4

THE BBGKY KINETIC THEORY

We have seen how, in the case of the plasma at moderate and low densities, the philosophy which underlay the older kinetic theory has broken down. The reason for the breakdown is twofold: (1) the long range of the Coulomb force invalidates the model in which two particles interact at a time, in an infinitely short time interval; and (2) the introduction of a probabilistic description of the plasma was unsystematic and vague.

Most of the various attempts to overcome these difficulties have in common that they begin by postulating some sort of probability distribution over the possible states of the plasma. For the purely classical case, which we are considering, this probability distribution is most conveniently defined by giving a (nonnegative) probability density everywhere in the phase space of the many-particle system. The question of *how* the probability distribution, or "ensemble," is to be chosen is deliberately left unanswered. There is no hypothesis of assigning equal probabilities to equal phase-space volumes a priori, as there is in equilibrium statistical mechanics, and an ensemble set up to represent a given set of data will depend upon many things, including the degree of completeness that the data imply. In so far as possible, one hopes to derive a formalism which is *independent* of the details of the probability distribution, invoking only such general properties as differentiability, integrability, and perhaps analyticity. The validity of the procedure can ultimately be determined only by comparing its predictions with experiment.

The approach we shall describe is the so-called "BBGKY theory." It is by no means the only contender among the various theories of nonequilibrium statistical mechanics, but it is the one which appears to us to be the most systematic and the most powerful. To compare it with alternative approaches would take us far afield, and is better left to authors more acquainted with these than we are.

In the interests of clarity we shall develop the theory throughout the following chapters for a one-component plasma with no magnetic field interactions. However, in our later applications in Part III, we shall occasionally require the kinetic equations for more complex plasmas. We have therefore, in Chaps. 5 and 7, derived for future use some more general equations for such plasmas.

4.1 DERIVATION OF THE HIERARCHY; THE CLUSTER EXPANSION

We take as our system of interest an electron plasma moving in a uniform background of immobile positive charge. There are to be N electrons, and the ith one has six phase-space coordinates $X_i \equiv (\mathbf{q}_i,\mathbf{p}_i)$, where the $\mathbf{q}_i$ and $\mathbf{p}_i$ are canonical Hamiltonian variables. For simplicity, we take the particle interactions to be derivable from a scalar potential $\phi(\mathbf{q}) = \phi(|\mathbf{q}|) \equiv e^2/|\mathbf{q}|$, where $\mathbf{q}$ is the separation of any two electrons. The Hamiltonian of the system is then

$$H_N = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \sum_{i<j=1}^{N} \phi_{ij} + \sum_{i=1}^{N} [\Phi_{\text{wall}}(\mathbf{q}_i) + \Psi_{\text{ion}}(\mathbf{q}_i)] \tag{4.1}$$

where $\phi_{ij} = e^2/|\mathbf{q}_i - \mathbf{q}_j|$, Φ_{wall} represents some potential barrier at the boundary of the configuration-space volume of the system, and $\Psi_{\text{ion}}(\mathbf{q}_i)$ is the potential of the ith electron moving in the potential of the smeared-out positive background. Φ_{wall} serves only to turn around those particles striking the wall; it is a purely formal device, and will not appear in the eventual formulation of the theory.

The probability distribution $D_N(X_1 \cdots X_N;t)$ is normalized to 1: $\int D_N\, dX_1 \cdots dX_N = 1$, where the integration runs over the entire accessible region of phase space. By Liouville's theorem, D_N develops in time according to

$$\frac{\partial D_N}{\partial t} + \{D_N;H_N\} = 0 \tag{4.2}$$

where the *Poisson bracket* of any two quantities A and B, which depend on the $\mathbf{q}_i$ and $\mathbf{p}_i$, is defined by

$$\{A;B\} = \sum_{i=1}^{N} \left[\frac{\partial A}{\partial \mathbf{q}_i} \cdot \frac{\partial B}{\partial \mathbf{p}_i} - \frac{\partial A}{\partial \mathbf{p}_i} \cdot \frac{\partial B}{\partial \mathbf{q}_i}\right] \tag{4.3}$$

It should be noted that magnetic, or $O(v^2/c^2)$, interactions could be included in the formalism, as could interaction with an externally imposed magnetic field. A word of caution is perhaps in order: inclusion of magnetic interactions correctly sometimes involves going to a relativistic description. We are presently interested in keeping the formalism as simple as possible.

Let us also agree to treat only D_N's which are symmetric under the interchange of like particle coordinates. This amounts to a (voluntary) limitation to situations in which like particles are taken to be indistinguishable. In kinetic-theory discussions of the test-particle problem (4) one may lift this restriction of symmetry.

We consider that we are dealing with an isolated (as opposed to driven) system: $D_N(X_1 \cdots X_N; 0)$ is understood as given, and (4.2) determines its subsequent development. It is an easy matter to verify that the symmetry and normalization of D_N are preserved by (4.2).

We now define *reduced probability distributions* f_s/V^s, where V is the configuration-space volume of the system, by

$$\frac{f_s}{V^s} \equiv \int D_N \, dX_{s+1} \cdots dX_N \tag{4.4}$$

The f_s/V^s are probability distributions in the phase space of s particles. We can identify f_1 with the quantity which was defined as the probability distribution in one-particle phase space in Chaps. 1 to 3, although it is to be stressed that this is by no means the only way of making the meaning of f_1 exact.

Let us now write out the Poisson bracket in (4.2) in detail:

$$\frac{\partial D_N}{\partial t} + \sum_{i=1}^{N} \frac{\mathbf{p}_i}{m} \cdot \frac{\partial D_N}{\partial \mathbf{q}_i} - e \sum_{i=1}^{N} \mathbf{E}_i \cdot \frac{\partial D_N}{\partial \mathbf{p}_i} - \sum_{i=1}^{N} \frac{\partial \Phi_{\text{wall}}}{\partial \mathbf{q}_i} \cdot \frac{\partial D_N}{\partial \mathbf{p}_i} = 0$$

where

$$e\mathbf{E}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\partial \phi_{ij}}{\partial \mathbf{q}_i} - \frac{\partial \Psi_{\text{ion}}(\mathbf{q}_i)}{\partial \mathbf{q}_i} \tag{4.5}$$

is e times the electric field seen by the ith electron at position $\mathbf{q}_i$. Then, integrating (4.5) over the $X_{s+1} \cdots X_N$ subspace, using the definition of the f_s and the symmetry of D_N, and assuming that D_N vanishes outside some large but finite region of phase space, we have

$$\begin{aligned} \frac{\partial f_s}{\partial t} + \sum_{i=1}^{s} \frac{\mathbf{p}_i}{m} \cdot \frac{\partial f_s}{\partial \mathbf{q}_i} - \sum_{i=1}^{s} \sum_{\substack{j=1 \\ j \neq i}}^{s} \frac{\partial \phi_{ij}}{\partial \mathbf{q}_i} \cdot \frac{\partial f_s}{\partial \mathbf{p}_i} \\ - \frac{(N-s)}{V} \sum_{i=1}^{s} \int \frac{\partial \phi_{i,s+1}}{\partial \mathbf{q}_i} \cdot \frac{\partial f_{s+1}}{\partial \mathbf{p}_i} (X_1 \cdots X_s, X_{s+1}) \, dX_{s+1} \\ = \sum_{i=1}^{s} \left(\frac{\partial \Phi_{\text{wall}}(\mathbf{q}_i)}{\partial \mathbf{q}_i} + \frac{\partial \Psi_{\text{ion}}(\mathbf{q}_i)}{\partial \mathbf{q}_i} \right) \cdot \frac{\partial f_s}{\partial \mathbf{p}_i} \end{aligned} \tag{4.6}$$

In an effort to calculate only intensive (as opposed to extensive, or volume-proportional) effects, we remove the boundary of the plasma

to infinity,

$$N \to \infty$$
$$V \to \infty$$

but in such a way that $n_0 \equiv N/V$, the mean particle number density, remains finite. In this limit, the Φ_{wall} terms vanish at all finite $\mathbf{q}_i$, and the equation for f_s can be written compactly as

$$\frac{\partial f_s}{\partial t} + \{f_s; H_s\} = n_0 \int \left\{ \sum_{i=1}^{s} \phi_{i,s+1}; f_{s+1} \right\} dX_{s+1} \tag{4.7}$$

In (4.7), H_s is the Hamiltonian of s particles,

$$H_s = \sum_{i=1}^{s} \frac{\mathbf{p}_i^2}{2m} + \sum_{i<j=1}^{s} \phi_{ij} + \sum_{i=1}^{s} \Psi_{\text{ion}}(\mathbf{q}_i)$$

Equation (4.7) is formally a linear partial differential equation for f_s in terms of f_{s+1}. In so far as the class of D_N's we are considering is concerned, solution of the coupled chain (4.7) is equivalent to the solution of the Liouville equation, and no simplification has been achieved yet.

For the infinite system, $\Psi_{\text{ion}}(\mathbf{q}_i)$ is only a constant, and so will be dropped from the equations hereafter, in the interests of simplicity. One rather ticklish point should be borne in mind as we go along: sometimes it is convenient to derive an electric field as an auxiliary variable directly from Poisson's equation,

$$\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{E} = 4\pi\rho$$

where ρ is the electrical charge density. The uniform background *does* contribute to this ρ and must balance the electron density in any field-free equilibrium. However, its inclusion does not modify the dynamics expressed in (4.7), so we do not carry it along in the following treatment.

Evaluating the Poisson brackets and writing out the first two members of the hierarchy, we have

$$\frac{\partial f_1}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial f_1}{\partial \mathbf{q}_1} = n_0 \int \frac{\partial \phi_{12}}{\partial \mathbf{q}_1} \cdot \frac{\partial f_2}{\partial \mathbf{p}_1} dX_2 \tag{4.8}$$

$$\frac{\partial f_2}{\partial t} + \left(\frac{\mathbf{p}_1}{m} \cdot \frac{\partial}{\partial \mathbf{q}_1} + \frac{\mathbf{p}_2}{m} \cdot \frac{\partial}{\partial \mathbf{q}_2} \right) f_2 - \left(\frac{\partial \phi_1}{\partial \mathbf{q}_1} \cdot \frac{\partial}{\partial \mathbf{p}_1} + \frac{\partial \phi_{12}}{\partial \mathbf{q}_2} \cdot \frac{\partial}{\partial \mathbf{p}_2} \right) f_2$$
$$= n_0 \int \left(\frac{\partial \phi_{13}}{\partial \mathbf{q}_1} \cdot \frac{\partial f_3}{\partial \mathbf{p}_1} + \frac{\partial \phi_{23}}{\partial \mathbf{q}_2} \cdot \frac{\partial f_3}{\partial \mathbf{p}_2} \right) dX_3 \tag{4.9}$$

It should be noted that we have not yet used the fact that ϕ_{ij} is the

Coulomb interaction; (4.8) and (4.9) apply for an *arbitrary* two-body scalar interaction and might, for example, include a departure from the $1/r^2$ force at short distances.

Expectation values of most of the measurable quantities are calculable in terms of f_1 and f_2. If there were some scheme by which f_1 and f_2 could be calculated without knowing $f_3, f_4, \ldots$, [i.e., if we could break the chain of equations represented by (4.7)], it is apparent that a vast and practical simplification would have been achieved. It should be stated unequivocally that there is not yet even one non-equilibrium situation where a clean proof of the correctness of such a procedure has been given. There do exist good reasons to hope that it is sometimes possible, however, and if it is *not*, plasma physics stops here.

If the system were isolated and in thermal equilibrium, D_N would be constant and uniform on the energy shell in the N-particle phase space, as $N \to \infty$. It is one of the better-known results (2) of classical equilibrium statistical mechanics that it is useful, as $N \to \infty$, to consider writing the f_s in the form

$$\begin{aligned} f_1(X_1) &= f_1(X_1) \\ f_2(X_1X_2) &= f_1(X_1)f_1(X_2) + P(X_1X_2) \\ f_3(X_1X_2X_3) &= f_1(X_1)f_1(X_2)f_1(X_3) + f_1(X_1)P(X_2X_3) + f_1(X_2)P(X_3X_1) \\ &\quad + f_1(X_3)P(X_2X_1) + T(X_1X_2X_3) \end{aligned} \tag{4.10}$$

This is called the *Mayer cluster expansion* (2) and is described in any standard text on statistical mechanics. Up to terms of $O(1/\sqrt{N})$, it turns out that f_1 is the Maxwell distribution,

$$f_1 \text{ (Maxwell)} = \left(\frac{1}{2\pi mKT}\right)^{3/2} e^{-p^2/2mKT}$$

In the theory of probability, two random variables x and y are said to be *uncorrelated*, or independent, if the joint probability $f_2(X,Y)$ that $x = X$ simultaneously with $y = Y$ can be factored:

$$f_2(X,Y) = f_1(X)f_1(Y)$$

This statement means that the *relative* probability that $x = X$, $y = Y_1$ to that for $x = X$, $y = Y_2$ does not depend on X. If this is *not* possible, then the *correlation function* for the variables x and y is defined by

$$P(X,Y) = f_2(X,Y) - f_1(X)f_1(Y) \tag{4.11}$$

Thus P is that part of f_2 which is not factorable.

Thinking loosely, we may imagine the positions in phase space of two particles to be random variables, describable by some joint prob-

ability distribution f_2. This can be true only in some average sense, of course; the positions of particular particles in particular experiments are well-defined delta functions, in classical physics.

The expansion (4.10) is principally useful in situations where the successive pair, triplet, etc., correlation functions P, T, . . . are in some sense "small," or relatively "unimportant," compared with the factorable parts of $f_2, f_3, \ldots$. To see how this might come about, let us imagine that the particles are neutral molecules, with forces which go to zero above a few angstroms separation. Suppose, moreover, that the density of the gas is very low, so that the average kinetic energy of a particle is much greater than its average potential energy. Then it suggests itself that if $|\mathbf{q}_1 - \mathbf{q}_2| \gg$ the range of the forces, "particle 1" will not greatly care where "particle 2" is. However, if particle 2 gets close enough to particle 1, its motion will affect that of particle 1, and the positions in phase space can no longer be prescribed independently. But at low densities, the fraction of the total phase space over which P is nonvanishing, relative to $f_1 f_1$, is very small, and roughly speaking, this statement becomes more accurate as the gas becomes less dense. The notion that the effect of P is negligible compared with $f_1 f_1$ and that the ternary correlation T is "small," relative to Pf_1, seems quite well motivated for this case. The structure of (4.8) and (4.9) rightly suggests that this smallness might become the basis for a *nonequilibrium* perturbation expansion as well. This idea, in one form or another, has occurred to all the authors listed in Ref. 1. Most of the original work was concerned with molecular gases, but our primary concern here is with the modification of the theory for the plasma case. The authors with whose work we shall be principally concerned in the next few chapters are Guernsey (3), Rostoker and Rosenbluth (4), Lenard (5), Balescu (6), and Dupree (7). Since our interest here is more in a clear exposition of the theory than in assigning proper program credits, we do not always say explicitly which contribution is due to which author.

The situation is not so clear-cut for Coulomb interactions, since ϕ_{ij} extends to infinity in this case; one might guess that the pair correlation would have to be given equal footing with the one-body distribution. However, it turns out that over almost all the two-particle phase space (the "hole" over which the statement does not apply has a configuration-space volume $\sim g^2 n_0^{-1}$), the quantity $P/f_1 f_1$ is of order $g = 1/nL_D^3$, where $L_D^2 = k_D^{-2} = KT/4\pi n_0 e^2$, K being Boltzmann's constant, T the temperature. As we have already seen (Table 3.1), g is often quite small. This is true *despite* the long range of the Coulomb force.

The explanation given in Chap. 2 was one of "shielding." Associated with any charge, we said, is a (statistical) cloud of charge of the opposite sign, which screens the potential at large distances, converting its effective form from $1/r$ to $(1/r)e^{-k_D r}$. A careful consideration of the

equilibrium thermodynamical situation (Appendix A) reveals that the exponentially damped term again appears. Stated in the more precise pair-correlation language, for $|\mathbf{q}_1 - \mathbf{q}_2| \gg e^2/KT$, the equilibrium pair correlation is

$$P_{eq} \approx \frac{-e^2}{KT} \frac{e^{-k_D|\mathbf{q}_1-\mathbf{q}_2|}}{|\mathbf{q}_1 - \mathbf{q}_2|} f_1(\mathbf{p}_1) f_1(\mathbf{p}_2) \tag{4.12}$$

where f_1 is a Maxwell distribution. The two-body distribution is therefore, for equilibrium,

$$f_2(X_1X_2) \approx f_1(\mathbf{p}_1) f_1(\mathbf{p}_2) \left[1 - \frac{e^2}{KT} \frac{e^{-k_D|\mathbf{q}_1-\mathbf{q}_2|}}{|\mathbf{q}_1 - \mathbf{q}_2|} \right] \tag{4.13}$$

which rapidly approaches the form $f_1(\mathbf{p}_1)f_1(\mathbf{p}_2)$—the uncorrelated (i.e., noninteracting) form—if k_D^{-1} becomes appreciably less than $|\mathbf{q}_1 - \mathbf{q}_2|$. Therefore, except for a tiny hole cut out of the two-particle phase space, it is an excellent approximation in equilibrium to treat P/f_1f_1 as a small quantity.†

The hope is that such statements will also apply to nonequilibrium situations. So far, this is only a hope, for any situations except those close to thermal equilibrium, as has been stressed by Meeron [8]. Let us bear this restriction in mind throughout the following chapters.

The approach of Rostoker and Rosenbluth is essentially a formal expansion procedure for (4.8) and (4.9), in which $T(X_1X_2X_3)$ is regarded as of higher than first order in g (though not necessarily of second order), P is regarded as of first order, and f_1 as of zeroth order. One keeps only terms of $O(1)$ and $O(g)$, and the result is a closed system of equations for f_1 and P. Thus we make a *nonequilibrium* cluster expansion of (4.8) and (4.9), of the form (4.10):

$$\frac{\partial f_1}{\partial t} + \frac{\mathbf{p}_1}{m} \cdot \frac{\partial f_1}{\partial \mathbf{q}_1} = n \int \frac{\partial \phi_{12}}{\partial \mathbf{q}_1} \cdot \frac{\partial f_1(X_1)}{\partial \mathbf{p}_1} f_1(X_2)\, dX_2 + n_0 \int \frac{\partial \phi_{12}}{\partial \mathbf{q}_1} \cdot \frac{\partial P(X_1X_2)}{\partial \mathbf{p}_1}\, dX_2 \tag{4.14}$$

† So as not to generate unnecessary confusion, we have at this point made the analogy between the plasma case and the case of the rarefied molecular gas seem closer than it actually is, for there is one important difference. Both situations can be viewed as perturbation expansions, but the dimensionless expansion parameter is different. In the molecular gas case, the short range of the forces means that the pair correlation between two particles vanishes except over a region of configuration space of the order of a^3, where a is the range of the force; thus the dimensionless expansion parameter is n_0a^3. However, in the region where P is nonzero, P/f_1f_1 is in no sense "small." For the plasma, on the other hand, the assumption is that the shielding of the Coulomb force so weakens the interaction between particles that $|P/f_1f_1| \ll 1$ for nearly all the two-particle phase space. In both cases the effect is to render the collision integral [the right-hand side of Eq. (4.18)] a "small" quantity.

$$\left(\frac{\partial}{\partial t}+\frac{\mathbf{p}_1}{m}\cdot\frac{\partial}{\partial \mathbf{q}_1}+\frac{\mathbf{p}_2}{m}\cdot\frac{\partial}{\partial \mathbf{q}_2}\right)P(X_1X_2)$$
$$=\left(\frac{\partial\phi_{12}}{\partial \mathbf{q}_1}\cdot\frac{\partial}{\partial \mathbf{p}_1}+\frac{\partial\phi_{12}}{\partial \mathbf{q}_2}\cdot\frac{\partial}{\partial \mathbf{p}_2}\right)[f_1(X_1)f_1(X_2)+P(X_1,X_2)]$$
$$+n_0\left[\int dX_3 f_1(X_3)\frac{\partial\phi_{13}}{\partial \mathbf{q}_1}\cdot\frac{\partial}{\partial \mathbf{p}_1}+(1\leftrightarrow 2)\right]P(X_1X_2)$$
$$+n_0\left[\frac{\partial f_1}{\partial \mathbf{p}_1}\cdot\int\frac{\partial\phi_{13}}{\partial \mathbf{q}_1}P(X_2X_3)\,dX_3+(1\leftrightarrow 2)\right]$$
$$+n_0\int\left[\frac{\partial\phi_{13}}{\partial \mathbf{q}_1}\cdot\frac{\partial}{\partial \mathbf{p}_1}+(1\leftrightarrow 2)\right]T(X_1X_2X_3)\,dX_3 \quad (4.15)$$

We shall use the symbol $(1\leftrightarrow 2)$ to mean the immediately preceding expression, with indices 1 and 2 interchanged.

If $g=1/n_0L_D^3=(4\pi n_0e^2/KT)^{3/2}/n_0$ is to be "small," we must specify which quantities are to be held finite for the meaning of the expansion to be precise for *all* the terms in (4.14) and (4.15). If the *thermal velocity* $\sim\sqrt{KT/m}$ is of $O(1)$, then $g=(n_0e^2/m)^{3/2}n_0^{-1}\times O(1)$. If the *plasma frequency* $\sqrt{4\pi n_0e^2/m}$ is also to be held finite, then $1/n_0$ must be a small, or $O(g)$, quantity. Therefore e^2/m must be $O(g)$. If the charge-mass ratio of an electron is to be held finite, then e/m must be $O(1)$. It finally follows that e and m must be $O(g)$ quantities.

We have shown that, if we keep the thermal velocity, the plasma frequency, and the acceleration imparted to an electron by a finite electric field all finite, then e, m, and $1/n_0$ (sometimes called "discreteness parameters") must all be treated as $O(g)$ quantities, with $P/f_1f_1=O(g)$ and $\mathbf{q}$, $\mathbf{p}/m$ both $O(1)$. These assignments have an element of arbitrariness in them and are physically, rather than mathematically, motivated. The (*unproved*) assumption which allows us to close the system is that T is of higher than first order in g.

Rewriting (4.14) and (4.15) under these assumptions, we get

$$\frac{\partial f_1}{\partial t}+\frac{\mathbf{p}_1}{m}\cdot\frac{\partial f_1}{\partial \mathbf{q}_1}-n_0\left[\int\frac{\partial\phi_{13}}{\partial \mathbf{q}_1}f_1(X_2)\,dX_2\right]\cdot\frac{\partial f_1}{\partial \mathbf{p}_1}$$
$$=n_0\int\frac{\partial\phi_{12}}{\partial \mathbf{q}_1}\cdot\frac{\partial P(X_1X_2)}{\partial \mathbf{p}_1}\,dX_2 \quad (4.16)$$

$$\frac{\partial}{\partial t}P(X_1X_2)+\left[\frac{\mathbf{p}_1}{m}\cdot\frac{\partial}{\partial \mathbf{q}_1}+\frac{\mathbf{p}_2}{m}\cdot\frac{\partial}{\partial \mathbf{q}_2}\right]P(X_1X_2)$$
$$=\left[\frac{\partial\phi_{12}}{\partial \mathbf{q}_1}\cdot\frac{\partial}{\partial \mathbf{p}_1}+(1\leftrightarrow 2)\right]f_1(X_1)f_1(X_2)$$
$$+n_0\left[\int dX_3 f_1(X_3)\frac{\partial\phi_{13}}{\partial \mathbf{q}_1}\cdot\frac{\partial}{\partial \mathbf{p}_1}+(1\leftrightarrow 2)\right]P(X_1X_2)$$
$$+n_0\left[\frac{\partial f_1}{\partial \mathbf{p}_1}\cdot\int\frac{\partial\phi_{13}}{\partial \mathbf{q}_1}P(X_2X_3)\,dX_3+(1\leftrightarrow 2)\right] \quad (4.17)$$

Most of the literature is being written in terms of velocity-space, rather than momentum-space, distributions, so we set

$$f_1(\mathbf{q},\mathbf{p})\, d\mathbf{p} = f(\mathbf{x},\mathbf{v})\, d\mathbf{v}$$
$$P(X_1X_2)\, d\mathbf{p}_1\, d\mathbf{p}_2 = P(\mathbf{x}_1\mathbf{v}_1\mathbf{x}_2\mathbf{v}_2)\, d\mathbf{v}_1\, d\mathbf{v}_2$$

and rewrite (4.16) and (4.17) in the form which we shall use most:

$$\frac{\partial f}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f}{\partial \mathbf{x}_1} - \frac{n_0}{m}\left[\int d\mathbf{x}_2\, d\mathbf{v}_2 \frac{\partial \phi_{12}}{\partial \mathbf{x}_1} f(\mathbf{x}_2\mathbf{v}_2)\right] \cdot \frac{\partial f(\mathbf{x}_1\mathbf{v}_1)}{\partial \mathbf{v}_1} = \frac{n_0}{m}\int \frac{\partial \phi_{12}}{\partial \mathbf{x}_1} \cdot \frac{\partial P(\mathbf{x}_1\mathbf{v}_1\mathbf{x}_2\mathbf{v}_2)}{\partial \mathbf{v}_1}\, d\mathbf{x}_2\, d\mathbf{v}_2 \quad (4.18)$$

$$\frac{\partial P}{\partial t} + \left(\mathbf{v}_1 \cdot \frac{\partial}{\partial \mathbf{x}_1} + \mathbf{v}_2 \cdot \frac{\partial}{\partial \mathbf{x}_2}\right) P = \frac{1}{m}\left[\frac{\partial \phi_{12}}{\partial \mathbf{x}_1} \cdot \frac{\partial}{\partial \mathbf{v}_1} + (1 \leftrightarrow 2)\right] f(\mathbf{x}_1\mathbf{v}_1) f(\mathbf{x}_2\mathbf{v}_2) + \frac{n_0}{m}\left[\int d\mathbf{x}_3\, d\mathbf{v}_3 f(\mathbf{x}_3\mathbf{v}_3) \frac{\partial \phi_{13}}{\partial \mathbf{x}_1} \cdot \frac{\partial}{\partial \mathbf{v}_1} + (1 \leftrightarrow 2)\right] P + \frac{n_0}{m}\left[\frac{\partial f(\mathbf{x}_1\mathbf{v}_1)}{\partial \mathbf{v}_1} \cdot \int \frac{\partial \phi_{13}}{\partial \mathbf{x}_1} P(\mathbf{x}_2\mathbf{v}_2\mathbf{x}_3\mathbf{v}_3)\, d\mathbf{x}_3\, d\mathbf{v}_3 + (1 \leftrightarrow 2)\right] \quad (4.19)$$

Equations (4.18) and (4.19) have proved far too difficult to solve. A large amount of the effort of theoretical plasma physics has been devoted to finding approximate schemes for doing so, and some of these attempts are described in the following chapters. First we shall describe an even simpler (but still not generally solvable) situation: the limit $g = 0$, in which all terms in (4.19) are trivially zero and the right-hand side of (4.18) vanishes. What is left of (4.18) is then called the *Vlasov equation*, or sometimes the *collisionless Boltzmann equation*, or *correlationless kinetic equation*. The nomenclature is unfortunately far from standard.

Exercise. Show that $f = f_m$, Maxwell's distribution, and $P = P_{eq}$, as given by Eq. (4.12), is a time-independent solution of (4.16) and (4.17), or equivalently, (4.18) and (4.19). Can you find any others?

Show that P_{eq} can be written as

$$P_{eq} = \int d\mathbf{k}_1 \int d\mathbf{k}_2\, e^{i(\mathbf{k}_1\cdot\mathbf{x}_1 + \mathbf{k}_2\cdot\mathbf{x}_2)} f_m(\mathbf{v}_1) f_m(\mathbf{v}_2) \chi(\mathbf{k}_1)\delta(\mathbf{k}_1 + \mathbf{k}_2)$$

where

$$\chi(k) = -\frac{1}{8\pi^3 n_0} \frac{k_D^2}{k^2 + k_D^2} \quad (4.20)$$

REFERENCES

1. The BBGKY theory has been described in many places. The following are, in a sense, the "original" references:
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The following are only a very few of the authors who have made contributions to the BBGKY theory of plasmas. We have not intended to assign credit to particular authors for particular innovations in all cases. The reader is urged to consult the original sources and form his own opinions.

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5

THE VLASOV EQUATION; ELECTRON PLASMA OSCILLATIONS

The limit $g \to 0$ (or equivalently, e, m, $1/n_0 \to 0$) as described at the end of the last chapter simplifies (4.18) and (4.19) considerably. The system in that limit reduces to the Vlasov equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f}{\partial \mathbf{x}_1} - \frac{n_0}{m} \int d\mathbf{x}_2 \, d\mathbf{v}_2 \frac{\partial \phi_{12}}{\partial \mathbf{x}_1} f(\mathbf{x}_2, \mathbf{v}_2) \cdot \frac{\partial f(\mathbf{x}_1, \mathbf{v}_1)}{\partial \mathbf{v}_1} = 0 \qquad (5.1)$$

Equation (5.1) is deceptively simple-looking. An enormous amount of effort has been spent on it in the last few years, but the only fully general and yet tractable solutions which exist are perturbation-theoretic ones. We shall first describe some of its general properties, and then investigate these linear solutions in some detail.

Several purposes are served by going into such detail. First, Eq. (5.1) and its generalizations to include positive-ion motions and magnetic forces have been the basis of many of the practical studies of high-temperature plasma physics. Second, the sort of mathematics that appears when one sets about solving the linearized version of (5.1) arises over and over again throughout the subject. The least painful way to introduce this type of calculation (which relies heavily on complex function theory) is a thorough study of Landau's solution to the linearized version of (5.1). When read in the light of a clear understanding of Landau's paper, much of what appears superficially complicated in modern plasma theory becomes actually quite simple.

5.1 GENERAL PROPERTIES OF VLASOV'S EQUATION (1)

It is instructive to write the electric field in (5.1) explicitly:

$$\frac{\partial f}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial f}{\partial \mathbf{x}_1} - \frac{e}{m}\mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}_1} = 0 \tag{5.2}$$

where

$$eE(\mathbf{x}_1,t) = n_0 \int d\mathbf{x}_2\, d\mathbf{v}_2 \frac{\partial \phi_{12}}{\partial \mathbf{x}_1} f(\mathbf{x}_2,\mathbf{v}_2,t) \tag{5.3}$$

It is clear that (5.2) is simply a statement that the time derivative of f is zero, computed along a particle trajectory, the equations of which are

$$\begin{aligned} \frac{d\mathbf{x}(t)}{dt} &= \mathbf{v}(t) \\ \frac{d\mathbf{v}(t)}{dt} &= -\frac{e}{m}\mathbf{E}(\mathbf{x}(t),t) \end{aligned} \tag{5.4}$$

In the language of partial differential equations, the orbits (5.4) are the *characteristics* of the partial differential equation (5.2) or (5.1). The general solution of (5.2), therefore, is any function of the constants of the motion (which may contain a time dependence) described by (5.4). The thing that makes the problem so difficult is that $\mathbf{E}$ itself is not given, but must be *self-consistently* determined by (5.3) at every instant in time. The identification of the solution of (5.1) with a function of constants of the particle motion has not proved an algebraically useful device, to any great extent.

That (5.1) is *time-reversible* may readily be seen by noting its invariance under the transformation

$$\begin{aligned} \mathbf{v}_1 &\to -\mathbf{v}_1 \\ \mathbf{x}_1 &\to \mathbf{x}_1 \\ t &\to -t \end{aligned}$$

That it also conserves what we defined as entropy for the molecular gas in Chap. 1 is readily seen by observing that

$$\begin{aligned} \frac{\partial}{\partial t}\int_V f \ln f\, d\mathbf{x}_1\, d\mathbf{v}_1 &= \int_V \frac{\partial f}{\partial t} d\mathbf{x}_1\, d\mathbf{v}_1 + \int_V \ln f \frac{\partial f}{\partial t} d\mathbf{x}_1\, d\mathbf{v}_1 \\ &= \int_V \left(-\mathbf{v}_1 \cdot \frac{\partial f}{\partial \mathbf{x}_1} + \frac{e}{m}\mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}_1}\right) d\mathbf{x}_1\, d\mathbf{v}_1 \\ &\quad + \int_V \ln f \left(-\mathbf{v}_1 \cdot \frac{\partial f}{\partial \mathbf{x}_1} + \frac{e}{m}\mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}_1}\right) d\mathbf{x}_1 d\mathbf{v}_1 \\ &= 0 \end{aligned}$$

where we have performed integrations by parts and assumed that f obeys periodic boundary conditions over some arbitrarily large but finite volume in $(\mathbf{x}_1,\mathbf{v}_1)$ space, which we have called V. If the positive background were allowed to move also, one could show that (5.1) conserves energy and momentum, as well as particle number.

We may show that if $f > 0$ initially, all $\mathbf{x}_1$, $\mathbf{v}_1$, it stays ≥ 0 for all time. For if f were initially >0 everywhere in the $\mathbf{x}_1$, $\mathbf{v}_1$ space and went negative at some time t, at some particular point $\mathbf{x}$, $\mathbf{v}$, the following conditions would have to apply at this point:

$$(a) \qquad \frac{\partial f}{\partial t} < 0$$

$$(b) \qquad \frac{\partial f}{\partial \mathbf{x}} = 0$$

$$(c) \qquad \frac{\partial f}{\partial \mathbf{v}} = 0$$

That these are manifestly inconsistent can be seen from (5.2).

So much for the general properties of Vlasov's equation. We turn now to Landau's perturbation-theoretic solution of it.

5.2 LANDAU'S SOLUTION (2)

Observe first that any $f(\mathbf{x}_1,\mathbf{v}_1,t) = f_0(\mathbf{v})$ only is a solution of (5.1) for which

$$\int f_0(\mathbf{v}_2) \frac{\partial \phi_{12}}{\partial \mathbf{x}_1} d\mathbf{v}_2 \, d\mathbf{x}_2 = 0$$

f_0 does *not* have to be a Maxwell distribution, in keeping with the complete reversibility of (5.1). The electron number density, $n_0 \int f_0(\mathbf{v}) \, d\mathbf{v}$, it is well to bear in mind, must be balanced by the "uniform, immobile, positive background" we have been assuming in these calculations. This point, while not apparent from (5.1), has its mathematical origin in the fact that the limit $N \to \infty$, $V \to \infty$ has to be performed in such a way that the $\int f_0(\mathbf{v}_2)(\partial \phi_{12}/\partial \mathbf{x}_1) \, d\mathbf{v}_2 \, d\mathbf{x}_2$ remains finite, otherwise. This requires a special shape of the surface at infinity, which is clearly something we should like to avoid on physical grounds. A neutralizing positive background removes the restriction, since there *is* no net "charge at infinity" contributing to the electric field. This positive background contributed nothing of significance to the Hamiltonian, so we did not bother to carry it along formally. It should be stressed that the formalism *could* be developed, treating the protons on an equal footing with the electrons. Some general results for multicomponent systems are collected in later sections.

Landau's approach was to study spatially varying perturbations on a uniform equilibrium, using (5.1), assuming

$$f(\mathbf{x}_1,\mathbf{v}_1,t) = f_0(\mathbf{v}_1) + f^{(1)}(\mathbf{x}_1,\mathbf{v}_1,t) \tag{5.5}$$

keeping only first-order terms in $f^{(1)}$. The resulting equation for $f^{(1)}$ is linear, and can be solved by a combination of Fourier and Laplace analysis.