MHD Turbulence II

- Anisotropic Cascades and Critical Balance - A closer look.
- Extending the 4/5 Law.
- Selective Decay and Relaxation.
- 2D MHD - A Study in Turbulent Relaxation.

(1) Anisotropic Cascades and Critical Balance - A closer look.

Recall: I-k Phenomenology:

\[ E \equiv \frac{Z(l,k)^2}{f_{in}(k)} \]  
\[ \frac{1}{f_{in}(k)} \equiv \frac{Z(l,k)^2}{f^2} \]

\[ \tau_a \sim \frac{l}{V_0} \]

\[ E(k) \sim \text{Re}_k \sim k^{-3/2} \]

and with strong \( B_0 \):

\[ E \sim \frac{Z(l_1)^2 Z(l_2)}{f_{in}^2 \log V_1} \]
so W.T.T. Alfvénic cascade:

$$ E(v_{\perp}, k_{\parallel}) \sim (E_{V_{A}})^{1/2} / k_{\parallel}^{3/2} k_{\perp}^{1/2} \sim \frac{\text{"hard"}}{\omega k_{\perp}} $$

However, note:

$$ \rho_{\perp}(l_{\parallel}) \sim \rho_{V_{A}}(l_{\parallel}) \sim (E_{\text{kin Val}})^{1/4} l_{\parallel}^{3/2} $$

$$ \Rightarrow $$

$$ \frac{d\rho_{\perp}}{dl_{\parallel}} \sim \frac{1}{E_{\text{kin Val}}^{1/4}} $$

But recall:

- Alfvén wave:

$$ \omega \sim k_{\parallel} V_{A} $$

derived from:

$$ \partial_{t} A = B_{0} \nabla_{\perp} \phi + \cdots $$

$$ \partial_{t} \nabla_{\perp} \phi = B_{0} \nabla_{\parallel} V_{\parallel} + \cdots $$

$$ \partial_{\parallel} = \partial_{z} + \frac{dB_{\perp}}{B_{0}} \cdot \nabla_{\perp} $$

linear \quad nonlinear
Ratio \[ \frac{\text{Nonlinear}}{\text{Linear}} = \frac{ku}{d} = \frac{\frac{\partial B}{\partial z}}{\frac{B_0}{A_1}} \]

\[ ku \sim \frac{\partial B}{\partial z} \]

\[ ku \to \text{parallel auto correlation length} \]

See stochastic fields discussion of Phys. 235 2016

Point: \[ R \sigma_2 \cdot C + dB_z \cdot \overline{D_1 \cdot C} = 0 \]

\[ \overline{D_2 \cdot C + dB_z \cdot D_1 \cdot C} = 0 \]

\[ ku < I \to C \text{ evolves by many kicks in } A_1 \to \text{diffusion} \]

\[ \to \text{ in WIT wave interactions are diffusive in character.} \]

\[ ku > I \to C \text{ scattered } \Delta_1 \text{ in one step} \]

\[ \to \text{ for transport in random media } \to \text{ periodicity} \]
\[ \nabla \cdot \mathbf{C} + \mathbf{U} \cdot \nabla \mathbf{C} = 0 \]

\[ \nu = \frac{\nu_{\text{rms}}}{\Delta} \]

So we have a concern:

- **Physics of MHD turbulence** understood in terms of Alfvén wave interactions.

- But scaling of WFT spectrum suggest that wave character last as cascade progresses

\[ \nu \propto \frac{\nu_{\text{rms}} [C \nu_v V]}{\nu_{\text{rms}}^2} \]

\[ \nu \propto \Delta \quad \text{as} \quad \Delta \rightarrow 0 \]

i.e. How high can \( \nu \) go and still be consistent with physics of Alfvén wave cascade

\[ \Rightarrow \text{Critical Balance Conjecture} \]

(GS 95, HPP '78)
MHD inertial range in strong field will set at $k \sim U_1$.

c.e. $\Rightarrow \vec{d} B_1 \cdot D_1 \sim \frac{Z(e_d)}{l_1} B_0 D_0$.

$\Rightarrow \frac{Z(e_d)}{l_1} \sim k U_0 a$

$\Rightarrow \frac{T_{an}}{T_{Tr}} \Rightarrow \frac{T_{Tr}}{T_{Teddy}} \Rightarrow \frac{T_{Teddy}}{T_{an}}$.

i.e. all timescales equalize.

$\Rightarrow k U_1 \sim$ maximum $k U_0$ and still retain Alfvénic character.

Recall:

- $\omega T_{an} \sim \omega_{ac} \text{ wave}$

\begin{align*}
&\Rightarrow \frac{\omega}{\Delta \omega_{ac}} \left( \frac{1}{\omega} - \frac{1}{\omega_{ac}} \right) \\
&\Rightarrow \frac{\omega}{\Delta \omega_{ac}} \\
&\Rightarrow \frac{1}{\Delta \omega_{ac}} + \frac{1}{\Delta \omega_{ac}} + \frac{1}{\Delta \omega_{ac}} \\
&\Rightarrow \delta \left( \frac{1}{\omega} - \frac{1}{\omega_{ac}} \right)
\end{align*}

- SST - Renormalized Theory

\begin{align*}
T_{an} \sim T_{bn} \\
\Rightarrow \frac{1}{\Delta \omega_{ac}} + \frac{1}{\Delta \omega_{ac}} + \frac{1}{\Delta \omega_{ac}} \\
&\Rightarrow \delta \left( \frac{1}{\omega} - \frac{1}{\omega_{ac}} \right)
\end{align*}
So, renormalized wave interaction
theory.

$$\Theta_{k, k', k''} = \frac{\Delta \omega_{k'} + \Delta \omega_{k''} + \Delta \omega_{k''}}{\left(\omega_{k'} - \omega_{k''} - \omega_{k''} + \frac{\Delta \omega_{k'} + \Delta \omega_{k''} + \Delta \omega_{k''}}{2}\right)^2}$$

recover both limits.

Now, $$\Theta_{k, k', k''}$$ clearly sets for.

So, can re-write phenomenologically transfer balance as:

$$\epsilon \sim \frac{1}{\sqrt{p_1^2 \mathcal{T}(l_1)}} \frac{Z(l_1)^2 Z(l_1^2)}{Z(l_1)^4}$$

$$\frac{1}{T_{c, c} l_1} = \frac{1}{\sqrt{\left[\left(\frac{v_n}{\epsilon_2}\right)^2 + \left(\frac{Z(l_1)}{\epsilon_2}\right)^2\right]}}$$

comparable at large $\lambda$.

by analogy with $\Theta_{k, k', k''}$.\
\( \frac{1}{l_1} > \frac{1}{l_2} \Rightarrow \text{ W.T.T.} \)

\( \frac{1}{l_1} < \frac{1}{l_2} \Rightarrow \text{ S.T.T.} \)

\( e \sim \frac{1}{l_1} \left( \frac{Z(l_1)^2 Z(l_2)}{Z(l_2)/l_1} \right) \)

\( = \frac{Z(l_1)^3}{l_1} \)

and \( \left( \frac{Z(l_1)}{l_1} \right) \sim \left( \frac{\varepsilon l_1}{l_2} \right)^{1/3} \) — Back to \( l_1 = 1 \) !

Point: \( <Z(x)\rangle \sim e^{\frac{2\pi}{h} k_x \cdot \frac{l}{l_2}} \) — GS spectrum

but different physics! — Softer than W.T.T.

\( \frac{Z(l_1)}{l_1} \) vs. \( \text{kin Va} \)

\( \frac{(\varepsilon l_1)^{1/3}}{l_1} \sim \frac{\varepsilon^{1/3}}{l_2^{2/3}} \) — Rate increases as \( l_2 \) \( \downarrow \)

\( \text{contrasts constant \( V_{\text{inVA1}} \)} \)

\( \frac{Z(l_2)}{l_1} \sim \frac{\varepsilon^{1/3}}{l_3^{3/3}} \) \( \Rightarrow \) \( \frac{\varphi B \cdot D_1}{B_2} \)
then \( k_{\|} \sim t \Rightarrow \)
\[
\frac{\varepsilon^{1/3}}{t^{2/3}} \sim k_{\|}
\]

\[
\Rightarrow k_{\|} \sim \varepsilon^{1/3} \tilde{k}_{\perp}^{2/3}
\]

- Critical balance is a hypothesis

- Plausible answer to question of "how maintain Alfvénic cascade in state of strong (i.e. non-weak) turbulence?"

- Anisotropy of spectrum supported by simulations (cf. Galtier).

But

- hypothesis, only

*Computational support semi-quantitative.

- \( 5/3 \) vs \( 3/2 \) etc. still ongoing.
A word about trials.

In wave turbulence cascade, must satisfy:

\[ h = A + Z \]

\[ \omega_n = \omega_0 \pm \omega_Z \quad \text{(WTT)} \]

Conditions satisfied by:

\[ Q = 0 \]

(i.e. \( Z \) is a cell, driven by beats)

\[ k_n = p_{\perp} \]

\[ k_{n+1} = p_{\perp} + Z \]

and \( \omega_{n+1} = \omega_0 \pm \Theta \)

- deformation of Alfvénic wave packet directly related to its interaction with 2D part of wave packet travelling in opposite direction.

- interaction persists w.r.t. by transfer of energy time limit.
(ii) 4/5 Law - See Lecture I

(iii) Cascade and Relaxation

⇒ Selection Decay

Recall: \( \{ \text{Taylor Relaxation} \} \)

\( C^0 \)

(2D - "Taylor of Flat End"

Argument: \( \oint \vec{B} \times \vec{B} \) minimized

subject to constraint of

\( \oint \vec{B} \times \vec{A} - \vec{B} \) conserved,

\[ J_n = J \cdot B / B^2 \Rightarrow \text{const.} \]

(2D \( J/A \Rightarrow \text{const.} \)).

Arguments heuristic:

- Power counting (1.5)
- Track Fields

Now - Dissipation at small scale

\( M, Y \)

- Expect energy transfer to small scale.
Inverse cascade of magnetic helicity would set up "selective decay" scenario

- Magnetic energy scattered to small scales and dissipated, relaxation

- Magnetic helicity (inverse cascade) avoids dissipation. Constraint as survives.

C.f. Fritch (75), Pouquet et al. (76)
(added)
See also: Montgomery

Why, where from?

- Primarily: Statistical Mechanics
- C.f.: Fritch '75, though not transparent.

Easier -> "Taylor in Flatland"
Recall: Relaxation
\[
\{ \text{Minimizes } \left< B^2 \right> \}
\]
\[
\text{Conserving } \left< A^2 \right>.
\]

Does this follow from Selective Decay?

\Rightarrow \text{ Explore Absolute Equilibrium}

\[
\begin{array}{c}
\text{finite box} \\
\text{km in} \quad \text{km out}
\end{array}
\]

\text{- remove forcing, dissipation etc}
\text{- input exciting}

For 2D MHD (ignoring cross helicity):

\[
A \Rightarrow X_c
\]
\[
\Rightarrow \text{Model amplitude}
\]

\[
E_m = \sum_{i=1}^{N} k_i X_i^2
\]
\[
H = \sum_{i=1}^{N} X_i^2 - \left< A^2 \right>
\]
\[ \phi \rightarrow Y_i' \]

\[ E_k = \sum_{i=0}^{n} k_i \cdot Y_i'^2 \]

Now, \[ H \rightarrow \lambda \]

\[ E = E_{\text{ext}} + E_{\text{int}} \rightarrow \lambda > \text{constant} \]

are conserved. We can write this closed system is given by the micro-canonical ensemble/distribution:

\[ P(X, Y) = C \exp \left[ -\sum_{i=1}^{N} \left( x_i + \beta k_i^2 \right) X_i^2 + \Theta k_i^2 \right] \]

and can integrate out \( Y_i \) (KE) part so:

\[ P(X) = C \exp \left[ -\sum_{i=1}^{N} \left( x_i + \beta k_i^2 \right) X_i^2 \right] \]

then:

\[ \langle A^2(k) \rangle = \int dX_i \cdot X_i^2 \cdot P(X,e) \]

\[ = \frac{1}{\left( x + \Theta k_i^2 \right)} \]

\[ \langle B^2(k) \rangle = \left[ k_i^2 / \left( x + \Theta k_i^2 \right) \right] \]
\[ A^2 \]

\[ \text{observe immediately:} \]

\[ A^2 \quad \text{wants remain at large scale} \]

\[ B^2 \]

\[ \text{"B}^2\text{ approaches equipartition"} \]

\[ k \rightarrow 0 \]

\[ k \rightarrow \infty \]

\[ \Rightarrow \text{A}^2 \text{ distribution most populated at large scales, decays at small scales.} \]

\[ \Rightarrow \text{B}^2 \text{ distribution most populated at smaller, approaches equipartition at small scale.} \]

\[ \Rightarrow \text{suggests A}^2 \text{ populates large scales, B}^2 \text{ approaches equipartition,} \]

\[ \Rightarrow \text{suggestive of inverse cascade of A}^2 \text{ along with forward cascade of energy.} \]
supports defective decay hypothesis
as foundation for "taylor's flatland".

similar story for magnetic
helicity, though more laborious.

N.B. For 2D Fluid:

\[ E = \int d^2x \left( \partial \phi \right)^2 \quad \text{-- energy} \]

\[ \mathcal{L} = \int d^3y \left( \partial \phi \right)^2 \quad \text{-- enstrophy} \]

\[ \mathcal{L}_i = \kappa_i \hat{x}_i \]

\[ \mathbf{v} = \tilde{\mathbf{x}} \hat{c} \]

\[ P(x) = c \exp \left[ -\sum_{i=0}^{\nu} \left( x + 2 \lambda_i k_i^2 \right) x_i^2 \right] \]

So \[ \mathcal{E}(k) = \left\langle \mathcal{L}^2(k) \right\rangle = \frac{\nu}{\left( k + 2 \lambda k^2 \right)} \]

\[ \mathcal{L}(k) = \kappa^2 \left( k + 2 \lambda k^2 \right) \]

similar suggestion of dual cascade
and minimum enstrophy state.
→ Is this story true?

→ What does dynamics tell us?

Consider interactions in 2D MHD.

Observed:

- Reduced MHD

\[
\frac{\partial \Psi}{\partial t} + \frac{1}{\mu} \varphi \times \frac{1}{\mu} \frac{\partial \Psi}{\partial t} = B_0 \frac{\partial \varphi}{\partial z} + \eta \nabla^2 \Psi
\]

- 2D MHD

\[
\frac{\partial \Psi}{\partial t} + \frac{1}{\mu} \varphi \times \frac{1}{\mu} \frac{\partial \Psi}{\partial t} = n D_\perp^2 \Psi
\]

so with strong \( B_0 \):

\[ \langle A \cdot B \rangle \rightarrow \langle \Psi \rangle B_0 \]

so mean \( \langle \Psi \rangle \) in 2D captures magnetic helicity dynamics in strongly magnetized system.

For \( \langle A^3 \rangle \) transfer, consider closure of \( \Delta t \langle A^3 \rangle \) equation, much akin to wave kinematics, though closure required.

See: Diamond, Hughes, Kim (posted).
Can write (see DHK):

\[ 2 \frac{1}{\hbar} \left[ \frac{\partial}{\partial \mathbf{k}} \langle A^2 \rangle + T(k) \right] = - \mathcal{F}_{\mathbf{k}} \frac{\partial A(k)}{\partial \mathbf{k}} - \mu \langle B^2 \rangle_\mathbf{B} \]

\[ \langle \mathbf{D} \cdot \langle \mathbf{V} A^2 \rangle \mathbf{A} \rangle \]

Flux

\[ \mathcal{F} = \int_0^b \mathcal{F}_n = \int_0^b \langle \mathbf{P}(\mathbf{u}) \mathbf{u}^2 \mathbf{A} \rangle \]

\[ - \frac{\langle \mathbf{k}^2 \rangle}{k + |\mathbf{k}|} \langle A^2 \rangle \mathbf{A} \mathbf{A} \]

\[ \sum_l \langle A^2 \rangle \mathbf{A} \mathbf{A} \]

\[ \sum \langle D \cdot E \rangle \mathbf{E} \] - \langle \Phi^2 \rangle_\mathbf{B}

1. coherent damping, incoherent emission
2. akin to scattering of passive scalar, \( \mathcal{S} \) small scale, chop-up,
3. conserve \( \langle \mathbf{P}^2 \rangle \) upon \( \sum \) together.

\[ \sum \]
2. Coherent damping/growth - from back reaction \( (\Phi^0)^2 \) into \( \text{John's Law} \).

reshuffle \( \langle A^2 \rangle \) to larger scale. Sign \( h^2 \) vs \( k^2 \).

\[ \sum_n \text{ conserve } \]

\[ \langle A^2 \rangle \]

Correspondence to condensation and waves (currents) attracting.

1 + 2 \rightarrow \text{net effective vorticity sign.}

\[ \eta, \text{ too.} \] \( \text{Negative vorticity} \)

\[ \Delta E_k > E_m \Rightarrow \text{shuffled to smaller scale.} \]

\[ E_m < E_k \Rightarrow \langle A^2 \rangle \text{ transferred to larger scale.} \]

Transfer need not be local.
In dynamics, evolution is complex.

N.B. Recall Flux expulsion:

\[ \frac{V_{AS}^2 \cdot R_m}{V^2} < 1 \rightarrow A \text{ passing } B \text{ expelled} \]
\[ > 1 \rightarrow J \times B \text{ disrupts vortex expulsion} \]

\[ B_0^2 < \frac{c_0 v^2}{R_m} \]

but \( < B^2 > \gg B_0^2 \) upon stretching.

Zeldovich: weak \( B_0 \) is sufficient.

\[ \frac{\partial A}{\partial t} + v \cdot \nabla A = - \nu \frac{\partial}{\partial x} \frac{\partial A}{\partial x} + \eta \nabla^2 A \]

\( A \) and avg. \( \phi \)

\[ \eta \frac{\partial}{\partial x} < B^2 > = < J \cdot \bar{A} > \frac{\partial}{\partial x} \]

\( \frac{\partial}{\partial x} < B^2 > = \frac{\eta}{H} B_0^2 \)

\[ = \frac{1}{\nu} v \cdot \partial_x B_0^2 = R_m B_0^2 \nu \]
\[ \frac{<B^{m^2}>}{K_m} < \frac{<N^{m^2}>}{K_m} \quad \checkmark \quad \] 

Questions still open!

Taylor conjecture remains a conjecture!