Wakes

- Physics Ideas → flow created by response flow to separation
- Link:
  - Drag → wake flow
- Width: - laminar
  - Turbulent
- Scalings
- Deficit and punched line.

- Discontinuity Stability
  \( \text{H similarity} \)
Wakes

- Region behind moving body of departure from potential flow. Wake is rotational.

- Wake is consequence of body experiencing drag (or flow drag over body)

- Region of wake is limited in angular extent.

Message of wake: A little viscosity forces a global adjustment in flow structure.

Why Ed vs Re curve again Wakes?
Wake Flow

\[ \text{Stokes, } Re < \frac{1}{B} \text{, laminar, drag crisis} \]

\[ \text{Re} \]

\[ F_t = C_D \rho A U^2 \]

Drag coefficient

\[ \text{Flow not turbulent, but inertia is relevant.} \]

Further:

\[ x \gg R, \text{ are region of wake} \]

\[ \text{If body speed } U, \text{ then if frame where body stationary} \]

\[ \text{As } x+y, \text{ with } k_x < 0 \]

\[ \text{(slower in wake)} \]

\[ U \text{ noticeably different from zero in limited region.} \]
How limited? 
As laminar, 1 
signal propagation is diffusion only.

How does wake form?

- No slip boundary condition slows down fluid flowing past body
- Discontinuity results on surface
- Viscosity smooths out discontinuity.

N.D. If turbulent, then turbulent mixing (O.O) smooths discontinuity faster than viscous mixing.
B) Waves - Sample Physics

Wake is:
- region of departure from potential flow behind object moving thru water and experiencing drag

Wake is inexorably coupled to drag

Message of wakes:
A little force results in global adjustment of flow structure

Dry - [thinking on plane] where object at rest, drag results from loss of flow momentum to object.

\[ \frac{\dot{m}}{\rho} \rightarrow 0 \]
\[ t \to \text{wake is region of flow where loss of momentum is evident.} \]

\[ \text{c.e.,} \]

- If potential flow (no drag)

\[ \text{symmetry upstream downstream in displacement of fluid element} \]

- With no-slip b.c., viscosity, turbulence, etc.

\[ \frac{u}{v} \rightarrow \frac{u}{v} \]

\[ U \rightarrow U + V \]

\[ \rightarrow \text{wake} \]

results from evolution of discontinuity.

\[ 0 \rightarrow \frac{-4\varphi}{u=0} \quad \text{unstable} \rightarrow \text{turb.} \]
- origin of wake is no-slip b.c. + viscosity + turbulence after separation

- "hole"

- but flow is unstable!

- how high in Re can one go?

- line sep.

- \( \mathbf{V}_{tan} = 0 \)

- vertical motion

- \( \mathbf{U} = \mathbf{0} \times \mathbf{V} = \mathbf{0} \)

- boundary of wake traced by fluid particles:

  - passing close to body

  - scattered by diffusion (and turbulent mixing)

  - \( \mathbf{U} \) expansion
Note:

- In general, wake multi-component Kelvin waves. For example, Kelvin wave due to screw bubbles. (Slit flow)

- Here consider spherical cow of wake problems

\[ F = 
\]

- No surface effects

→ How calculate wake structure?

\[
\text{Force of Drag} = \begin{cases} 
\text{Rate of Net Momentum Loss} \\
\text{Flow}
\end{cases}
\]
Simply put

deck

\[ \text{Rate Momentum Loss} = \frac{\int_{x} A P(x) \, dx + A P(0)}{\text{Total}} = \Phi_{x} \]

\[ P_{\text{tot}}(0) = P + \rho + \frac{\rho U^2}{2} \]

Bernoulli's Principle

\[ P_{\text{tot}}(x) = P + \rho + \frac{\rho (U+V)^2}{2} \]

\[ P_{\text{tot}} + \rho + \frac{\rho (U+V)^2}{2} \]

\[ A \sim \pi w(x)^2 \quad w = \text{width at } \]

\[ \text{conical symmetry} \quad \times \text{downstream} \]

\[ F_x \approx w(x)^2 \left[ \left( \frac{P}{2} + \frac{\rho (U+V)^2}{2} \right) + \left( \rho + \frac{\rho U^2}{2} \right) \right] \]

\[ \text{Punched out} \rightarrow \text{straight streamlines} \]
Formally

\[ F_i = \int \Pi_i \, df_i \]

\[ = \int (A' + A') \, df_i + \sigma (U_i + U_i) (U_i + V_i) \, df_i \]

\[ \cos \theta \, df_i = 0 \]

\[ \rightarrow \text{For} \]

\[ F_i = \int (A' - A') (P' + \rho U_i U_i) \, dy \, dz \]

- outside \( \rightarrow \) Bernoulli
- inside, \( V_x / \text{area} \)
  - Pressure \( \text{unchanged} \) \( \rightarrow \) start streamline
\[ F_x = -w(x)^2 \left[ 1 + \frac{8\mu^2}{2} + 2\rho UV_x - \frac{\gamma}{2} - \frac{\beta}{2} \right] \]

\[ \sim -\rho U V_x w(x)^2 \]

**n.b. why \( p(x) \sim p(x) \)?

**\( V_x > 0 \)

**\( F_x > 0 \)

\[ \rightarrow H \]

Now, need \( w(x) \) to get \( V_x \)!

\[ \rightarrow \text{Observe:} \]

- Problem now reduced to one of scale (within)
- Waves are self-similar

\[ \rightarrow w \sim x^\alpha \]

- Waves can be laminar (or turbulent)
i) Laminar \[ \frac{UR}{r} < 1 \]

Now
\[ \frac{\partial}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \nu \frac{\partial^2 u}{\partial x^2} = -\frac{\partial p}{\partial x} \]

St. state \[ u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial u}{\partial y} - \nu \frac{\partial^2 u}{\partial x^2} = \frac{\partial p}{\partial x} \]

Oseen defined laminar as turbulent

\[ \vec{v} \cdot \nabla \vec{v} \rightarrow 0 \]

\[ \vec{v} : \nabla = 0 \]

Now walls

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} \]

and

\[ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} \]

Scaling

Take \[ dx \sim 1/x \rightarrow \text{downstream distance} \]

\[ dy \sim 1/w \rightarrow \text{scale} \]

Obv. \[ \frac{1}{\sqrt{\frac{y^2}{x/y}}} \exp \left[ -\frac{y^2}{x/y} \right] \]

\[ w \sim \sqrt{x/y} \]
\[
\left( \frac{5}{x} - \frac{r}{W^2} \right) y - \frac{p}{w^2} \\
\left( \frac{U}{k} - \frac{r}{W^2} \right) v_x \sim -\frac{p}{x} \\
\frac{\partial v}{\partial y} = 0 \Rightarrow \frac{v_x}{x} \sim \frac{v_y}{y} \\
as \ p \text{ negligible (will show)} \Rightarrow \\
\frac{u}{x} \sim \frac{r}{W^2} \\
\downarrow \\
W \sim (r x / u)^{1/2}
\Rightarrow \text{diffusive spreading of momentum by } v \\
\downarrow \\
\sim (r t)^{1/2} \\
\text{with } t \sim x / u.\]
\[ w \sim \left( \frac{x}{R} \right)^{1/2} \left( \frac{y R}{u} \right)^{1/2} \]

\[ \frac{w}{R} \sim \left( \frac{x}{R} \right)^{1/2} \frac{1}{Re^{1/2}} \]

\[ \frac{v x - \tau_1}{\rho u w^2} \]

diffuse

\[ \rightarrow \text{skin friction B.L. thickness} \]

\[ \rightarrow \text{in case you are wondering:} \]

\[ P \sim \frac{v v_x}{w^2} \quad \text{(if \text{\textbullet\textbullet\textbullet})} \]

\[ \text{(drop u\text{\textbullet\textbullet\textbullet})} \]

and

\[ \frac{v_x}{x} \sim \frac{v_y}{w} \]

\[ \Rightarrow P \sim \rho r \frac{v x}{x} \]

and

\[ P / P x \sim \frac{v v_x}{x^2} \ll \frac{v v_x}{w^2} \]

\[ \text{drop P.} \quad \text{and safely} \quad \frac{v v_y}{w^2} \]
by analogy with H.T. gas
\[ \nabla \cdot \mathbf{V} \rightarrow -\nabla_x \mathbf{V}^2 \]
\[ \mathbf{V}_\text{r} \sim \mathbf{V}_\text{mix} \]

\( \text{ii) Turbulent} \)

\[ \mathbf{V}_x + \mathbf{V}_y \rightarrow \mathbf{V}_y \frac{\mathbf{V}_x^2}{\mathbf{V}_y} = -\frac{\partial P}{\partial x} \]

\( \text{ignore} \)

\( \text{wave spreads by advection, not diffusion} \)

\( \mathbf{V}_y \sim \text{turbulent velocity} \)

\( \text{Take wake turbulence (isoviscous)} \)

so \( \mathbf{V}_x \sim \mathbf{V}_y \)

\[ W \sim \mathbf{V}_y \frac{x}{y} \]

\[ W \sim \frac{x \sqrt{V_x}}{U} \]

but from drag:

\[ \mathbf{V}_x \sim \frac{F_y}{\rho U w^2} \]

\[ \Rightarrow \]
\[ W \sim x \frac{F_t}{\rho u^2} \sim x \left( \frac{F_t}{\rho u^2 w^2} \right) \]

\[ W^3 \sim \frac{F_t x}{\rho u^2} \]

\[ W \sim \left( \frac{F_t}{\rho u^2} \right)^{1/3} \times \frac{1}{13} 
\sim (C R^2)^{1/3} \times \frac{1}{13} \]

then, comparing widths:

- **Laminar:** \[ \frac{W}{R} \sim \left( \frac{x}{R} \right)^{1/2} \quad Re^{-3/2} \]
  \[ Re = \frac{UR}{\nu} \]

- **Turbulent:** \[ \frac{W}{R} \sim \left( \frac{x}{R} \right)^{1/3} \quad C_0^{1/3} \]

Interestingly, Laminar wake expands with downstream \underline{length} more rapidly.
Why?

\[ \text{turbulence can relax IV behind object (due separation) rapidly and faster than } \frac{V}{r}. \text{ Thus surrounding flow penetrates the dead water region more rapidly, less wake expansion.} \]

Also observe: Wake Re drops with \( x \).

\[ \text{Re} \sim \frac{W V_y}{V} \sim \frac{W V_x}{V} \sim \frac{F_d}{\mu IV} \]

(y direction) \( \text{wake flow Re} \)

(opr) \[ \text{Re} \sim \frac{F_d}{\mu IV} \]

\[ \begin{align*}
C_n & \sim 1 \\
& \sim \left( \frac{UR}{V} \right)^{1/3} (R/X)^{1/3}
\end{align*} \]
\[ \text{Re}(x) \sim \text{Re}_c (R/X)^{1/3} \]

and \( \text{Re}(x) \to 1 \) at
\[ x \sim R \left( \text{Re}_c \right)^3 \]

distance behind (not where)
turbulent wake transitions to
laminar.

i.e. akin to: transition from turbulent mixing to viscous mixing

N.B. [In wake, vertical/axialional region
can expand into irrotationatl region, but never reverse!]

... would really violate H-Thm...
Later discussion

**Wake** - Supplemented

Revisit turbulent wake using turbulent viscosity, i.e.

\[ W \sim (\frac{\nu x}{u})^{1/2} \quad (\nu \to 0) \]

\[ \to (\frac{A x}{u})^{1/2} \]

i.e., is width of turbulent wake set by turbulent diffusivity following Blasius law?

but \( D_t \sim W^2 \) \& turbulent viscosity at mixing length level

\[ \sim W \left( \frac{E_1}{\rho u^2 w^2} \right) \]

\[ \sim \frac{E_1}{\rho u w} \sim \text{const}/W \]

\[ \to \]

\[ W \sim \left( \frac{E_1 x}{\rho u^2 w} \right)^{1/2} \]

\[ W^{3/2} \sim \left( \frac{E_1}{\rho u^2} \right)^{1/2} x^{1/2} \sim C_0 R^{2/3} x^{1/2} \]

\[ W \sim (C_0)^{1/3} R^{2/3} x^{1/3} \sim C_0^{1/3} R \times x^{1/2} \]
\[
\frac{w}{R} \sim C_0^{1/3} (x/R)^{1/3}
\]

Now, \( D_f \sim \sqrt{w} \)

\[
\sim \frac{\sqrt{w}}{w}
\]

\[
\sim \frac{\rho u w^2}{\rho w} \quad \sim \frac{\rho u w}{\rho w} \quad \sim \frac{Q}{w} \sim \frac{Q}{w} (x/R)^{1/3}
\]

The point is that turbulent viscosity mixing drops downstream relatively constant viscosity mixing.

- Follows from \( \sqrt{w} \sim \frac{Q}{w} \)

- Explains why turbulent wake spreads more slowly than laminar wake.
Some Observations for Wake Flows

- Note: replace \( A \) with \( \Delta t \)

\[
F_x = -p U \int v_x \, dy \, dz \quad \text{Wake}
\]

Now \( Q = \int v_x \, dy \, dz \) \( \text{Deficit Flux} \)

- Difference with
- Fluid flow thru
- Fluid flow thru wake area
- Miss Flow due wake
- Deficit

\[
Q = \text{const.} \quad \text{c.f. Ex/1/4}
\]

- But if encircle body

\[
\int v_x \, dy \, dx = \int v_x \, dy \, dz + \int v_x \, dx \, dz \\
\text{outside wake} \quad \text{outside wake}
\]

\[
V_x \approx 1/\sqrt{R} \\
\text{as} \quad p \int v \, da = 0 \quad \text{i.e. continuity!}
\]

Now total \( V \to \) \{Velocity field\}

\[
\begin{align*}
\int v \, da & = \text{const.} \quad \text{const.} \\
\int v \, da & \approx 1/\sqrt{R} \\
\text{inside wake} & \quad \text{outside}
\end{align*}
\]

Vertical Wake flow + Potential Flow
so, must have \( \nabla \cdot \mathbf{v} = \frac{Q}{\pi} \) flow

\[ \int \mathbf{v} \cdot d\mathbf{a} = \frac{Q}{\pi} \]

to compensate.

then, for area at \( r \):

\[ v \pi r^2 \sim \frac{Q}{\pi} \]

\[ \Rightarrow v \sim \frac{Q}{r^2} \]

\[ \Phi \sim \frac{Q}{r} \]

\[ \text{monopole} \]

\[ \text{global adjustment in potential flow} \]

due wake/viscous

(\text{localized})

Message:

A little \( r \) force \( \Phi \) global adjustment in flow structure.

Note: \( \delta \)-dominant far from body \( \delta \) - pot flow \( \mathbf{v} \sim \frac{1}{r} \) dipole

- Wake consequence of \( \mathbf{v} \)