of the small scales of the velocity field from global Gaussian behaviour can be quantitatively expressed in terms of the skewness and flatness

$$S \sim \langle \psi^3\rangle / \langle \psi^2\rangle^{\frac{3}{2}}, \quad F \sim \langle \psi^4\rangle / \langle \psi^2\rangle^2, \tag{4.8}$$

where ψ is some component of the velocity gradient. The contribution to the velocity gradient in scales $\sim l_n$ is $\sim \delta v(l_n)/l_n$, so that the largest velocity gradients are found in the smallest scales. Assuming that the viscosity imposes a sufficiently sharp cut-off, we can then, within a numerical factor, evaluate any moment of the velocity gradient using its inertial-range expression at the dissipation scale l_d. The skewness and flatness are given by the dimensionless structure functions $a_3(l_d)$ and $a_4(l_d)$. From (3.11), (3.12), (4.5) and (4.6) we obtain

$$S \sim R^{3(3-D)/2(1+D)}, \quad F \sim S^2. \tag{4.9), (4.10}$$

The relation to experimental results and to the scaling laws of critical phenomena will be discussed in § 7.

The fluctuating 'dissipation' at point $\mathbf{r}$ is defined as

$$\epsilon(\mathbf{r}) = \nu \psi^2(\mathbf{r}), \tag{4.11}$$

where ψ is again any component of the velocity gradient. For homogeneous isotropic turbulence the mean of $\epsilon(\mathbf{r})$ is, within a numerical factor, equal to the mean dissipation $\bar{\epsilon}$ per unit mass. Clearly $\epsilon(\mathbf{r})$ is a dissipation-range variable, in the sense that its primary contribution comes from the smallest scales $\sim l_d$. Experimentally, however, it is found that the dissipation has spatial correlations extending over inertial-range distances (see the end of this section). This strongly suggests that the dissipation takes place in structures which have one characteristic size $\sim l_d$ but are otherwise quite extended, such as rods, sheets or more complicated self-similar objects with non-integral dimensions. Let us evaluate the dissipation correlation function

$$\langle \epsilon(\mathbf{r})\, \epsilon(\mathbf{r}+\mathbf{l})\rangle$$

for the β-model, following essentially the original Novikov–Stewart (1964) argument (see also Monin & Yaglom 1975, § 25). Let $l \sim l_n \gg l_d$. The only way that $\epsilon(\mathbf{r})$ and $\epsilon(\mathbf{r}+\mathbf{l})$ can be correlated is if $\mathbf{r}$ and $\mathbf{r}+\mathbf{l}$ simultaneously belong to an active m-eddy of size $l_m \geqslant l_n$. We may therefore write

$$\langle \epsilon(\mathbf{r})\, \epsilon(\mathbf{r}+\mathbf{l})\rangle = \langle \epsilon\rangle^2_{\text{cond}} \Pr\{\mathbf{r} \text{ and } \mathbf{r}+\mathbf{l} \text{ belong to an } m\text{-eddy}\}, \tag{4.12}$$

where $\langle \epsilon\rangle_{\text{cond}}$ is the conditional mean of the dissipation $\epsilon(\mathbf{r})$ given that $\mathbf{r}$ is in an active m-eddy. Because of the energy cascade $\langle\epsilon\rangle_{\text{cond}}$ is also equal to the energy transfer per unit mass from active m-eddies to active $(m+1)$-eddies:

$$\langle\epsilon\rangle_{\text{cond}} \sim v_m^3/l_m. \tag{4.13}$$

Owing to the conditional averaging $\langle\epsilon\rangle_{\text{cond}}$ differs from $\langle\epsilon\rangle$ by a factor β_m. The probability of having both $\mathbf{r}$ and $\mathbf{r}+\mathbf{l}$ in an m-eddy of size greater than l is of the same order as the probability of having $\mathbf{r}$ in an m-eddy, namely β_m. We thus obtain

$$\langle\epsilon(\mathbf{r})\,\epsilon(\mathbf{r}+\mathbf{l})\rangle \sim \sum_{m=0}^{n} (v_m^3/l_m)^2\,\beta_m. \tag{4.14}$$

Using (3.4) we see that the largest value comes from $m = n$, so that our final result is

$$\langle\epsilon(\mathbf{r})\,\epsilon(\mathbf{r}+\mathbf{l})\rangle \sim \bar{\epsilon}^2(l/l_0)^{-(3-D)}. \tag{4.15}$$

The exponent of the dissipation correlation function is usually denoted by μ. For the β-model

$$\mu = 3-D. \tag{4.16}$$

Experimentally a power-law dependence seems to work quite well and defines an exponent $\mu \approx 0{\cdot}5$, giving $D \approx 2{\cdot}5$ (see Gibson *et al.* 1970; Monin & Yaglom 1975, chap. 25).

5. Extension to other spatial dimensions

So far we have considered only the three-dimensional case. For a space of dimension $d > 3$ all our arguments remain essentially unchanged, and the results are the same when expressed in terms of β. The self-similarity dimension D, still defined by (3.9), is now related to β by

$$\beta = 2^{D-d}, \tag{5.1}$$

so that wherever $\mu = 3-D$ appears it should be replaced by $\mu = d-D$. In order for the cascade to be terminated by viscosity, $D > d-4$ is required (cf. remarks 3.1 and 3.2). This can be violated for positive D if $d > 4$. The significance of this cross-over dimension of 4, introduced by Mandelbrot (1976), is not very clear.

The two-dimensional case deserves special attention. Because of vorticity conservation there is both an enstrophy (mean-square vorticity) cascade to small scales and an inverse energy cascade to large scales (Kraichnan 1967; Batchelor 1969; Pouquet *et al.* 1975). For the enstrophy cascade we refer the reader to Kraichnan (1967, 1971), where it is shown that the energy spectrum follows a k^{-3} law with logarithmic corrections. Intermittency is probably present in the enstrophy cascade but is not expected to change the power law from k^{-3} (Kraichnan 1975).† The presence of highly non-local interactions makes the ideas of the β-model of doubtful interest in this case. The inverse cascade, however, is local and intermittency corrections to the $\frac{5}{3}$ law can not be ruled out. The closest thing to the β-model would be a cascade of the kind shown in figure 4 which becomes less and less space filling with *increasing* scale size. Let us assume that after n octave-steps the fraction of space filled with active n-eddies is

$$\beta_n = 2^{-n(2-D)} = (l_n/l_0)^{D-2}. \tag{5.2}$$

† One can actually prove rigorously that a two-dimensional enstrophy inertial range cannot have a spectrum steeper than $k^{-\frac{11}{3}}$ (Sulem & Frisch 1975, corrected in Pouquet 1978).

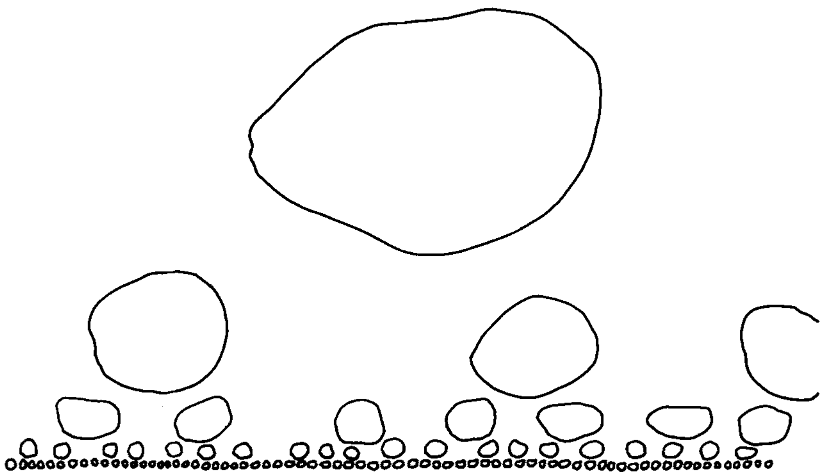

FIGURE 4. An intermittent inverse cascade.

Repeating the calculation in § 3, we obtain

$$E(k) \sim \bar{\epsilon}^{\frac{2}{3}} k^{-\frac{5}{3}} (kl_0)^{\frac{1}{3}(2-D)}. \tag{5.3}$$

So intermittency corrections to the two-dimensional inverse cascade will, if they exist, decrease the $\frac{5}{3}$ exponent. This has already been noticed by Kraichnan (1975).

Non-integer values of the space dimension d (not to be confused with the self-similarity dimension) have been considered by Nelkin (1975) and by Frisch *et al.* (1976). Within a second-order closure theory, Frisch *et al.* find a cross-over dimension $d_c \approx 2{\cdot}03$ where the direction of the energy cascade reverses. Intermittency corrections to the inertial-range exponent for the energy spectrum may thus change sign on crossing d_c. Whether there is any dimension or range of dimensions for which intermittency corrections vanish remains an open question.

Finally, we consider the one-dimensional case. For this case the incompressible Navier–Stokes equation is meaningless (the velocity would be uniform) so instead we take the Burgers equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \nu \frac{\partial^2 v}{\partial x^2}. \tag{5.4}$$

There are no great mysteries left in connexion with the Burgers equation: in the limit of zero viscosity the solution has, for long times, a sawtooth structure. The interesting aspect is that in a sense the Burgers case is the most intermittent. The dissipative structures are just isolated shocks which form a set of Hausdorff dimension $D = 0$. If we calculate the energy spectrum from the β-model with

$$\beta = 2^{D-d} = 2^{-1}, \tag{5.5}$$

we obtain

$$E(k) \propto k^{-2}, \tag{5.6}$$

which is indeed the correct spectrum for the random sawtooth solution of the Burgers equation.

6. Relation to previous hierarchical intermittency models

Doubts about the universal validity of the K41 theory started with a remark by Landau that there could be strong fluctuations in the dissipation on scales $\sim l_0$ (see Landau & Lifshitz 1959; Kraichnan 1974, p. 309). As pointed out by Kraichnan (1974), this remark did not deal with the essential feature that fluctuations amplify as the scale size decreases. This was first included by Oboukhov (1962), who introduced the random variable ϵ_l, which is the dissipation $\frac{1}{2}\nu(v_{i,j}+v_{j,i})^2$ averaged over a region of size l. Oboukhov proposed that the K41 theory still holds in the modified form

$$\delta v(l) \sim \epsilon_l^{\frac{1}{3}} l^{\frac{1}{3}}. \tag{6.1}$$

The left-hand side of (6.1) is a kind of conditional average of the velocity difference across a region of size l conditional upon ϵ_l having a given value. Kolmogorov (1962), assuming a lognormal distribution of ϵ_l, was able to calculate the structure functions

$$\langle|\delta v(l)|^p\rangle. \tag{6.2}$$

He obtained the result given in (4.3) and (4.7).

The basic idea expressed by (6.1) seems likely to be correct. It states that in a volume of size $\sim l$ velocity differences $\sim \delta v(l)$ will produce a transfer of energy to smaller scales of the order of

$$\epsilon_l \sim \delta v^3(l)/l.$$

In a time of the order of the eddy turnover time $l/\delta v(l)$ this energy will be dissipated somewhere in the same region. When we say 'the same region' we imply 'relative to a given (moving) eddy'. As for Kolmogorov's lognormality assumption, this has been seriously questioned by Kraichnan (1974) and Mandelbrot (1974) and shown to be only one of many possibilities.

Novikov & Stewart (1964) have put forward a probabilistic model of intermittency which is closely related to our β-model. In fact it gives the same results; in particular it replaces (4.7) with (4.4). Instead of working with dynamical variables, however, Novikov & Stewart constructed a model of the dissipation involving nested cubes. In cubes $\sim l_0$ the dissipation is taken as uniform. In each succeeding generation only a certain fraction of the available cubes are taken to contain dissipation, the remainder being taken to be empty. A generalization of the Novikov–Stewart model was introduced by Yaglom (1966) and later studied in detail by Mandelbrot (1974, 1976), who called it 'weighted curdling' as opposed to 'absolute curdling' in the Novikov–Stewart model. In absolute curdling the dissipation in an n-eddy is concentrated in only a fraction of the $(n+1)$-eddies whereas in weighted curdling the $(n+1)$-eddies completely fill the available space but each $(n+1)$-eddy has its dissipation weighted by a random factor W of unit mean value. Absolute curdling is recovered when W has a Bernoulli distribution. A kind of weighted curdling in a more dynamical context has also been considered by Kraichnan (1974). We could easily modify the β-model to allow for weighted curdling, but this would essentially reproduce Mandelbrot's (1976) results. Let us mention only that weighted curdling leads to a correction B to the $\frac{5}{3}$ law which is less than $\frac{1}{3}\mu$. In particular, lognormal curdling, which, we stress again, has no special merits, leads to $B = \frac{1}{9}\mu$.

The advantage of the β-model over the Novikov–Stewart model is that it deals with dynamically relevant variables such as nonlinear energy transfer instead of the dynamically irrelevant local dissipation. This allows us to consider more subtle questions such as the possible intermittency of the inverse energy cascade in two dimensions. It also allows us to make quantitative Kraichnan's (1972) suggestion that the energy spectrum will be modified by intermittency owing to changes in the eddy turnover time $t_n \sim l_n/v_n$. Most important, it allows experimental tests of the model by giving definite predictions for higher-order statistical quantities such as the skewness, flatness and velocity structure functions.

An interesting open question is whether on dynamical grounds one should favour weighted rather than absolute curdling. Mandelbrot (1976) notes that for the Burgers equation only absolute curdling gives precisely the correction needed to change the $k^{-\frac{5}{3}}$ spectrum to a k^{-2} spectrum (see § 5). This, however, is hardly conclusive since the Navier–Stokes equation, contrary to the Burgers equation, is not stable against small perturbations, and is therefore likely to generate much more randomness. For the Burgers equation absolute curdling is just a succession of nested intervals converging to positionally well-defined shocks. For the Navier–Stokes equation in the limit of zero viscosity, the probable singularities are likely to appear downstream of regions of high initial vorticity, but their precise location or structure seems to be impossible to predict.

Finally, it must be mentioned that several authors have proposed non-hierarchical models of intermittency consisting of shear layers whose thickness is the Kolmogorov microscale but which are coherent over much larger distances (Corrsin 1962; Tennekes 1968; Saffman 1968). Kraichnan (1974, § 3) explains why such models are unlikely candidates.

7. Relations among measurable exponents

A characteristic feature of fully developed turbulence is the existence of scaling laws. Second- and higher-order statistical quantities are found experimentally to exhibit power-law behaviour with presumably universal exponents. A deductive theory of turbulence based on the Navier–Stokes equation should be able to derive all the exponents. In the absence of an appropriate theoretical framework such a programme is very far from completion. A less ambitious and perhaps quite useful programme can be considered at a more phenomenological level. Perhaps there are simple relations among the exponents as, for example, in the β-model, where everything is expressible in terms of the self-similarity dimension D, or equivalently $\mu = 3-D$. Such relations among exponents can be studied experimentally and can give considerable insight into the necessary structure of an eventually successful theory.

It is interesting to recall that such relations are found in the now reasonably deductive theory of equilibrium critical phenomena (Wilson & Kogut 1974; Ma 1976), a subject which has some formal relation to fully developed turbulence (Nelkin 1973, 1974, 1975). In fact a phenomenological scaling theory of critical phenomena preceded a dynamical theory, and was a very important guide to the eventual structure of the dynamical theory. The analogy with critical phenomena in its strongest form (Nelkin 1975) gives the same relations among exponents as does the β-model, in particular see (3.8), (3.11), (4.10), (4.11) and (4.16). The critical-phenomena analogy does not, in its present stage of development, allow the calculation of the higher-order structure functions, but it would be of interest to extend it in this direction.

In the β-model there is only one independent exponent, but in critical phenomena there are at least two. However, the presence of an energy cascade imposes a supplementary constraint without a parallel in critical phenomena, namely that for $\nu \rightarrow 0$ the mean dissipation has a finite limit. This means that the 'critical' exponent γ governing the divergence of the enstrophy as $\nu \rightarrow 0$ is constrained to have the value $\gamma = 1$. If it were not for intermittency corrections this constraint would trivially determine all the scaling laws, namely by the K41 theory.

This point is seen clearly in dynamical models where this constraint is relaxed. Recently Bell & Nelkin (1977) have considered a cascade model in which a free parameter allows the energy cascade to be in either direction. When the cascade is towards large k, the K41 theory is recovered. When the cascade is towards small k, the large k behaviour still obeys scaling laws analogous to those for critical phenomena, but no energy cascades to large k in the limit of zero viscosity. Related results have been obtained by Frisch *et al.* (1976). They considered a second-order closure model analytically continued between two and three dimensions. For $2 < d < d_c \approx 2{\cdot}03$ the energy cascade is towards small wavenumbers, but the large wavenumber behaviour is still a power-law behaviour even though nothing is cascading. This type of behaviour, which seems intuitively surprising in turbulence theory, seems natural in the context of critical phenomena.

Neither the β-model of intermittency nor the lognormal model should be taken too seriously. It is, however, worth mentioning that experiments can probably determine which to choose. With the currently accepted value of μ, the correction to the $\frac{5}{3}$ law is $\frac{1}{3}\mu \approx 0{\cdot}17$ in the β-model and *three times smaller* in the lognormal model. A correction as large as $0{\cdot}17$ should be experimentally measurable. Another respect in which the two models differ considerably is for the higher-order structure functions of (4.3). The β-model exponents are given by (4.4) and the lognormal-model exponents by (4.7). These exponents agree only for $p = 3$ and for $p = 6$. For $p = 3$ this is a simple consequence of the von Kármán–Howarth equation, as already noted. The case of the sixth-order structure function is more interesting and leads us to the following.

Conjecture. The sixth-order structure function $\langle \delta v^6(l) \rangle$ is related to the dissipation correlation function by

$$\langle \delta v^6(l) \rangle / l^2 \sim \langle \epsilon(\mathbf{r})\, \epsilon(\mathbf{r} + \mathbf{l}) \rangle. \tag{7.1}$$

This relation is easily checked for both the β-model and the lognormal model. To see that it is probably model independent, notice that in view of homogeneity we can write (Novikov 1971)

$$\langle \epsilon(\mathbf{r})\, \epsilon(\mathbf{r} + \mathbf{l}) \rangle = \tfrac{1}{2} d^2 (l^2 \langle \epsilon_l^2 \rangle)/dl^2 \sim \langle \epsilon_l^2 \rangle, \tag{7.2}$$

where ϵ_l is the (random) dissipation averaged over a volume of size l. Therefore (7.1) can be rewritten as

$$\left\langle \frac{\delta v^3(l)}{l} \frac{\delta v^3(l)}{l} \right\rangle \sim \langle \epsilon_l^2 \rangle. \tag{7.3}$$

Now $\delta v^3(l)/l$ is the (random) energy transfer (per unit mass) from scales $\sim l$ to smaller scales, so that (7.3) just states that the volume-averaged transfer and dissipation have identical variances. Energy transfer in a volume of size l need not equal the energy dissipation ϵ_l in the same volume (except in the mean). As noticed by Kraichnan (1974, p. 309), it is even in principle possible that transfer fluctuates and dissipation does not. But this situation seems to be ruled out if the inertial range is intermittent: in that case there are active regions with enhanced transfer which will give rise to enhanced dissipation downstream after at most the local eddy turnover time. It is therefore reasonable to assume that transfer and dissipation have the same statistics (within numerical factors). Careful measurement of the sixth-order structure function would give a useful check on these ideas relating energy transfer and energy dissipation.

8. Concluding remarks

We have presented a simple dynamical cascade model which allows a variety of measurable scaling exponents to be expressed in terms of a single parameter, the self-similarity dimension D of Mandelbrot. The β-model gives the results obtained earlier in more abstract, non-dynamical contexts in explicit, easily understandable terms. The conclusions of the model are stated in terms that can be tested by experiment and discriminated from the lognormal model. Although the β-model is much too simple to be literally true, the possibility that the relations (4.3)–(4.5) among measurable exponents suggested by this geometrical model are in fact more generally valid (Nelkin & Bell 1978) can not be excluded.

To go further, a genuine dynamical theory starting from the Navier–Stokes equations is needed. It is unlikely that the inertial-range scaling laws can be obtained correctly using only the obvious symmetries and conservation laws of the Navier–Stokes equation, such as energy conservation (Kraichnan 1974). So far no analytical tool has been found which allows these scaling laws to be determined theoretically although there have been many speculations that the renormalization-group theory developed for critical phenomena (Wilson & Kogut 1974; Ma 1976) could be appropriate (Martin 1972, private communication; Nelkin 1973; see also Rose & Sulem 1978). It is possible that renormalization-group ideas, which have allowed the calculation of scaling exponents for critical phenomena, will eventually also succeed for intermittent-scale similar turbulence. The presently existing applications of the renormalization group to turbulence are not concerned with inertial ranges (Forster, Stephen & Nelson 1976, 1977; Rose 1977). In the meantime, phenomenological models can give important hints as to the necessary structure of an eventual dynamical theory.

Finally, we should like to point out that, besides phenomenological and theoretical work, much more understanding of intermittency can be gained from further numerical and experimental work, particularly on questions of geometry. Numerical simulations (Orszag & Patterson 1972) cannot achieve sufficiently high Reynolds numbers to display an inertial range. But spottiness of the high vorticity regions should be noticeable even at moderate Reynolds numbers. This requires, however, a method of handling the data quite different from the usual procedure: instead of taking averages to get information on spectra, transfer, skewness, etc., one should try to plot directly the components, or at least the amplitude, of the vorticity. The number of data points to be graphically represented can be greatly reduced if conditional sampling of the high vorticity regions is made. On the experimental side one should also try to get more data on the small scales. Practically all existing methods involve sampling along a line, so that we have only very indirect information about the geometry of the dissipative structures. What is really needed is a method by which these small scales can be directly visualized, say by a light-scattering experiment. The difficulty is that these regions have very small velocities, so that standard velocimetric procedures have a poor signal-to-noise ratio. But since these regions have high vorticities, we suggest the development of 'strophometric' methods, i.e. measurements of the vorticity or more generally of the velocity gradient. The technique introduced by W. W. Webb and D. H. Johnson, which involves light scattering from highly anisotropic Tobacco Mosaic viruses sensitive to velocity gradients, may be an interesting step in this direction (Johnson 1975).

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Appendix. Derivation of the Mandelbrot inequality

We show that the Mandelbrot inequality (1.1), which becomes an equality for the β-model, can be derived from the Navier–Stokes equation under one strong but physically plausible assumption. We assume that the viscous cut-offs k_d governing the spectra of velocity and dissipation fluctuations are the same within a numerical factor which does not depend on the Reynolds number. This is sufficient to derive (1.1).

Let us introduce some notation. For convenience the large scale l_0 and the mean rate of energy dissipation $\bar{\epsilon}$ are taken equal to one. The energy spectrum is

$$E(k) \sim \begin{cases} k^{-(\frac{5}{3}+B)} & \text{for} \quad k < k_d, \\ 0 & \text{for} \quad k > k_d. \end{cases} \tag{A 1}$$

The dissipation spectrum, which is the Fourier transform of the dissipation correlation function, is

$$E_\epsilon(k) \sim \begin{cases} k^{-1+\mu} & \text{for} \quad k < k_d, \\ 0 & \text{for} \quad k > k_d. \end{cases} \tag{A 2}$$

Since the mean dissipation has a finite limit for $\nu \to 0$, we can determine k_d from

$$\nu \int_0^\infty k^2 E(k)\, dk \sim \nu k_d^{(\frac{4}{3}-B)} \sim \bar{\epsilon} = 1. \tag{A 3}$$

By evaluating the dissipation correlation at zero separation we obtain (where ψ is any component of the velocity gradient)

$$\nu^2 \langle \psi^4 \rangle \sim \int_0^\infty E_\epsilon(k)\, dk \sim k_d^\mu. \tag{A 4}$$

The next step requires a dynamic argument concerning the enstrophy balance from the Navier–Stokes equation. Consider the vorticity equation

$$\partial \boldsymbol{\omega}/\partial t + (\mathbf{v} \cdot \boldsymbol{\nabla})\, \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\, \mathbf{v} + \nu \nabla^2 \boldsymbol{\omega}, \tag{A 5}$$

take the scalar product of this with $\boldsymbol{\omega}$ and average using stationarity, homogeneity and isotropy. In a stationary state the rate of enstrophy production by the nonlinear terms will balance the rate of enstrophy dissipation. There will be an additional term due to the random forcing necessary to maintain the stationary state, but this term is negligible at very high Reynolds number (see Orszag 1977). The enstrophy balance can be expressed in the form

$$\langle \psi^3 \rangle \sim \nu \int_0^\infty k^4 E(k)\,dk \sim \nu k_d^{(\frac{10}{3}-B)}. \tag{A 6}$$

Equation (A 6) can be used directly to express the rate of divergence of the skewness with increasing Reynolds number in terms of the correction B to the $\frac{5}{3}$ law (Nelkin 1975). Since this relates two quantities which are difficult to measure with any accuracy, we take a slightly different point of view here. We use the Schwarz inequality

$$\langle \psi^3 \rangle^2 \leqslant \langle \psi^2 \rangle \langle \psi^4 \rangle \tag{A 7}$$

and rewrite (A 3) in the form

$$\langle \psi^2 \rangle \sim \int_0^\infty k^2 E(k)\,dk \sim k_d^{(\frac{4}{3}-B)} \sim \nu^{-1}. \tag{A 8}$$

Combining (A 4), (A 5), (A 7) and (A 8), we finally obtain

$$k_d^{\frac{3}{2}B} \lesssim k_d^{\frac{1}{2}\mu}. \tag{A 9}$$

Since (A 9) must hold for all large values of k_d, the Mandelbrot inequality (1.1) follows. In order for this inequality to become an equality the Schwarz inequality (A 7) must become an equality to within a numerical factor. This is equivalent to the assumption that the skewness is proportional to the square root of the flatness for large Reynolds number. The reader should be warned that the existence of a single cut-off, however reasonable it may appear, need not be correct. Indeed, in certain probabilistic models of intermittency the viscous cut-off fluctuates greatly from realization to realization (Mandelbrot 1976).

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