A Quick Look at Closures

- Turbulence, so far:
  - satisfied from "physicist's perspective"?
  - scalings - rooted in phenomenology?!?

- Mixing length models - also rooted in phenomenology? \( \frac{v_T}{\Delta} = \frac{u_{11}}{L} \)

- Where have Navier-Stokes equations gone?

- Might one:
  - derive eddy viscosity
  - derive \( k-\varepsilon \) spectrum

- From some systematic procedure starting from NSE?

- Apply to more complex problems - MHD, stratified turbulence, etc.

- Framework
References on Closure:

- Kraichnan 59 → Basic of DIA
- Kraichnan 61 → Random Coupling Model
- Kraichnan 76 → Test Field Model
- Forstner Nelson, Stephen 77 → Forced Burgers Turbulence

- Hunt 70 → Rapid Disturbance Theory
Spectral Equation

\[ \frac{\partial}{\partial t} \langle \tilde{v}^2 \rangle_k + 2 \sum_{k'k''} \langle \tilde{v}^2 \rangle_{k'} \langle \tilde{v}^2 \rangle_{k''} = x_k^2 + 2 \sum_{k'k''} x_{k'}^2 \langle \tilde{v}^2 \rangle_{k} \langle \tilde{v}^2 \rangle_{k'} \]

Random stirring

made-coupling-induced stirring - nonlinear noise

(\( \gg x_k \) in inertial range)

- structure is that of Langevin equation with noise and drag renormalized.

\[ \frac{d \tilde{v}}{dt} + \mu \tilde{v} = \tilde{f} \]

\( \mu = \frac{6\pi a}{\rho} \)

\[ \frac{d E_k}{dt} + r T \langle \tilde{v}^2 \rangle_k = \delta T \]

\( \delta T \) - turbulent viscosity

\[ \sum_k T_k = 0 \]

- energies - does renormalized theory respect primitive equations?
\[ \sum_{k} T_k = \sum_{k} \sum_{k'} 2 (k + k')^2 \Theta_{k,k'} \left\langle \gamma^2 \right\rangle_{k} \left\langle \gamma^2 \right\rangle_{k'} \]

\[ - \sum_{k} \sum_{p \neq 2} 2 (p+2)^2 \Theta_p \left\langle \gamma^2 \right\rangle_p \left\langle \gamma^2 \right\rangle_p \quad \text{RPA:} \quad \Theta_{p+2} \rightarrow \Theta_{p} \rightarrow \Theta_{p+2} \rightarrow \Theta_{p} \rightarrow \cdots \]

Comment: Equilibrium closure as C(A) + H thms \rightarrow Projection to equipartition spectra (stat. mech).

N.B.: Upon summation, coherent damping conserves energy vs. incoherent emission.

\[ (\text{re-label}) \]

\[ = 0 \]

\[ \text{i.e.: cascade as sequence of coherent damping} \rightarrow \text{incoherent emission} \rightarrow \text{coherent damping} \rightarrow \cdots \text{all band models.} \]

Closure zoology: based upon use of coupled response ftn, spectral eqns.

\[ \text{i.e. } \delta \text{ response ftn} \rightarrow \text{depends on } \delta \text{ of spectra } \left\langle \gamma^2 \right\rangle_k \]

\[ \frac{\partial \left\langle \gamma^2 \right\rangle}{\partial t} \text{ depends on } C_{kk}, \quad L_k \]

etc.

QIA: solve coupled equations for \( \delta y / \delta t \) and \( \left\langle \gamma^2 \right\rangle_k \)
EDONM: parametrize $G$ in terms $<D^3>_k$, yielding spectral equation.

Eddy viscosity models: $\frac{\partial u}{\partial t}$ equation.

Weak Turbulence: neglect $C_{n+1}$ in $L_{n+1}$.

Comments on closures:
- consistent with conservation laws, albeit trivially.
- based upon assumed weak coupling, $\frac{\partial}{\partial t}$ hypothesis (The Swindle Occurs Here).

$N \sim C_{n+1} V_{n+1} + (C) V_k V_{k'}$, etc.

$W_{\text{tried}} = \Sigma (W_{\text{decomp}})_{\text{tried}}$

decomoln.

- no restriction on shape of interacting
  tried, i.e. $\rightarrow$ confusion of sweeping $\left\{ \text{streams} \right\}$

$p + 2 = k$

$\frac{\partial}{\partial t}$ $\frac{\partial}{\partial x}$ $\frac{\partial}{\partial y}$, etc. $\rightarrow$ sweeping?

- reprise of the O.I.A. and the O.I.A. propagator for N.S.T.

- stochastic oscillator models ⇒ general structure

- random coupling model and the problem of realizability

Reprise

Recall O.I.A. ⇒ coupled equations for propagator

spectrum

Interesting to note:

⇒ essential physics is nonlinear scrambling in triad coherence (i.e. sets coherence time)

- spectrum

sets response propagator

sets scrambling time ⇒ cascade dynamics
Useful to note for later that for $N > 1$, EQA
D.I.A. for propagator evaluation gives:

$$\frac{df}{dt} = g \mathbf{k}^2 g_{kk} \mathbf{k}^4$$

modular viscosity

$$\frac{df}{dt} \propto \text{propagator}$$

\[ \frac{k + p + z}{2} = 0 \]

non-Markovian structure

$$= -\frac{k}{2} \int \frac{dP}{Z} \left[ E(2) \left( \frac{dE}{dS} + \frac{dE}{dS} \right) \right]$$

coupling background energy

coefficient decay-wave response

self-correlation of propagator

(from closure)

spectral

Can simplify using:

a) $E(2)$ largest at small $z \to$ energy containing range

b) and $p + z = k \to |p| \sim |k| \gg z$ (selection rule)

c) $N_2(z) \approx N_2(0)$ \[ \text{i.e., large odd} \]

long lifetimes, treat as slow relative to high $k$ response.

So...
\[
\frac{\partial \phi(x, t)}{\partial t} + \nu k^2 \phi(x, t) = -k^2 \nu^2 \int g_k(y, x - s) \phi(s) \, ds = 0
\]

\( \Rightarrow \) non-Markovian - convolution

\[
h^2 \nu^2 \psi = \frac{1}{2} \int \delta(x - y) \psi(y) \, dy \quad \text{effective strain/sweeping (D)}
\]

time

Can solve via Laplace Transform (n.b. convolution!)

So:

\[
g(k, t) = e^{-w k^2 t} \int (2k \nu k) / k \nu t
\]

Some observations:

(i) Sweeping vs strain \( \rightarrow \) physical of eddy lifetime \( \rightarrow \) vs \( \psi \)?

(ii) \( \delta k, \delta H \) oscillators \( \psi \) - physical meaning?

(iii) Ultimately gives \( E(k) \sim k^{-\frac{3}{2}} \) not \( E(k) \sim k^{-5/3} \)
Closures and Renormalization

- **Overview**
  - W.D. McComb, "The Physics of Fluid Turbulence"
  - Renormalization: A Guide for Beginners

- **Object of closure to derive equations for observables of turbulence from Navier-Stokes Eqn. - dynamics not just geometry.**
  - Contrast fractality
  - Observables typically: response function, spectrum, not full RDF...

- **Procedure is perturbative/RPT (c.f. QLT, mean field theory)**

- **Closure methodology usually involves:**
  - a) RPA/weak coupling approximation
    - Generic NL model eqn.

\[
\begin{align*}
\langle a^2 \rangle_n & \sim \langle a a a a \rangle \\
\frac{\partial}{\partial t} \langle E(k) \rangle + \delta_{k} \langle E(k) \rangle & + \sum_{n} c_{n} a_{n} \tilde{a}_{n}^{*} a_{n+1}^{*} = 0
\end{align*}
\]

\[
(\langle a_{n}^{2} \rangle)^{2} = E(k)
\]
and moment hierarchy \( \frac{\partial}{\partial t} \langle a^3 \rangle \sim \langle aaaaa \rangle \sim \langle a^2 \rangle \langle a^2 \rangle \)

- application of RPA to \( \langle a^4 \rangle \)
- on \( \langle a^4 \rangle \sim \langle C C aaaaa \rangle \)
  \( \sim 10 \langle a^2 \rangle \langle a^2 \rangle \)
  (condm coupling)

b) to renormalization

\( \langle a^3 \rangle \sim T_C \langle a^3 \rangle \langle a^2 \rangle \)

What controls this?

- if simple perturbation theory
  \( \rightarrow \) this physics?

\( 1/\mu \sim \nu k^2 \), necessarily

\( \Rightarrow T_C \sim (\nu k^2)^{-1} \rightarrow \infty \), relative to
  (inertial range time scale)

so

\( \frac{\partial}{\partial t} \langle a^3 \rangle \sim T_C \langle a^2 \rangle \langle a^2 \rangle \)

transfers unphysically large, due to long
  correlation times (also unphysical)
mindless perturbation theory yields unphysically long correlations \[ \frac{\partial \langle a^2 \rangle}{\partial t} \rightarrow E \Rightarrow 0 \] results in unphysical \( E \rightarrow \) 'realizability problem' \( \Rightarrow \) model

must "renormalize" \( T_v \rightarrow (\nu k^2)^{-1} \) (i.e. treat time-scale self-consistently) so that model coherence consistent with inertial range scrambling rate!

Example: Burgers\/Kaz Equation

\[ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} - \nu \frac{\partial^2 v}{\partial x^2} = f \]

stochastic forcing

n.b.: perturbative closure will completely miss shock formation physics, \[ [\text{pdf}(v')] \text{ asymmetry} \]

a.) Response Function

\[ \frac{\partial v_k}{\partial t} + \frac{c_k}{2} \sum_{k'} V_{k-k'} V_{k+k'} + \gamma k^2 v_k = f_k(t) \]

Eddy viscosity!

now, seek \( \frac{\partial v_k}{\partial t} \Rightarrow \text{response function for mode } k \).

Key physics: space/time scales
for $Re \ll 1$,

$$\frac{\partial V_k}{\partial t} + \nu k^2 V_k + i k \sum_{n} V_{n-1} V_{n+1} = f_k(t)$$

$$(i\omega + \nu k^2) V_k = f_{k,0}$$

$$R_k = \frac{dV_k}{d\xi} = \sqrt{-i\omega + \nu k^2}$$

$\Rightarrow$ time scale set by viscosity !?

for $Re \gg 1 \Rightarrow$ id est $\Rightarrow$ need faster time scale.

\[\text{\textbf{need extract effective time-scale from}}\]

- physics is time scale of nonlinear scrambling/coupling $\Rightarrow$ non-linear response $\Rightarrow$ how calculate?

\[\text{\textbf{c.e.}}\]

$$\frac{\partial V_k}{\partial t} + \nu k^2 V_k + C_k V_k = f_k(t)$$

\[\text{\textbf{c.e. seek response mode of test wave interacting with rest of turbulent spectrum...}}\]

- reflects $ik \sum_{n} V_{n-1} V_{n+1}$

- phase coherent with $f_k$

$$C_k V_k \Rightarrow \frac{i k}{2} \sum_{n} V_{n-1} V_{n+1}$$

so, in lowest order

$$C_k \sim |V|^2 \text{ (c phase independent)}$$
Now to calculate $C_{kk'}$

$$(-\omega^2 + \nu k^2) V_{k} + \frac{i k}{\omega} \sum_{k'} \frac{V_{k'}}{\omega_{1} - \omega_{1}} \frac{V_{k+k'}}{\omega_{2} + \omega_{2}} = f_{k}$$

$V_{k+k'} \rightarrow V_{k+k'}^{(2)} \Rightarrow$ $V$ driven by direct best interaction of $V_{k}$, $V_{k'}$

(hence DFF)

$$(-\omega^2 + \nu k^2) V_{k} + \frac{i k}{\omega} \sum_{k'} \frac{V_{k'}}{\omega_{1} - \omega_{1}} \frac{V_{k+k'}}{\omega_{2} + \omega_{2}} = f_{k}$$

there: $\frac{i k}{\omega} \sum_{k'} \frac{V_{k'}}{\omega_{1} - \omega_{1}} \frac{V_{k+k'}}{\omega_{2} + \omega_{2}} \equiv C_{kk'} \frac{V_{k}}{\omega_{1}}$ $(S+)$

so, when calculated:

$$\frac{\partial f_{k}}{\partial V_{k}} = \frac{1}{(-\omega^2 + \nu k^2 + C_{kk'} \omega)}$$

Limits $\nu \rightarrow 0$

reflects inertial range scrambling

dressed viscosity

test field hypothesis

Now, to calculate: $NL$ scrambling $\Rightarrow$ (self-consistent)

$$(-i(\omega + U_{1}) + \nu \frac{(k+1)^2}{2} + C_{k+k'} \frac{1}{\omega_{1} + \omega_{2}}) \frac{V_{k+k'}}{\omega_{2} + \omega_{2}} = -c(k+1)\frac{V_{k}}{\omega_{1}}$$

$$-c(k+1)\left( V_{k} \cdot V_{k} + V_{k'} \cdot V_{k'} \right) = -c(k+1)(V_{k} \cdot V_{k})$$
Now, define

\[
L^{-1}_{k+k'} = \frac{-i(\omega + \omega') + v(k+k')^2 + C_{k+k'}}{\omega + \omega'} \quad \text{(renormalized propagator)}
\]

\[
V^{(a)}_{k+k'} = L_{k+k'} \left( -i(k+k') \right) \frac{V_{\omega_i} V_{\omega_j}}{\omega_{\omega_i} \omega_{\omega_j}}
\]

so self consistently,

\[
C_{\omega_i} V_{\omega_j} = i k \sum_{\omega_{\omega_i}} V_{\omega_{\omega_i}} L_{k+k'}(\varepsilon^{i}) (k+k') \frac{V_{\omega_{\omega_i}} V_{\omega_{\omega_j}}}{\omega_{\omega_{\omega_i}} \omega_{\omega_{\omega_j}}}
\]

\[
= \left( i k^2 \sum_{k' \omega_{\omega_i}} |V_{\omega_{\omega_i}}|^2 L_{k+k'}(1 + k') \right) \frac{V_{\omega_{\omega_i}} V_{\omega_{\omega_j}}}{\omega_{\omega_{\omega_i}}}
\]

so

\[
\frac{dN_{\omega_i}}{dN_{\omega_j}} = \frac{1}{-i\omega + v k^2 + C_{\omega_{\omega_i}}} \quad \text{\{renormalize response function\}}
\]

\[
C_{\omega_i} = \frac{v_k h^2}{2} \equiv \frac{v^2}{2} \sum_{k' \omega_{\omega_i}} |V_{\omega_{\omega_i}}|^2 L_{k+k'}(1 + k')
\]

\[
\text{\{\text{renormalized turbulent viscosity\}} \rightarrow \text{nonlinear scrambling rate}}
\]

\[
\nu = \nu + \nu_{\omega_i} \omega_i.
\]
About $\nu_{ij}$:

- at long wavelength ($l > l_i$) and low frequency ($\omega < \omega_i$) it resembles the quasilinear limit, Markovian

$$\nu_{ij} \rightarrow \nu^{T} = \sum_{i,j} \frac{\nu_{ij} \nu_{jk} \omega_{ij}}{\omega}$$

effective transport coefficient $\sim$ sets N.Y. turbulent time scale.

$$\nu^{T} \sim \langle \nu^{2} \rangle \nu_{c} \sim \nu_{ms} \nu_{c}$$

$\nu_{c} \sim \nu_{c}$

$\nu \rightarrow F = F_{E, E}$

$\nu_{ij} \rightarrow$ need Zwanzig-Mori theory

- important to note:

$$\nu_{ij} \rightarrow \nu^{T} = \sum_{i,j} \left[ \nu_{ij} \left( \frac{\nu_{ij} \omega_{ij}}{\omega^2 + (\nu_{ij} \omega_{ij})^2} \right) \right]$$

$\nu^{T}$ also renormalized due to self-consistency

$\nu^{T} \rightarrow$ random Doppler shift

$\nu^{T}$ also minimum for QLT, with resonance, i.e.,

$D = \frac{2}{m^2} \sum_{k, \pi} |K_{k} |^2 \pi \rho(\omega - k\nu)$

$\rightarrow$ to estimate dissipation, c.f., contrast QLT with resonance, i.e.,

$D = \frac{2}{m^2} \sum_{k, \pi} |K_{k} |^2 \pi \rho(\omega - k\nu)$
\[
\begin{aligned}
\gamma^2 \approx & \frac{1}{L^2} V_{\text{rms}}^2 \\
\gamma & \approx \frac{1}{L} V_{\text{rms}}
\end{aligned}
\]

- \( \frac{V_{k_x}}{\omega} \) vs. \( V_{k_y}^T \)

\( k_\omega \to 0 \) if \( k < k', \omega < \omega' \)

\( \rightarrow \) Markovian limit \( \rightarrow \) no memory (ale' F.P.E.)

\( \text{Feffer-Philen Eq.} \)

i.e. consider interaction of 'test wave' \( k', \omega' \) with background \( k, \omega \)

\[ k_\omega \approx k' \approx \omega' \]

\( \rightarrow \) for \( R', x' \ll R, \lambda \)

\( \rightarrow \) interaction appears as random memory-less kick, as in walk.

for \( R'_x \sim R, \lambda \)

\( \rightarrow \) interaction is one of mutual slashing, etc.

i.e. test wave "feels" space-time history of turbulence background.
\[
\text{eddy viscosity}
\]

\[
\nabla \cdot \nu \nabla \nu \rightarrow - \nu \nabla^2 \nu
\]

\[
\nu \nabla^2 \nu \rightarrow \int dx' dr' \ C(x-x', t-r) \ \nu(x', r)
\]

why "renormalization"?

\[\text{QED} \quad \frac{1}{\beta-m_0} \rightarrow \text{Fermion propagator (bare)}\]

\[\beta-m_0 + \Sigma \rightarrow \frac{1}{\beta-m} \quad \text{(renormalized)}\]

\[\text{due electron interaction with vacuum polarization cloud (ambient fluctuations)}\]


turbulence:

\[\frac{1}{-i\omega + \nu \kappa^2} \rightarrow \nu \text{ propagator} \]

\[\rightarrow \text{bare (collisional) viscosity}\]
renorm. \[ \rightarrow \left[ -\partial^2 + (\nu + \nu_s) k^2 \right] \rightarrow v \text{ propagator} \]

renormalized viscosity (dressing)

\[ \sum \rightarrow \nu_s \]

interaction of mode/eddy with turbulence (dressing)

D.I.A. is procedure for calculation of self-energy.
Aside: Candidate Time Scales for Model Interaction

1. $\nu k^2 \to$ viscous damping rate
2. $\delta n$ \rightarrow nonlinear energy transfer rate
3. $\left| \frac{\partial \phi}{\partial x} \right| \to$ wave-(resonant particle) autocorrelation

4. $|D_{W\mathcal{W}}| \to$ wave-wave autocorrelation rate set by mis-match dispersion
5. $\Delta W_k \rightarrow$ nonlinear scrambling rate

Examples:

1. Weak Turbulence Theory \rightarrow Wave-Wave (includes weak wave turbulence) $4 < 2, 5$
2. Wave-Particle \rightarrow $3 < 2, 5$, $\frac{1}{2} \times 10^5$
(2) N.S.T. $\rightarrow$ no resonance, $Re \gg 1$

$\sigma_j \sigma_j \sigma_j \rightarrow 0$

$\sigma_j \sigma_j \sigma_j < \sigma_j \sigma_j \sigma_j \Rightarrow$ denormalization needed.
Spectral Equation

\[ \frac{d}{dt} \langle v^2 \rangle + \frac{d}{dx} \langle v^2 v \rangle + \frac{d}{dx} \langle (\partial_x v)^2 \rangle = \langle \overline{f^2} \rangle \]

\[ \langle \frac{10}{(\partial_x v)^2} \rangle \rightarrow \text{NL conserves energy (to boundary terms)} \]

4. have energy balance:

\[ \frac{d}{dt} \langle v^2 \rangle = \langle \overline{\partial_x^2 f} \rangle - \nu \langle (\partial_x v)^2 \rangle \]

- net K.E. / source (forcing)
- viscous dissipation
- S

in \( k \):

\[ \frac{d}{dt} \langle \overline{v^2} \rangle_k = S_k - \nu k^2 \langle v^2 \rangle_k + \frac{T_k}{\Delta} \]

inertial range interaction

where \( \sum_k T_k = 0 \rightarrow \text{NL transfer conserves energy} \)
i.e. expect $T_k$ is sum of two cancelling terms (upon summation) as is anti-symmetric in $k$.

Now: Renormalized theory must respect symmetry conservation laws of original, primitive $\omega_n$.

$$T_k = \frac{1}{3} \left( \frac{d^2}{dx} \frac{d^3}{dx} \right) \left( V_k, V_k^{(3)} \right) \text{ coherent mode coupling}$$

$$= \frac{2}{3} \sum_k \left( V_k, V_k, V_{k+k}^{(3)} \right)$$

$$= -2 \sum_{p=2}^{p+2} \sum_{p+2}^{p+2} \left( V_p, V_p, V_p^{(3)} \right) \text{ incoherent mode coupling}$$

i.e. coherent:

$$\nabla \left( C_k \frac{\nabla}{\nabla} \right) \rightarrow \text{ some as renormalized response function}$$

$$C_k \left( \nabla^2 \right) \rightarrow \text{ dissipation of } \left( \nabla^2 \right)\text{ due to turbulent viscosity}$$

i.e. incoherent:

$$\nabla \left( \nabla \right) \frac{\nabla}{\nabla} \rightarrow \text{ nonlinear noise on } k \text{ via mode coupling}$$
Now, must treat best virtual mode self-consistently. 

\[ \text{mode self-consistency } \Rightarrow \text{include NL mixing in time response} \]

\[ \text{self-consistent field} \]

\[ \mathcal{V}_k(t) = \text{const} \]  

\[ \mathcal{V}_{k+k'}^{(2)} = \mathcal{V}_k^{(2)} + \mathcal{V}_{k+k'}^{(2)} + \mathcal{V}_k^{(2)} \mathcal{V}_{k+k'}^{(2)} \]

\[ = -6(\pi)(k+k') \left[ \mathcal{V}_{k'} \mathcal{V}_k \right] \]

\[ \mathcal{V}_{k+k'}^{(2)} = -6(\pi)(k+k') \int_{k+k'}^{+1} \mathcal{V}_{k'} \mathcal{V}_k \, d\tau \]

\[ \mathcal{V}_{k+k'}^{(2)} = -6(\pi)(k+k') \int_{k+k'}^{+1} \mathcal{V}_{k'} \mathcal{V}_k \, d\tau \]

\[ T_k^C = 2(\pi) \sum_{k'} \mathcal{V}_k^{(2)} \mathcal{V}_{k+k'}^{(2)} \mathcal{V}_{k+k'}^{(2)} \mathcal{V}_k^{(2)} \]

\[ = 2 \sum_{k+k'}^\infty (k+k')^2 \mathcal{V}_k^{(2)} \mathcal{V}_{k+k'}^{(2)} \mathcal{V}_{k+k'}^{(2)} \mathcal{V}_k^{(2)} \]

\[ \text{need model of temporal self-coherence!} \]

\[ \mathcal{V}_k^{(2)} = \text{const} \]

Now, take:

\[ \langle \mathcal{V}_k(t) \mathcal{V}_k(t+\tau) \rangle = \text{const} \tau \]

\[ \langle \mathcal{V}_k(t) \rangle = \text{const} \tau \]

\[ \text{response time} \]

\[ \langle \mathcal{V}_k(t) \mathcal{V}_k(t+\tau) \rangle = 1 \] for convenience.
\[ I^c_k = 2 \sum_{k} (k+\xi)^2 \int_0^\infty dq \exp \left[ - \left( C_{k+\xi} + C_k + C_\xi \right) k \right] \]

\[ \left\langle \mathcal{V}^2 \right\rangle_k \]

\[ \left\langle \mathcal{V}^2 \right\rangle_\xi \]

\[ \left\langle \mathcal{V}^2 \right\rangle_{k+\xi} \]

\[ \text{slow time modulated} \]

\[ \Theta_{k,k',k+\xi} = \int_0^\infty dt \exp \left[ - \left( C_{k+\xi} + C_k + C_\xi \right) k \right] \]

\[ \text{vibronic coherence time} \rightarrow \text{set by modal decorrelation rates} \]

Similarly,

\[ I^I_k = 2 \sum_{\xi} (\xi+\xi)^2 \frac{\Theta_{\xi,\xi}}{\xi+\xi} \left\langle \mathcal{V}^2 \right\rangle_\xi \]

\[ \left\langle \mathcal{V}^2 \right\rangle_\xi \]

\[ \left\langle \mathcal{V}^2 \right\rangle_k \]

\[ \left\langle \mathcal{V}^2 \right\rangle_{k+\xi} \]

\[ \text{energy equation becomes:} \]

\[ \frac{\partial}{\partial t} \left\langle \mathcal{V}^2 \right\rangle_k + \chi \mathcal{V}^2 \left\langle \mathcal{V}^2 \right\rangle_k + T_k = S_k \]

\[ T_k = 2 \sum_{k'} (k+\xi)^2 \frac{\Theta_{k',k+\xi}}{k+\xi} \left\langle \mathcal{V}^2 \right\rangle_{k'} \left\langle \mathcal{V}^2 \right\rangle_k \]

\[ - 2 \sum_{\xi} (\xi+\xi)^2 \frac{\Theta_{\xi,k+\xi}}{\xi+\xi} \left\langle \mathcal{V}^2 \right\rangle_\xi \left\langle \mathcal{V}^2 \right\rangle_k \]