PHYSICS 220 : GROUP THEORY
PROBLEM SET #2

[1] For the group C_{3v}, find the matrices $D^{\rm reg}(R)$ and $D^{\rm reg}(\sigma')$.

[2] The tetrahedral group T is a nonabelian group of order 12 describing the symmetries of the regular tetrahedron. To fix notation, imagine the tetrahedron is lying flat on one of its four triangular faces with the vertex labeled 1 above the surface. The three vertices in contact with the surface are labeled $2, 3, 4$ in counterclockwise fashion as one looks down from above.

(i) Express all 12 elements of T as permutations $\sigma \in S_4$.

(ii) T has four classes: E, $3C_2$, $4C_{3+}$, and $4C_{3-}$. Figure out what these classes mean and assign your permutations to these classes.

(iii) Fill in all the missing entries in the character table below.

(iv) Decompose all products $\Gamma_i \otimes \Gamma_j$ into IRREPs.

T	E	$3C_2$	$4C_{3+}$	$4C_{3-}$
Γ_1	1	1	1	1
Γ_2	$\times$	1	ω	ω^2
Γ_3	$\times$	1	ω^2	ω
Γ_4	$\times$	$\times$	$\times$	$\times$

Table 1: Portion of character table for the tetrahedral group T.

[3] The character table for the alternating group A_5 is given in Tab. 2. There are five classes, which we can call E and $C_{a,b,c,d}$. The table columns are labeled by representatives from each class, with $(12)(34) \in C_a$, $(123) \in C_b$, $(12345) \in C_c$, and $(12354) \in C_d$. Note that $(12345) = (45)(12354)(45)$ in S_5, but (45) is odd, so these two five-cycles are not in the same class within A_5. The quantities s and t are given by $s = \frac{1}{2}(1 - \sqrt{5})$ and $t = \frac{1}{2}(1 + \sqrt{5})$. Bonus factoid: A_5 is the symmetry group of the icosahedron, and is sometimes called the *icosahedral group*.

(i) Find out the number of elements in each of the five classes. As a hint, here's how to figure out N_a, the number of elements in C_a. We ask how many $\sigma \in A_5$ are of the form $(ij)(kl)$ with i, j, k, l all distinct. This means that one symbol among the five $\{1, 2, 3, 4, 5\}$ is left out. There are five ways to do this. Now pick one of the remaining four. It can be in a 2-cycle with any of the other three, so three possibilities there. The remaining two symbols must form their own 2-cycle. So $N_a = 5 \cdot 3 = 15$. Now work out the rest.

(ii) Verify the orthogonality of the rows of the character table.

(iii) Decompose all direct product representations $\Gamma_i \otimes \Gamma_5$.

A_5	E	$(12)(34)$	(123)	(12345)	(12354)
Γ_1	1	1	1	1	1
Γ_2	3	-1	0	s	t
Γ_3	3	-1	0	t	s
Γ_4	4	0	1	-1	-1
Γ_5	5	1	-1	0	0

Table 2: Character table for the alternating group A_5.

[4] Read the material on group presentations in §1.3.4. The presentation of D_n is $\langle\, r, \sigma \,|\, r^n, \sigma^2, (r\sigma)^2 \,\rangle$.

(i) Show that $\{r^k, r^{-k}\}$ form a class for any k. How many distinct such classes are there for even n? For odd n?

(ii) Show that $\sigma_k \equiv \sigma r^{k-1}$ is a reflection satisfying $\sigma_k^2 = 1$. Show that $r^{-1}\sigma_k r = \sigma_{k+2}$, and that if n is odd, all σ_k together form a class, but that if n is even, this breaks into two classes.

(iii) Remembering to include the identity, what is the total number of conjugacy classes in D_n for n even? For n odd?

[5] Consider the projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{E, \sigma, \tau, \sigma\tau\}$ where $D(\sigma) = -iX$, $D(\tau) = -iY$, and $D(\sigma\tau) = -iZ$, where $\{X, Y, Z\}$ are the three Pauli matrices.

(a) Find all the cocycles.

(b) Show that this lifts to a conventional representation of the quaternions Q. Elicit the corresponding exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{\ \psi\ } Q \xrightarrow{\ \pi\ } \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1 \quad . \qquad (1)$$