Physics 141 Lecture 1 and Lab 1
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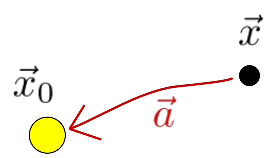
1 One Particle Central Force

1.1 Acceleration Vector

Suppose a single particle of mass m is at position $\vec{x}$, and is attracted gravitationally to a single fixed particle of mass M at position $\vec{x}_0$. There is a force vector on $\vec{x}$, denoted $\vec{a}$.



The magnitude of acceleration is $\|\vec{a}\| = \frac{GM}{\|\vec{x}_0 - \vec{x}\|^2}$ The direction of acceleration is *from* $\vec{x}$, *to* $\vec{x}_0$, so it points in the direction of $\vec{x}_0 - \vec{x}$. One way to remember the sign is to think of putting $\vec{x}$ at the origin. We have the magnitude and direction of the acceleration, so we can find the vector itself:

$$\vec{a} = GM \frac{\vec{x}_0 - \vec{x}}{\|\vec{x}_0 - \vec{x}\|^3}$$

The quantity GM has units of length cubed per time squared, and it's the quantity that is easily experimentally measurable. G is known to four or five digits, but $GM_\odot$ (G times the mass of the sun) is known to ten or eleven digits! GM is called the standard gravitational parameter for a given body.

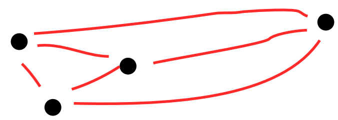
As an aside, how do you do the same process if instead of one free body you have N free bodies interacting with each other? Sum the forces!

$$\vec{a_i} = \sum_{j=1, j \neq i}^{N} GM_j \frac{\vec{x}_j - \vec{x}_i}{\|\vec{x}_j - \vec{x}_i\|^3}$$

The distance from particle i to particle j is the same as the distance from j to i, and we ignore the $j = i$ case, so there are actually only (N choose 2) $= N(N-1)/2$ calculations. Try plugging in $N = 10^{10}$, the particle number in some modern simulations. Not even modern supercomputers can run the brute force method[1]!

[1] A back of the envelope estimate tells me, with a few tens of teraflops of computing power, it would take months to half a year to do finish all of these distance calculations = one timestep. The thing is you need thousands of timesteps to get a useful result.

$$\binom{N}{2} = \frac{N(N-1)}{2} \quad \text{distances}$$



1.2 Numerics

Let's run a simulation of our one body orbiting around a sun using **Euler's method**. I'm going to omit vector arrows, but keep in mind that x, v, and $a(x)$ are all vectors here. We want to solve the second order ODE $\ddot{x} = a(x)$. This is equivalent to solving the two first order ODEs $\dot{x} = v$, $\dot{v} = a(x)$. We have initial conditions v_0 and x_0 at $t = 0$. Let's use a fixed timestep δt, and let x_n denote the position of the particle at time $n\delta t$.

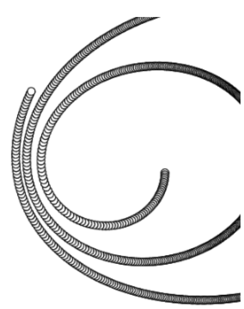
One way to approximate the solution is using the Euler scheme:

$$\dot{x} \approx \frac{x_{n+1} - x_n}{\delta t} = v_n \qquad \dot{v} \approx \frac{v_{n+1} - v_n}{\delta t} = a(x_n)$$

Then...

$$x_{n+1} = x_n + \delta t \cdot v_n \qquad v_{n+1} = v_n + \delta t \cdot a(x_n)$$

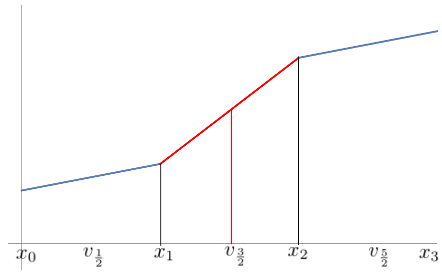
As we saw in the javascript applet in class, this is a pretty bad method. We should get an elliptical orbit, but the result isn't elliptical at all!!



For a function $f(t)$, the quantity $\frac{1}{\delta t}(f(t+\delta t) - f(t))$ is a pretty bad approximation to the derivative at t, and the error is proportional to δt. You can prove this by using Taylor expansions:

$$\frac{1}{\delta t}(f(t + \delta t) - f(t)) = f'(t) + O(\delta t)$$

There has to be a better way, and the better way is the **leapfrog method**. We can get something accurate to second order in δt if we use a bit of intuition: when we find the slope of a line segment, we're really finding a good approximation to the slope at the midpoint of the line segment.



You can prove the following formula if you Taylor expand both sides about t:

$$\frac{f(t+\delta t)-f(t)}{\delta t} = f'\left(t+\frac{\delta t}{2}\right) + O(\delta t^2)$$

Then we have the following approximations:

$$\dot{x} \approx \frac{x_{n+1}-x_n}{\delta t} = v_{n+1/2} \qquad \dot{v} \approx \frac{v_{n+1/2}-v_{n-1/2}}{\delta t} = a(x_n)$$

which we turn into the Leapfrog numerical scheme:

$$v_{n+1/2} = v_{n-1/2} + \delta t \cdot a(x_n) \qquad x_{n+1} = x_n + \delta t \cdot v_{n+1/2}$$

Leapfrog is a great algorithm for four reasons:

1. It's second-order (whereas Euler is only first-order),
2. It only uses one force evaluation per timestep (force evaluations are expensive!),
3. It is time-reversible (unlike Euler),
4. It's *symplectic*, meaning it conserves phase space volume.

The power of the method is demonstrated by the Javascript implementation we went over class.