Weak Turbulence Theory of Velocity Space Diffusion and the Nonlinear Landau Damping of Waves

Wallace M. Manheimer, and Thomas H. Dupree

Citation: The Physics of Fluids **11**, 2709 (1968); doi: 10.1063/1.1691878
View online: https://doi.org/10.1063/1.1691878
View Table of Contents: http://aip.scitation.org/toc/pfl/11/12
Published by the American Institute of Physics

Weak Turbulence Theory of Velocity Space Diffusion and the Nonlinear Landau Damping of Waves

WALLACE M. MANHEIMER AND THOMAS H. DUPREE
Department of Physics, Department of Nuclear Engineering
and
Research Laboratory of Electronics, Massachusetts Institute of Technology
Cambridge, Massachusetts
(Received 12 February 1968)

The particle–wave interaction to arbitrary order in perturbation theory is investigated. One finds that to every order the ensemble average distribution function obeys a diffusion equation. The diffusion constant specifies how the energy and momentum of particles change in time. By conservation of energy and momentum between waves and particles, the growth or damping rates of waves to any order can be determined. The growth rates for nonlinear instabilities and the damping rates for nonlinear Landau damping are explicitly calculated.

I. INTRODUCTION

Collisionless plasmas, both in space and in the laboratory, are rarely in thermal equilibrium. Generally, the plasma coexists with a spectrum of turbulent waves. The time evolution of a particle distribution function is then governed by the manner in which the individual particles interact with these turbulent waves. Clearly, this interaction is in some way responsible for anomalous diffusion and turbulent heating of laboratory plasmas, and for stochastic acceleration of high-energy particles in astrophysical plasmas.

The simplest kinetic equation for the distribution function $f(\mathbf{v}, t)$ in the presence of turbulent waves is the quasilinear diffusion equation.[1-5] This can be derived by using the Fokker–Planck equation. Here, one assumes the turbulence is sufficiently weak so that a particle orbit differs only slightly from a straight-line orbit. This deviation is then expressed as a power series in the turbulent field strength. The first two terms lead to the familiar quasilinear diffusion equation.

We will show that to any order in perturbation theory, the Fokker–Planck equation reduces to a diffusion equation

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \mathsf{D} \cdot \frac{\partial f}{\partial \mathbf{v}}. \tag{1}$$

A method of calculating D to any order will be derived. The diffusion constant to fourth order in turbulent field strength will be explicitly calculated for a variety of turbulent spectra.

Once D is calculated one can determine not only how particle energy changes with time, but also how wave energy changes. For example, we will see that diffusion to fourth order may be interpreted as an interaction of two waves with frequency and wavenumber $(\omega, \mathbf{k})$, $(\omega', \mathbf{k}')$ with resonant particles whose velocity is such that

$$(\mathbf{k} \pm \mathbf{k}')\cdot\mathbf{v} - (\omega \pm \omega') = 0. \tag{2}$$

The dynamics of this interaction may be such that net energy is lost by the waves. If this is the case, the process is called nonlinear Landau damping. On the other hand, the dynamics may be such that the waves gain energy; then, we have a nonlinear instability. The instability may be such that both waves grow. In this case the instability is often called explosive. In the other type of two-wave instability, one wave may grow while the other damps, in which case the growth is much slower. In any of these cases, the growth or damping rates may be calculated in a very simple way from the fourth-order velocity space diffusion constant. We shall present calculations of damping rates for nonlinear Landau damping and of growth rates for nonlinear instabilities.

Using a Hamiltonian formalism, we will show how growth rates may be calculated to any order in perturbation theory in terms of the appropriate velocity space diffusion constants. Finally, we will show how diffusion to higher orders provides a means by which a plasma can tap "free energy"[6] which is not available to it in the linear theory.

---

1 W. E. Drummond and D. Pines, Ann. Phys. (N.Y.) **28**, 478 (1964).
2 A. A. Vedenov, E. P. Velekov, and R. Z. Sagdeev, Nucl. Fusion Suppl. Pt. 2, 465 (1962).
3 D. B. Chang, Phys. Fluids **7**, 1980 (1964).
4 P. A. Sturrock, Phys. Rev. **141**, 186 (1966).
5 I. Bernstein and F. Englemann, Phys. Fluids **9**, 937 (1966).
6 C. S. Gardner, Phys. Fluids **6**, 839 (1963).

II. REDUCTION OF THE FOKKER–PLANCK EQUATION TO A DIFFUSION EQUATION

In any turbulent plasma one assumes there are random forces on the individual particles. If the autocorrelation time of the stochastic acceleration is small compared with the time scale on which the ensemble average distribution function $f(\mathbf{v}, t)$ changes, the equation for $f(\mathbf{v}, t)$ is a Fokker–Planck equation,[7,8]

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} = \frac{1}{2}\frac{\partial^2}{\partial v_i\,\partial v_j}\frac{\langle \Delta v_i\,\Delta v_j\rangle}{\Delta t} f - \frac{\partial}{\partial v_i}\frac{\langle \Delta v_i\rangle}{\Delta t} f. \quad (3)$$

If we assume that the acceleration comes from a random force field dependent upon instantaneous particle position and velocity, the Langevin equation becomes

$$\frac{d^2\mathbf{x}}{dt^2} = \frac{d\mathbf{v}}{dt} = \mathbf{F}(\mathbf{x}, \mathbf{v}, t). \quad (4)$$

For the turbulent plasmas which we will be discussing, F represents the random electric and magnetic fields in the turbulent plasma.

Here we will deal with cases in which the turbulence is sufficiently weak that the orbit of a particle differs little from an unperturbed orbit. That is, the deviation from an unperturbed orbit, $\Delta\mathbf{x}$, $\Delta\mathbf{v}$ during the autocorrelation time of $\mathbf{F}(t)$ must be small compared with the scale on which F fluctuates in $\mathbf{x}$ and $\mathbf{v}$. We will see that this means that $\Delta\mathbf{x}$ must be small compared with a typical wavelength of the random fields.

We will now examine the condition under which the Fokker–Planck equation reduces to a diffusion equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} = \frac{\partial}{\partial v_i} T_{ij}\frac{\partial f}{\partial v_j} \quad (5)$$

to every order in orbit perturbation theory. As will become apparent, this condition depends on the details of the orbit of the particle in six-dimensional phase space under the random acceleration $\mathbf{F}$.

We will determine a condition for Eq. (5) to apply by comparing the acceleration which a particle feels as a function of time to the acceleration which it would feel if it moved along a straight-line orbit. If at time $t = 0$, a particle's position is $(\mathbf{x}, \mathbf{v})$, its acceleration as a function of time will be denoted $\mathbf{a}(t; \mathbf{x}, \mathbf{v})$. The diffusion tensor and dynamical friction of Eq. (3) may be written as

$$\begin{aligned}\frac{\langle \Delta v_i\,\Delta v_j\rangle}{2\,\Delta t} &= \frac{1}{2\,\Delta t}\cdot\int_0^{\Delta t} dt_1\int_0^{\Delta t} dt_2\langle a_i(t_1; \mathbf{x}, \mathbf{v})a_j(t_2; \mathbf{x}, \mathbf{v})\rangle \\ &= \frac{1}{2}\left[\int_0^t \langle a_i(t; \mathbf{x}, \mathbf{v})a_j(\tau, \mathbf{x}, \mathbf{v})\rangle\, d\tau + \int_0^t \langle a_j(t; \mathbf{x}, \mathbf{v})a_i(\tau, \mathbf{x}, \mathbf{v})\rangle\, d\tau\right] \\ &= \tfrac{1}{2}[D_{ij} + D_{ji}]\end{aligned} \quad (6a)$$

and

$$\frac{\langle \Delta v_i\rangle}{\Delta t} = \int_0^{\Delta t}\langle a_i(t; \mathbf{x}, \mathbf{v})\, dt\rangle. \quad (6b)$$

Since the tensor indices serve only as indices of summation in Eq. (3), the first term on the right can be rewritten as $(\partial^2/\partial v_i\partial v_j)\, D_{ij}f$.

The acceleration which this particle would feel if it moved along an unperturbed trajectory is simply $\mathbf{F}(\mathbf{x} + \mathbf{v}t, \mathbf{v}, t)$. The functions $\mathbf{a}$ and $\mathbf{F}$ can be related to each other as follows:

$$\mathbf{F}(\mathbf{x} + \mathbf{v}t, \mathbf{v}, t) = \mathbf{a}[t; \mathbf{x} + \boldsymbol{\theta}(t), \mathbf{v} + \boldsymbol{\psi}(t)]. \quad (7)$$

For any time t, $\mathbf{x}' = \mathbf{x} + \boldsymbol{\theta}$ and $\mathbf{v}' = \mathbf{v} + \boldsymbol{\psi}$ are simply the initial positions a particle must have if it is to be at $(\mathbf{x} + \mathbf{v}t, \mathbf{v})$ at time t. The functions $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$ are assumed small compared to the scale length on which $\mathbf{F}$ and $\mathbf{a}$ fluctuate. Thus, the right-hand side of Eq. (7) can be expanded to give

$$\begin{aligned}\mathbf{F}(\mathbf{x} + \mathbf{v}t, \mathbf{v}, t) &= \mathbf{a}(t; \mathbf{x}, \mathbf{v}) + \boldsymbol{\theta}(t)\cdot\frac{\partial}{\partial \mathbf{x}}\mathbf{a}(t, \mathbf{x}, \mathbf{v}) \\ &\quad + \boldsymbol{\psi}(t)\cdot\frac{\partial}{\partial \mathbf{v}}\mathbf{a}(t, \mathbf{x}, \mathbf{v}) \\ &\quad + O\{[\mathbf{a}(\Delta\mathbf{x})]^2, \mathbf{a}(\Delta\mathbf{v})^2\} + \cdots .\end{aligned} \quad (8)$$

We can now write $\boldsymbol{\theta}(t)$, $\boldsymbol{\psi}(t)$ in terms of $\Delta\mathbf{x}(t; \mathbf{x}', \mathbf{v}')$ and $\Delta\mathbf{v}(t; \mathbf{x}', \mathbf{v}')$. These latter quantities are the changes from the unperturbed values of position and velocity during a time t of a particle which began at $\mathbf{x}'$, $\mathbf{v}'$. Thus,

$$\begin{aligned}\boldsymbol{\theta}(t) &= (\mathbf{v} - \mathbf{v}')t - \Delta\mathbf{x}(t; \mathbf{x}', \mathbf{v}'), \\ \boldsymbol{\psi}(t) &= -\Delta\mathbf{v}(t, \mathbf{x}', \mathbf{v}').\end{aligned} \quad (9)$$

The above result may easily be visualized in two-dimensional phase space as demonstrated in Fig. 1. Here the coordinates of the various points are $A = (\mathbf{x}, \mathbf{v})$, $B = [\mathbf{x} + \Delta\mathbf{x}(t; \mathbf{x}, \mathbf{v}), \mathbf{v} + \Delta\mathbf{v}(t; \mathbf{x}, \mathbf{v})]$, $C = (\mathbf{x} + \mathbf{v}t, \mathbf{v}) = [\mathbf{x}' + \mathbf{v}'t + \Delta\mathbf{x}(t; \mathbf{x}', \mathbf{v}'), \mathbf{v}' + \Delta\mathbf{v}'(t, \mathbf{x}', \mathbf{v}')]$, $D = (\mathbf{x}', \mathbf{v}') = (\mathbf{x} + \boldsymbol{\theta}, \mathbf{v} + \boldsymbol{\psi})$, $E = (\mathbf{x}' + \mathbf{v}'t, \mathbf{v}')$. The line AD represents the phase

[7] S. Chandrasekhar, Rev. Mod. Phys. 15, 1 (1943).
[8] M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. 17, 323 (1945).

space vector (θ, ψ), and the line EC represents the phase space vector $[\Delta\mathbf{x}(t, \mathbf{x}', \mathbf{v}'), \Delta\mathbf{v}(t, \mathbf{x}', \mathbf{v}')]$. Now, in Eq. (9), $\Delta\mathbf{x}$ and $\Delta\mathbf{v}$ can also be expanded exactly as was $\mathbf{a}$. The result is

$$\Delta\mathbf{x}(t; \mathbf{x}', \mathbf{v}') = \Delta\mathbf{x}(t; \mathbf{x}, \mathbf{v}) + \cdots O(\Delta\mathbf{x})^2, \qquad (\Delta\mathbf{v})^2$$

$$\Delta\mathbf{v}(t; \mathbf{x}', \mathbf{v}') = \Delta\mathbf{v}(t; \mathbf{x}, \mathbf{v}) + \cdots O(\Delta\mathbf{x})^2, \qquad (\Delta\mathbf{v})^2. \tag{10}$$

Inserting Eqs. (9) and (10) in Eq. (8), one obtains

$$\mathbf{F}(\mathbf{x} + \mathbf{v}t, \mathbf{v}, t) = \mathbf{a} + (\mathbf{v} - \mathbf{v}')t\cdot\frac{\partial}{\partial\mathbf{x}}\mathbf{a} - \Delta\mathbf{x}\cdot\frac{\partial}{\partial\mathbf{x}}\mathbf{a} - \Delta\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{v}}\mathbf{a} + \cdots, \tag{11}$$

where the arguments of the variables on the right are now understood to be $(t; \mathbf{x}, \mathbf{v})$. Ensemble averaging each side of Eq. (11) and integrating over t from 0 to Δt yields

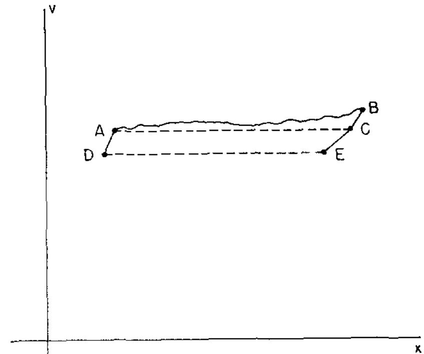

FIG. 1. Orbits in two-dimensional phase space.

$$\int_0^{\Delta t} \langle \mathbf{F}(\mathbf{x} + \mathbf{v}t, \mathbf{v}, t)\rangle\, dt = \langle\Delta\mathbf{v}\rangle - \int_0^{\Delta t} dt \left\langle \frac{\partial}{\partial\mathbf{x}}\cdot(\Delta\mathbf{x}\,\mathbf{a}) + \frac{\partial}{\partial\mathbf{v}}\cdot(\Delta\mathbf{v}\,\mathbf{a}) - \mathbf{a}\frac{\partial}{\partial\mathbf{x}}\cdot\Delta\mathbf{x} - \mathbf{a}\frac{\partial}{\partial\mathbf{v}}\Delta\mathbf{v} - (\mathbf{v} - \mathbf{v}')t\cdot\frac{\partial}{\partial\mathbf{x}}\mathbf{a}\right\rangle. \tag{12}$$

We will rewrite the third and fourth terms in the integral in terms of the Jacobian of the transformation[9] from $(\mathbf{x}, \mathbf{v})$ to $(\mathbf{x} + \mathbf{v}t + \Delta\mathbf{x}, \mathbf{v} + \Delta\mathbf{v})$ generated by the stochastic acceleration $\mathbf{F}$. By looking at the determinant form of the Jacobian, it is clear that

$$J = 1 + \frac{\partial}{\partial\mathbf{x}}\cdot\Delta\mathbf{x} + \frac{\partial}{\partial\mathbf{v}}\Delta\mathbf{v} + Q, \tag{13}$$

where Q has products of two or more terms such as

$$\frac{\partial(\Delta\mathbf{x}, \Delta\mathbf{v})}{\partial(\mathbf{x}, \mathbf{v})} = O[(\Delta\mathbf{x})^2, (\Delta\mathbf{v})^2].$$

Thus, only keeping terms in Eq. (12) which are linear in Δt, we have

$$\int_0^{\Delta t} \langle \mathbf{F}(\mathbf{x} + \mathbf{v}t, \mathbf{v}, t)\, dt\rangle = \langle\Delta\mathbf{v}\rangle - \frac{\partial}{\partial\mathbf{x}}\cdot\left\langle\int_0^{\Delta t} \Delta\mathbf{x}\,\mathbf{a}\, dt\right\rangle - \frac{\partial}{\partial\mathbf{v}}\cdot\left\langle\int_0^{\Delta t} \Delta\mathbf{v}\,\mathbf{a}\, dt\right\rangle + \int_0^{\Delta t} \langle\mathbf{a}(J - 1)\rangle\, dt. \tag{14}$$

Rewriting $\mathbf{v}$ in terms of the time integral of $\mathbf{a}$ and assuming the ensemble averaged quantities on the right have no spacial dependence, we obtain (in tensor notation)

$$\int_0^{\Delta t} \langle F_i(\mathbf{x} + \mathbf{v}t, \mathbf{v}, t)\rangle = \langle\Delta v_i\rangle - \frac{\partial}{\partial v_j}\int_0^{\Delta t} dt \int_0^t d\tau\langle a_j(\tau; \mathbf{x}, \mathbf{v})a_i(t; \mathbf{x}, \mathbf{v})\rangle + \int_0^{\Delta t} \langle a_i(J - 1)\rangle\, dt. \tag{15}$$

Thus we see that if the ensemble average acceleration along the particle's unperturbed orbit is zero, and if $J = 1$, the two terms on the right-hand side of Eq. (3) may be combined to the form

$$\frac{\partial f}{\partial t} + v\cdot\frac{\partial f}{\partial\mathbf{x}} = \frac{\partial}{\partial v_i} D_{ij}\frac{\partial f}{\partial v_j}, \tag{16}$$

where D_{ij} is defined by Eq. (6a).

The Jacobian will be equal to one if the transformation from $\mathbf{x}$, $\mathbf{v}$ to $\mathbf{x} + \mathbf{v}t + \Delta x$, $\mathbf{v} + \Delta v$ is a contact transformation. Thus if $\mathbf{F}$ is any random electric or magnetic field whose average value is zero, then J will be unity so Eq. (3) reduces to Eq. (16) or (6). This diffusion equation has been derived previously using a quantum-mechanical formulation,[10] and also has been derived classically to lowest order in orbit perturbation theory.[4]

[9] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1950), p. 245.

[10] V. N. Tsytovich, Usp. Fiz. Nauk **89**, 89 (1966) [Sov. Phys.—Usp. **9**, 370 (1966)].

Finally, let us remark that D_{ij} as defined in Eq. (6a) is not necessarily symmetric. To determine its symmetry properties, one must actually calculate them. In every case to be studied here, it will turn out that D_{ij} is indeed symmetric.

III. PARTICLE DIFFUSION TO ALL ORDERS

We will now restrict ourselves to the case of a turbulent, infinite homogeneous plasma where the stochastic acceleration is produced by electric fields of the form

$$\mathbf{E}(\mathbf{x}, t) = \frac{m}{e}\frac{d\mathbf{v}}{dt} = \sum_{\mathbf{k}}\sum_{p} \mathbf{E}(\mathbf{k}) \exp\left[i(\mathbf{k}\cdot\mathbf{x} - \omega t + \phi_{\mathbf{k}})\right]. \quad (17)$$

In Eq. (3.1), $\phi_{\mathbf{k}}$ is the random phase of the kth wave while $\omega = \omega(\mathbf{k})$ and is determined by the dispersion equation.

In what follows, ω will be assumed pure real; the case of complex ω will be treated in a later section.

We shall adopt a convention in which the wave vector $\mathbf{k}$ will be defined as being in the direction of the wave momentum. That is, $\mathbf{k}$ is in the direction of the wave phase velocity for positive energy waves, and is directed opposite to the phase velocity for negative energy waves. Therefore $\omega(\mathbf{k})$ in this convention is numerically positive for positive energy waves and is negative for negative energy waves. We will return in a later section to discuss how the wave energy is determined.

In accordance with this convention, the $\mathbf{k}$ summation in (17) is limited to $\mathbf{k}$ in the direction of wave momentum. Since the electric field in Eq. (17) must be real, to every complex term in Eq. (17), we must add a complex conjugate. These complex conjugate terms are included by introducing the second summation $\sum_p$ which sums over plus and minus k.

We shall now use the Fokker–Planck equation to find $\partial f(\mathbf{v}, t)/\partial t$ for the case in which the ensemble is specified by the random phases $\phi_{\mathbf{k}}$. The validity of the Fokker–Planck equation in this particular problem has been discussed much more fully elsewhere.[11] However, if the Fokker–Planck equation is valid, it was shown in the previous section that it reduces to a diffusion equation since $\langle \mathbf{E}(\mathbf{x} + \mathbf{v}t, t)\rangle = 0$ and $J = 1$. Thus, the problem of finding the Fokker–Planck coefficients reduces to the problem of finding the diffusion tensor. In order to simplify both the notation and the mathematics, we will assume the problem to be one dimensional (i.e., $\mathbf{E}$, $\mathbf{k}$, $\mathbf{v}$, and $\mathbf{x}$ will all be scalars). The generalization to three dimensions is straightforward and will be discussed later.

To find the diffusion constant, we must solve sense in some Newton's equation for the stochastic acceleration

$$\frac{dv}{dt} = F(t), \quad (18)$$

where $F(t)$ is determined from Eq. (17). Thus, once $F(t)$ is known, it becomes a simple matter[8] to write the diffusion constant

$$D = \frac{1}{2}\int_{-\infty}^{\infty} \langle F(\tau + t)F(t)\rangle\, d\tau, \quad (19)$$

where the integrand is assumed independent of t.

In this case $F(t)$ is given as a force field which is a function of space and time; thus, $F(t) = F[x(t), t]$. The problem is that, initially, $F(t)$ is not known since it depends on the, as yet unknown, orbit of the particle. In order to find $F(t)$, some approximation scheme for finding the particle orbit must be used. In this paper the approximation scheme will be orbit perturbation theory.

We will assume the turbulence is sufficiently weak so that the particle does not deviate very much from its straight-line orbit $x(t) = x + vt$, and that these deviations can be expanded in powers of the turbulent electric field strength, $F = F_1 + F_2 + F_3 + \cdots$ where F_n is of order E^n. For instance, the equation for F_1 is

$$F_1 = \frac{dv_1}{dt} = \sum_{k,p} E(k) \exp\left[i(kx + \phi_k)\right] \exp\left[i(kv - \omega)t\right].$$

The acceleration the particle feels from each wave is at the Doppler-shifted frequency $kv - \omega$. To solve for v_1 and x_1, one simply integrates from $t = 0$ to $t = \Delta t$. Doing so one finds that x_1 and v_1 also have Fourier components at $kv - \omega$ and are proportional to the first power of E. However, we should note that v_1 and x_1 also have Fourier components at zero frequency arising from the contribution at the lower limit of integration $t = 0$. Using the corrected orbit $x(t) = x + vt + x_1(t)$ on the right-hand side, Eq. (17) can be iterated further, yielding

$$F_2 = \frac{dv_2}{dt} = \sum_{k',p} \frac{eE(k')}{m}\left[ik'x_1(t)\right] \cdot \exp\left[i(k'x + \phi_{k'})\right] \exp\left[i(k'v - \omega')t\right], \quad (20)$$

$$F_3 = \sum_{k',p} \frac{eE(k')}{m}(ik'x_2) - \tfrac{1}{2}\sum_{k',p}\frac{eE(k')}{m}(k'x_1)^2, \quad \text{etc.}$$

[11] W. M. Manheimer, Ph.D. thesis, Massachusetts Institute of Technology (1967).

Inserting $x_1(t)$ above yields a contribution to F_2 at frequency $(k \pm k')v - (\omega \pm \omega')$ as well as a contribution at frequency $k'v - \omega'$. In either case F_2 is of second power in E and has products of exponentials of two random phases, ϕ_k and $\phi_{k'}$. Thus, instead of simply writing $F = F_1 + F_2 + F_3 + \cdots + F_n + \cdots$, we may generalize to write F as

$$F(t) = \sum_{k,p} B_1(k, v, t) \exp[i(kv - \omega)t] + \sum_{k,k',p} B_2(k, k', v, t) \exp\{i[(k + k')v - (\omega + \omega')]t\} + \cdots + \sum_{k_1,\cdots,k_n,p} B_n(k_1 \cdots k_n, v, t) \exp\{i[(k_1 + \cdots + k_n)v - (\omega_1 + \cdots + \omega_n)]t\} + \cdots, \tag{21}$$

where the term containing B_n will have contributions from F_n, F_{n+1}, $\cdots$. The summations over p are now summations over permutations of the signs of all k_i's. The contribution from F_n to B_n will be denoted $A_n(k_1 \cdots k_n, v)E(k_1) \cdots E(k_n) \exp[i(\phi_1 + \cdots + \phi_n)]$. B_n is written as a function of time since terms secular in time can arise in the perturbation expansion. Thus, we have a scheme for ordering $F(t)$ both in powers of the electric field strength and in frequency. When $F(t)$, given by Eq. (21), is inserted in Eq. (19), the diffusion constant will be similarly ordered in powers of E.

Equation (19) shows that D is the spectral density of the force field evaluated at *zero* frequency. The frequencies of the oscillating acceleration have been given in Eq. (21). Thus, diffusion can occur whenever one of the following criterion is satisfied:

$$\begin{aligned} \omega_1 - k_1 v &= 0, \\ (\omega_1 \pm \omega_2) - (k_1 \pm k_2)v &= 0, \\ &\vdots \\ (\omega_1 \pm \cdots \pm \omega_n) - (k_1 \pm \cdots \pm k_n)v &= 0, \\ &\vdots \end{aligned} \tag{22}$$

D_{2n} will be defined as the diffusion coefficient described by diffusion at the nth resonance in Eq. (22). Thus, according to Eq. (21), the dominant contribution to D_{2n} is given by

$$D_{2n} = \frac{1}{2}\int_{-\infty}^{\infty} d\tau \Big\langle \sum_{k_1,\cdots,k_{2n},p} A_n(k_1 \cdots k_n, v) \exp\{i[(k_1 + \cdots + k_n)v - (\omega_1 + \cdots + \omega_n)]t\} \cdot E(k_1) \cdots E(k_{2n}) \exp[i(\phi_1 + \cdots + \phi_{2n})] A_n(k_{n+1} \cdots k_{2n}, v) \cdot \exp\{i[(k_{n+1} + \cdots k_{2n})v - (\omega_{n+1} + \cdots + \omega_{2n})](t + \tau)\} \Big\rangle. \tag{23}$$

Using the fact that the wave phases are random,[12]

$$D_{2n} = \frac{n!}{2} \sum_{k_1,\cdots,k_n,p} |A_n(k_1 \cdots k_n, v)|^2 \cdot |E(k_1)|^2 \cdots |E(k_n)|^2 \quad 2\pi\, \delta[(k_1 + \cdots + k_n)v - (\omega_1 + \cdots + \omega_n)]. \tag{24}$$

We emphasize that D_{2n} is obtained by iterating the equation of motion only n times to obtain A_n and squaring. Thus, terms such as $\langle v_{2n-1}v_1\rangle/2\Delta t$ do not contribute to D_{2n}.

If a lower-order resonance coincides in velocity with the nth resonance, then diffusion to lower orders will be dominant, and it can be shown that Eq. (24) is not valid.[11] In deriving Eq. (24) that form of A_n which is symmetric with respect to interchanging any pair of k's is used. Thus, the $n!$ simply accounts for all possible permutations of collapsing the $2n$-fold summation over k into an n-fold summation. The delta function in Eq. (24) arises from taking the τ integral in Eq. (23) from $-\infty$ to $+\infty$.

When the turbulent field is given by Eq. (17), A_n is obtained by iterating the equation of motion n times, and in each iteration, keeping only those terms whose frequency of oscillation increases by $k_i v - \omega_i$ over the previous iteration. Thus, in taking each time integral involved in the iteration, we need keep only contributions from the upper limit.

Let us note that this proof is not restricted to

[12] In deriving Eq. (24) from Eq. (23), we should note that the integral over τ is actually from zero to Δt so that any secular contribution to A_1 will give a term going as Δt^2. Thus we keep only those terms with no secular behavior.

cases for which $F(t)$ is derived from Eq. (17). For example, the electric field may be given by

$$E(x,t) = \sum_{k_1,p} E(k_1) \exp[i(k_1 x - \omega_1 t + \phi_k)] + \sum_{k_1,k_2,p} E_2(k_1 k_2) \cdot \exp\{i[(k_1+k_2)x - (\omega_1+\omega_2)t + \phi_{k_1} + \phi_{k_2}]\} + \cdots + \sum_{k_1,\cdots,k_n,p} E_n(k_1 \cdots k_n) \cdot \exp\{i[(k_1+\cdots+k_n)x - (\omega_1+\cdots+\omega_n)t + \phi_{k_1} + \cdots + \phi_{k_n}]\} + \cdots, \quad (25)$$

where the ϕ_k's are random. Clearly, for the above electric fields, as long as E_n is of nth order in E_k, the particle orbit may be iterated as before, and the expansion for $F(t)$ will be exactly as given in Eq. (21), except that the individual B_i's will now be more complicated because of the additional terms in Eq. (25). For instance,

$$F_2(t) = \frac{dv_2}{dt} + \sum_{k,k',p} E_2(k,k) \cdot \exp\{i[(k+k')x + \phi_k + \phi_{k'}]\} \cdot \exp\{i[(k+k')v - (\omega+\omega')]t\}, \quad (26)$$

where dv_2/dt is given by Eq. (20). Additional terms such as the second term on the right of Eq. (26) are very important in nonlinear media, the reason being that any spectrum of turbulent waves given by Eq. (17) which exist in it initially will generate other electric fields at sum and difference frequencies and wavenumbers.

Thus, we have a method of calculating D to any order in perturbation theory. D_2 can be shown to be the ordinary quasilinear diffusion coefficient.[3,4] To further illustrate the method of calculation, let us obtain D_4 in the case where $F(t)$ is given by Eq. (17). Integrating dv_2/dt twice in time and keeping only the upper limit of integration each time gives a contribution to $x_1(t)$ [called $x_1'(t)$].

$$x_1'(t) = -\sum_{k,p} \frac{eE(k)}{m} \frac{\exp[i(kx+\phi_k)]}{(kv-\omega)^2} \cdot \exp[i(kv-\omega)t]. \quad (27)$$

Inserting Eq. (27) into Eq. (20) gives

$$A_2(k,k',v) = -\frac{1}{2}\left(\frac{e}{m}\right)^2 \exp[i(k+k')x] \cdot \left(\frac{k'}{(kv-\omega)^2} + \frac{k}{(k'v-\omega)^2}\right) \quad (28)$$

and, thus, from Eq. (24),

$$D_4 = \sum_{k,k',p} \frac{2\pi e^4}{4m^4} |E(k)|^2 |E(k)|^2 \cdot \left(\frac{k'}{(kv-\omega)^2} + \frac{k}{(k'v-\omega')^2}\right)^2 \cdot \delta[(k+k')v - (\omega+\omega')]. \quad (29)$$

In Eq. (29) the summation over p means that the wave energies and momenta (ω, k) in the delta function can either add or subtract. We will calculate the contribution to D_4 in the case where they subtract. Denoting it also as D_4, we get

$$D_4 = \sum_{k,k'} \left(\frac{e}{m}\right)^4 \pi |E(k)|^2 |E(k)|^2 \cdot \left(\frac{k-k'}{(kv-\omega)(k'v-\omega')}\right)^2 \cdot \delta[(k-k')v - (\omega-\omega')]. \quad (30)$$

In Eq. (30), consistent with the adopted convention, the summation over k (without $\sum_p$) is a summation over only those k vectors in the direction of wave momentum. Use has also been made of the delta function[13] in rewriting Eq. (30).

We will now look qualitatively at the conditions under which the preceding analysis is valid. First, in the integral over τ used to calculate D_{2n}, we have made the replacement

$$\sum_{k_1,\cdots,k_n,p} \frac{n!}{2} |A(k_1\cdots k_n,v)|^2 |E(k_1)|^2 \cdots |E(k_n)|^2 \cdot \int_0^{\Delta t} d\tau\, e^{i\alpha_n \tau} = \sum_{k_1,\cdots,k_n,p} \frac{n!}{2} |A(k_1\cdots k_n,v)|^2 \cdot |E(k_1)|^2 \cdots |E(k_n)|^2\, 2\pi\, \delta(\alpha_n), \quad (31)$$

where

$$\alpha_n = [(k_1+\cdots+k_n)v - (\omega_1+\cdots+\omega_n)].$$

Thus, Δt must be sufficiently large so that the upper and lower limits of the τ integral may be replaced with ∞. This means that for given v, the autocorrelation time of the nth-order force field, defined by

$$\tau_{ac}^{(n)} = \frac{\int_{-\infty}^{\infty} \langle F_n(t)F_n(t+\tau)\rangle\, d\tau}{\langle F_n(t)F_n(t)\rangle} \quad (32)$$

must be small compared with Δt. If $(k_1+\cdots+k_n)v - (\omega_1+\cdots\omega_n) = 0$ for some set of waves in

[13] Note that $k = k'$ for all v is a root of the delta function in Eq. (30). However, the contribution goes to zero as $(k-k')^2$ so the contribution vanishes.

the spectrum, then $\tau_{ac}^{(n)}$ is given roughly by

$$\{(k_1 + \Delta k_1 + k_2 + \cdots + k_n)v - [\omega_1(k_1 + \Delta k) + \omega_2 + \cdots + \omega_n]\}\tau_{ac}^{(n)} = \pi, \quad (33)$$

where Δk_1 is the spectral width of $|E(k_1)|^2$. Clearly, $\tau_{ac}^{(n)}$ depends not only on the spectral width, but also on the velocity v and the dispersion relation between ω and k. Estimations of $\tau_{ac}^{(n)}$ for various dispersion relations have appeared elsewhere.[11]

Thus, having determined that $\tau_{ac}^{(n)}$ is a lower limit for Δt, we may inquire about upper limits. Upper limits for Δt [or in effect $\tau_{ac}^{(n)}$] appear in two ways. First of all, the replacement of the Kolmogoroff equation with a Fokker–Planck equation is valid only if $f(v)$ does not change very much in one autocorrelation time, or if

$$D_{2n}\tau_{ac}^{(n)} \ll (v_T)^2 \quad (34)$$

for all n. In Eq. (34), v_T measures the interval in velocity space over which f changes significantly; v_T is typically a thermal velocity.

Secondly, if the theory in this section is to be valid, orbit perturbation theory must be valid for the time Δt. To get F_n, we used the orbit generated by $F = (F_1 + \cdots + F_{n-1})$, but not F_n itself because F_n is of lesser order than the terms we retain. However, a portion of F_n, $A_n' \sim n!\, A_n E(k_1) \cdots E(k_n)$ has almost zero frequency because for some waves in the spectrum

$$(\omega_1 + \cdots + \omega_n) + (k_1 + \cdots + k_n)v = 0. \quad (35)$$

Therefore, A_n' may have a large effect on the particle orbit, because the particle sees an almost constant force. During the time $\tau_{ac}^{(n)}$, A_n' will cause an rms deviation in the particle orbit of

$$\tfrac{1}{2}A_n'\tau_{ac}^{(n)2}. \quad (36)$$

This deviation must be small compared with the smallest wavelength in A_n', or otherwise, we will have a significant error in computing F_n. Thus,

$$\tfrac{1}{2}(A_n')_{\mathrm{rms}}\tau_{ac}^{(n)2} \ll \frac{1}{k}. \quad (37)$$

To first order, $A' \approx eE_{\mathrm{rms}}/m$ and Eq. (37) becomes

$$\tau_{ac} \ll \left(\frac{2m}{eE_{\mathrm{rms}}k}\right)^{\frac{1}{2}} \equiv \tau_{tr}, \quad (38)$$

where τ_{tr} is the trapping time of a particle in a wave of amplitude E_{rms} and wavenumber k. To nth order we will define the time $2/[(A_n')_{\mathrm{rms}}k]^{\frac{1}{2}}$ as the nth-order trapping time $\tau_{tr}^{(n)}$. Then a necessary condition for the validity of the analysis in this section is

$$\tau_{ac}^{(n)} \ll \tau_{tr}^{(n)} \quad (39)$$

as well as Eq. (34).

The analysis in this section has been carried out for a one-dimensional plasma. The one-dimensional theory is easily generalized to obtain the three-dimensional theory. The diffusion constant becomes a diffusion tensor D_{ij} and D_{2n} is nonzero in regions of velocity space defined by $(\mathbf{k}_1 \pm \cdots \pm \mathbf{k}_n)\cdot\mathbf{v} - (\omega_1 + \cdots + \omega_n) = 0$. To calculate D in three dimensions from D in one dimension, simply replace A_n by $\mathbf{A}_n$ in Eq. (23). Clearly, D_{2n} as defined in Eq. (20) and (23) becomes a symmetric tensor and can be written $\mathsf{D}_{2n} = \langle\Delta\mathbf{v}_n\ \Delta\mathbf{v}_n\rangle/(2\ \Delta t)$. For instance, had D_4 been calculated in three dimensions, Eq. (30) would have been

$$D_{4ij} = \sum_{\mathbf{k},\mathbf{k}'} \pi\left(\frac{e}{m}\right)^4 |\mathbf{E}(\mathbf{k})\cdot\mathbf{E}(\mathbf{k}')|^2 \cdot\frac{(k-k')_i(k-k')_j}{[(\mathbf{k}\cdot\mathbf{v}-\omega)(\mathbf{k}'\cdot\mathbf{v}-\omega)]^2} \cdot\delta[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}-(\omega-\omega')]. \quad (40)$$

Finally, let us say a few words about the size of the various contributions to D_n and where they contribute in velocity space. The case $n = 1$ is the well-known quasilinear diffusion constant and has been widely discussed in the literature.[1,2] Unfortunately, there are no precise rules by which one can in general relate the size of D_{2n} to the size of D_2. For instance, writing the ratio in the following form:

$$\frac{(D_{2n})_{\max}}{(D_2)_{\max}} = \left[\frac{\sum_k |E(k)|^2}{nmv_T^2}\right]^n \beta, \quad (41)$$

it can be shown that β has widely disparite values, depending on the specific problem at hand. If Eq. (17) represents plasma waves at frequency $\omega \approx \omega_p$, then for $n=2$ Eq. (30) shows that

$$\beta \sim \left(\frac{v_T}{\omega/k}\right)^2,$$

generally a very small number. If the driven field $E_2(k, k')$ is included in the problem, it has been shown that for a one-dimensional plasma $\beta \sim (kv_T/\omega_p)$.[6] In three dimensions we will show later that $\beta \sim (kv_T/\omega_p)$.[4] Thus, the driven field almost cancels dv_2/dt. On the other hand, if the ion dynamics are included,[14]

$$\beta \sim \left(\frac{kv_T}{\omega_p}\right)^2\left(\frac{m_e}{m_i}\right)^2.$$

[14] A. B. MacMahon and W. E. Drummond, Phys. Fluids 10, 1714 (1967).

It is also very difficult to draw any general conclusions about the regions in velocity space where the various D_{2n} make a contribution. The location of these regions depends on the detailed nature of the dispersion relation of the turbulent spectrum as well as the spectral density. It has been shown in three dimensions that for a spectrum of waves whose phase velocities lie on a closed surface surrounding the origin in velocity space, $D_2 \neq 0$ everywhere outside this surface.[5,15] In fourth order, if the beat velocity $[(\omega - \omega')/|\mathbf{k} - \mathbf{k}|^2](\mathbf{k} - \mathbf{k}')$ instead of the phase velocity lies on a closed surface around the origin, $D_4 \neq 0$ everywhere outside this surface. Later we will show that the beat velocities can enclose the origin even though the wave spectrum may be quite well columnated.

IV. REVIEW OF WAVE ENERGY AND MOMENTUM

In this section and the next two, the growth rates of waves to any order in perturbation theory will be calculated by requiring that momentum and energy density be conserved between waves and resonant particles. Before doing so, however, we must look into how energy and momentum density of waves are defined. Let us assume that an undamped wave at frequency ω_i and wavenumber $\mathbf{k}_i$, satisfying the linear dispersion relation, can be described in terms of a Hamiltonian density $H(\rho_i, \nabla\rho_i, \pi_i)$ where ρ_i is the appropriate field variable at wavenumber k_i, and π_i is the canonical momentum density

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\rho}_i}, \tag{42}$$

where $\mathcal{L}$ is the associated Lagrangian density and $\dot{\rho}_i = \partial\rho_i/\partial t$. The solution to the canonical equation of motion is

$$\rho_i(\mathbf{x}, t) = \rho_i \exp[i(\mathbf{k}_i\cdot\mathbf{x} - \omega_i t + \phi_k)] = \rho_i e^{i\theta}. \tag{43}$$

The energy density I and momentum density P associated with the wave are[16]

$$I = \omega N_{\mathbf{k}}, \tag{44}$$

$$\mathbf{P} = \mathbf{k}N_{\mathbf{k}}, \tag{45}$$

where $N_{\mathbf{k}}$ is the action density associated with the wave at $\mathbf{k}$ and is given by

$$N_{\mathbf{k}} = \int_0^{2\pi} d\theta \pi_i \frac{\partial n_i}{\partial\theta} = \frac{1}{\omega}\frac{\partial}{\partial\omega}[\omega\,\epsilon(k, \omega)]\frac{|\mathbf{E}(\mathbf{k})|^2}{4\pi}. \tag{46}$$

Let us now discuss the transformation properties of ω, $\mathbf{k}$, N, I, and P as one goes from the laboratory frame to one moving at velocity $\mathbf{v}$ with respect to it. If primed variables denote the moving frame, the transformations have been given[16] as

$$\begin{aligned} \omega' &= \omega - \mathbf{k}\cdot\mathbf{v}, \\ k' &= \mathbf{k}, \\ N' &= N, \\ \mathbf{P}' &= \mathbf{P}, \\ I' &= (\omega - \mathbf{k}\cdot\mathbf{v})N = \left(\frac{\omega - \mathbf{k}\cdot\mathbf{v}}{\omega}\right)I. \end{aligned} \tag{47}$$

From Eqs. (47) it is clear that a negative frequency refers to a negative energy wave, consistent with the convention adopted in the last section. In order to determine whether the wave has negative or positive energy, one must look at the expression for average wave energy given by

$$I(\mathbf{k}) = \frac{\partial}{\partial\omega}[\omega\,\epsilon(\mathbf{k}, \omega)]\frac{|\mathbf{E}(\mathbf{k})|^2}{4\pi}. \tag{48}$$

Thus, the sign of the wave energy is given simply as the sign of the quantity $(\partial/\partial\omega)[\omega\,\epsilon(\mathbf{k}, \omega)]$.

V. THE ACTION CHANGES OF INTERACTING MODES

We have seen how particle diffusion is calculated and in particular how n waves with wavenumber $\mathbf{k}_1 \cdots \mathbf{k}_n$ interact to diffuse particles with velocity

$$\left(\sum_{i=1}^{n} \mathbf{k}_i\right)\cdot\mathbf{v} = \sum_{i=1}^{n}\omega_i.$$

Now, let us examine how the energy of one of these n modes changes in such an interaction. The equations of energy and momentum conservation are

$$\sum_{i=1}^{n} \dot{I}(\mathbf{k}_i) = \int \frac{1}{2} n_0 m v^2 \frac{\partial}{\partial\mathbf{v}}\cdot\mathsf{D}_{2n}\cdot\frac{\partial f}{\partial\mathbf{v}}\,d\mathbf{v}, \tag{49a}$$

$$\sum_{i=1} \frac{\mathbf{k}}{\omega}\dot{I}(\mathbf{k}_i) = \int n_0 m v \frac{\partial}{\partial\mathbf{v}}\cdot\mathsf{D}_{2n}\cdot\frac{\partial f}{\partial\mathbf{v}}\,d\mathbf{v}, \tag{49b}$$

where n_0 is the average number density, $I(\mathbf{k}_i)$ is the energy density associated with the mode at $\mathbf{k}_i$, and D_{2n} is that term in Eq. (24) describing the diffusion from the modes at $\mathbf{k}_1, \cdots, \mathbf{k}_n$, or

$$\begin{aligned} \mathsf{D}_{2n} = {} & 2\pi(n!)^2\mathbf{A}(\mathbf{k}_1\cdots\mathbf{k}_n\mathbf{v})\mathbf{A}^*(\mathbf{k}_1\cdots\mathbf{k}_n\mathbf{v}) \\ & \cdot|\mathbf{E}(\mathbf{k}_1)|^2\cdots|\mathbf{E}(\mathbf{k}_n)|^2 \\ & \cdot\delta[(\mathbf{k}_1 \pm\cdots\pm \mathbf{k}_n)\cdot\mathbf{v} - (\omega_1 \pm\cdots\pm\omega_n)]. \end{aligned} \tag{50}$$

In the above we assume that one particular set of signs relative to $\mathbf{k}_1$ is chosen in the delta function. D_{2n} is then the contribution to the diffusion tensor for that set. In going from Eq. (24) to Eq. (50), we

[15] R. Z. Sagdeev and A. A. Galeev, International Center for Theoretical Physics, Trieste Paper IC/66/64 (1966), Chap. II.2.

[16] P. A. Sturrock, in *Sixth Lockheed Symposium on MHD*, D. Bershader, Ed. (Stanford University Press, Stanford, California, 1962), p. 47.

have gained an additional factor of $2n!$. This additional factor occurs because the wave vectors $\mathbf{k}_1 \cdots \mathbf{k}_n$ in Eq. (24) occur in $n!$ different arrangements, and because an interaction with one set of signs of the $\mathbf{k}$'s is equivalent to an interaction with all signs negative.

If Eqs. (49a) and (49b) were sufficient to determine the $\dot{I}_{\mathbf{k}}$'s, then the total growth rate of the wave at $\mathbf{k}_1$ could be obtained simply by adding up the contributions to $\dot{I}_{\mathbf{k}_1}$ from interactions of a wave at $\mathbf{k}_1$ with all possible sets of other waves at $\mathbf{k}_2 \cdots \mathbf{k}_n$. The trouble is that in general, Eq. (49) is not sufficient since there are n unknowns and only two equations. However, for $n = 1$ or 2, Eqs. (49) are sufficient to determine growth rates. But in the general case, more information is needed.

Let us assume the resonant particles constitute only a small fraction of the total number of particles. Then, we shall postulate that if the resonant particles are not present, the linear theory of the waves can be described with a Hamiltonian formalism as discussed in the last section. If this be the case, it turns out that Eq. (49) can be solved exactly. This is quite reasonable, for if a Hamiltonian formalism applies, the quantitization of the system is straightforward. Quantum mechanically, the waves become bosons (for instance, plasmons, photons, or phonons). In the quantum mechanical interaction of these bosons with resonant particles, selection rules insure that the number of bosons change by small integral numbers, usually plus or minus one. The correspondence principle guarantees that a similar relation must hold true in the classical limit. That is, the time rates of change of the action densities of the various modes, the classical analog of the occupation numbers, must be related by similar selection rules. We will now sketch a classical proof. Basically, all that is involved is a proof that the action density of any system is an adiabatic invariant. For a detailed discussion the reader is referred to various other sources.[17,18] Although the proof is somewhat laborious, the final result, Eq. (61), is very simple.

We assume that the n modes and resonant particles are described by some Hamiltonian density H which can be expressed as

$$\begin{aligned} H = &\sum_{i=1}^{n} H_i(\rho_i, \nabla\rho_i, \pi_i) \\ &+ H_{\text{part}}(\omega_1 \pm \cdots \pm \omega_n, \mathbf{k}_1 \pm \cdots \pm \mathbf{k}_n) \\ &+ H_m(\rho_1 \cdots \rho_n, \nabla\rho_1 \cdots \nabla\rho_n, \pi_1 \cdots \pi_n), \end{aligned} \quad (51)$$

[17] G. Schmidt, *Physics of High Temperature Plasmas* (Academic Press Inc., New York, 1966), p. 23.
[18] Reference 9, p. 288.

where H_i is the Hamiltonian density describing the mode at k_i in the linear picture. H_{part} is the Hamiltonian density of the resonant particles, $= n_r(p^2/2m) = n_r(p_\parallel^2/2m) + n_r(p_\perp^2/2m)$, where $\|$ and $\perp$ refer to directions parallel and perpendicular to the vector $\mathbf{k}_1 \pm \cdots \pm \mathbf{k}_n$, and n_r is the number density of resonant particles where $n_r \ll n_0$. H_{part} depends only on $\omega_1 \pm \cdots \pm \omega_n$, $\mathbf{k}_1 \pm \cdots \pm \mathbf{k}_n$ since $p_\parallel = m(\omega_1 \pm \cdots \pm \omega_n)/|\mathbf{k}_1 \pm \cdots \pm \mathbf{k}_n|$ and $p_\perp$ is not a function of the $\mathbf{k}$'s. H_m is the term causing the interaction between the n modes and resonant particles, and presumably is small. It is further assumed that H has no explicit dependence on space or time and that the frequencies of the modes $\omega_1 \cdots \omega_n$ are all independent.

It should be noted that the derivation to follow can be made to apply to the interaction of the waves $\mathbf{k}_1, \cdots, \mathbf{k}_n$ with an $(n+1)$th wave instead of the resonant particle.[19] The $(n+1)$th wave must have a frequency and wavenumber of $\omega_1 \pm \cdots \pm \omega_n$, $\mathbf{k}_1 \pm \cdots \pm \mathbf{k}_n$. This interaction is usually called resonant wave coupling.

The canonical equations of motion are

$$\frac{\partial H}{\partial \rho_i} - \sum_j \frac{\partial}{\partial x_j} \frac{\partial H}{\partial(\partial\rho_i/\partial x_j)} = -\dot{\pi}_i, \qquad \frac{\partial H}{\partial \pi_i} = \dot{\rho}_i. \quad (52)$$

If H_m is deleted from H, the solutions to Eq. (52) are

$$\begin{aligned} \rho_i(\mathbf{x}, t) &= \rho_i e^{i\theta_i}, \\ \theta_i &= \mathbf{k}_1 \cdot \mathbf{x} - \omega_i t + \phi_{\mathbf{k}_i}, \end{aligned} \quad (53)$$

for the n waves and

$$\mathbf{x} = \mathbf{x}_0 + (\mathbf{P}/m)t \quad (54)$$

for the resonant particle. They are simply the solutions to the linearized equations of motion. If H_m is now included, Eqs. (53) and (54) are only approximate solutions. We will assume that the exact solutions have time dependences other than that given by Eq. (53), but that these other time dependences are very slow compared with Eq. (53). For instance, for the n waves we will take

$$\rho_i = \rho_i(\theta_i, t), \qquad \pi_i = \pi_i(\theta_i, t), \quad (55)$$

where $\omega(\partial/\partial\theta) \gg \partial/\partial t$.

We shall relate the action change of all modes to the action change of the mode at $\mathbf{k}_1$. We wish to introduce a notation which permits us to take partial derivatives with respect to ω_i, θ_i with the sums

$$\sum_{i=1}^{n} \mathbf{k}_1 \quad \text{and} \quad \sum_{i=1}^{n} \omega_i$$

[19] P. A. Sturrock, Ann. Phys. (N. Y.) 9, 422 (1960).

held constant. To do so we shall consider ω_1 and $\mathbf{k}_1$ dependent variables and $\Omega = \omega_1 \pm \cdots \pm \omega_n$ and $\mathbf{K} = \mathbf{k}_1 \pm \cdots \pm \mathbf{k}_n$ independent variables. Hence, let us say

$$\begin{aligned} \omega_1 &= \Omega \mp \omega_2 \mp \cdots \mp \omega_n, \\ \mathbf{k}_1 &= \mathbf{K} \mp \mathbf{k}_2 \mp \cdots \mp \mathbf{k}_n, \\ \theta_1 &= \theta \mp \theta_2 \mp \cdots \mp \theta_n. \end{aligned} \tag{56}$$

Then, by virtue of Eq. (51) H_{part} can be expressed as a function of only Ω and $\mathbf{K}$, and is now independent of $(\omega_2 \cdots \omega_n, \mathbf{k}_2 \cdots \mathbf{k}_n)$. Thus, $\partial H_{\text{part}}/\partial\theta_i = 0$ for $i \neq 1$.

If ρ has the form of Eq. (55), the canonical equations of motion become

$$\frac{\partial H}{\partial \pi_i} = \dot{\rho}_i = \frac{\partial \rho_i}{\partial \theta_i}\frac{\partial \theta_i}{\partial t} + \frac{\partial \rho_i}{\partial t}, \tag{57a}$$

$$\frac{\partial H}{\partial \rho_i} - \sum_j \frac{\partial}{\partial x_j}\frac{\partial H}{\partial(\partial\rho_i/\partial x_j)} = -\dot{\pi}_i = -\frac{\partial \pi_i}{\partial \theta_i}\frac{\partial \theta_i}{\partial t} - \frac{\partial \pi_i}{\partial t}. \tag{57b}$$

Now, let us consider $(\partial H/\partial\theta_r)r \neq 1$. Since all θ_r are independent, in differentiating Eq. (51) with respect to θ_r, the only nonzero terms are the first, the rth, and the last. The result is

$$\begin{aligned} \frac{\partial H}{\partial \theta_r} &= \frac{\partial H_r}{\partial \theta_r} + \frac{\partial H_1}{\partial \theta_1}\frac{\partial \theta_1}{\partial \theta_r} + \frac{\partial H_m}{\partial \theta_r} + \frac{\partial H_m}{\partial \theta_1}\frac{\partial \theta_1}{\partial \theta_r} \\ &= \frac{\partial H}{\partial \rho_r}\frac{\partial \rho_r}{\partial \theta_r} + \sum_j \frac{\partial H}{\partial(\partial\rho_r/\partial x_j)}\frac{\partial}{\partial \theta_r}\left(\frac{\partial \rho_r}{\partial x_j}\right) \\ &\quad + \frac{\partial H}{\partial \pi_r}\frac{\partial \pi_r}{\partial \theta_r} + \frac{\partial \theta_1}{\partial \theta_r}\left[\frac{\partial H}{\partial \rho_1}\frac{\partial \rho_1}{\partial \theta_1} \right. \\ &\quad \left. + \sum_j \frac{\partial H}{\partial(\partial\rho_1/\partial x_j)}\frac{\partial}{\partial \theta_1}\left(\frac{\partial \rho_1}{\partial x_j}\right) + \frac{\partial H}{\partial \pi_1}\frac{\partial \pi_1}{\partial \theta_1}\right]. \end{aligned} \tag{58}$$

Now, let us make use of the fact that an x dependence appears only through a θ dependence, or $\partial/\partial\mathbf{x} = \mathbf{k}_1(\partial/\partial\theta_1)$. Then, the second term on the left-hand side of Eq. (57b) becomes

$$-\frac{\partial}{\partial \theta_i}\frac{\partial H}{\partial(\partial\rho_i/\partial\theta_i)}$$

and the middle term on the right-hand side of the second line of Eq. (58) becomes

$$\frac{\partial H}{\partial(\partial\rho_r/\partial\theta_r)}\frac{\partial^2}{\partial\theta_r^2}\rho_{\mathbf{k}_r}.$$

Now, let us average Eq. (58) over θ, i.e., integrate over θ from 0 to 2π, making use of the fact that

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{\partial}{\partial\theta}(\;) = 0.$$

This last relation follows from the fact that the function in the parentheses is assumed periodic in θ. The left-hand side of Eq. (58) then immediately averages to zero. The right-hand side can be integrated by parts over θ, yielding

$$0 = \int_0^{2\pi} d\theta_r\left[-\dot{\pi}_r\frac{\partial \rho_r}{\partial \theta_r} + \dot{\rho}_r\frac{\partial \pi_r}{\partial \theta_r} + \frac{\partial \theta_1}{\partial \theta_r}\left(-\dot{\pi}_1\frac{\partial \rho_1}{\partial \theta_1} + \dot{\rho}_1\frac{\partial \pi_1}{\partial \theta_1}\right)\right]. \tag{59}$$

To obtain the above result we have made use of the canonical equations. Now, one can make use of the expressions in Eq. (57) for $\dot{\pi}$ and $\dot{\rho}$ to obtain

$$\begin{aligned} 0 = \int_0^{2\pi} d\theta_r\Bigg\{&-\frac{\partial}{\partial t}\left(\pi_r\frac{\partial \rho_r}{\partial \theta_r}\right) + \frac{\partial}{\partial \theta_r}\left(\frac{\partial \rho_r}{\partial t}\pi_r\right) \\ &+ \frac{\partial \theta_1}{\partial \theta_r}\left[-\frac{\partial}{\partial t}\left(\pi_1\frac{\partial \rho_1}{\partial \theta_1}\right) + \frac{\partial}{\partial \theta_1}\left(\frac{\partial \rho_1}{\partial t}\pi_1\right)\right]\Bigg\}. \end{aligned} \tag{60}$$

The factor $\partial\theta_1/\partial\theta_r = \mp 1$ from Eq. (56).

Integrating Eq. (60) over θ then yields

$$0 = \dot{N}_r \mp \dot{N}_1, \tag{61}$$

where

$$\begin{aligned} N_r &= \frac{1}{2\pi}\int_0^{2\pi} d\theta_r \pi_r \frac{\partial \rho_r}{\partial \theta_r}, \\ &= \frac{1}{\omega_r}\frac{\partial}{\partial \omega_r}[\omega_r\,\epsilon(\mathbf{k}_r, \omega_r)]\frac{|\mathbf{E}(\mathbf{k}_r)|^2}{4\pi} \end{aligned} \tag{62}$$

is the action density associated with the wave at k_r. We have considered θ_i and t to be independent variables. Of course, this is not physically possible since θ_i and t are directly related. This independence of variables is valid when θ varies much faster than any other quantity of interest. Thus, it may vary while other functions of time (i.e., the wave amplitude) are constant. Let us also note that the sign in Eq. (61) is determined by the sign of $\partial\theta_{\mathbf{k}_1}/\partial\theta_{\mathbf{k}_r}$ which is determined by (56).

We have seen that the changes in action density are all related by (61). Also, we recall $I_{\mathbf{k}_1} = \omega_1 N_1$ and $\mathbf{P}_{\mathbf{k}_1} = \mathbf{k}_1 N_1$. These relations have been derived before when waves interact with other waves, but not for waves interacting with particles.[19] They are equivalent to the Manley–Rowe relations[20] applied to an infinite system which is not in a steady state.

It should be pointed out that once we restrict ourselves to examining the wave at $\mathbf{k}_1$ and $\mathbf{k}_r$, it does not make any difference what the waves couple to. If a wave at $(\omega_1 \pm \cdots \omega_n, \mathbf{k}_1 \pm \cdots \pm \mathbf{k}_n)$ were

[20] P. Penfield, *Frequency Power Formulas* (MIT Press, Cambridge, Massachusetts, 1962).

a normal mode of the system, the waves $\mathbf{k}_1, \cdots, \mathbf{k}_n$ could couple to it, and the change in action density of the wave at $\mathbf{k}_1$ would also be related to that at $\mathbf{k}_r$ by Eq. (61).

VI. GROWTH RATES TO ALL ORDERS

In the last section we have derived "selection rules" which must be obeyed in a wave–particle interaction. For instance, in the interaction of waves $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ with particles such that $(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3)\cdot\mathbf{v} - (\omega_1 + \omega_2 - \omega_3) = 0$, a plasmon is destroyed at ω_1 and ω_2 while one is created at ω_3 or vice versa. The only difference in the quantum-mechanical case is that the N's do not change by integral number. Thus, plasmons are created or destroyed according to the sign chosen for the frequency.

Using the selection rules which we have determined, we will now obtain the energy change in the mode $\mathbf{k}_1$ from the elementary interaction with the modes at $\mathbf{k}_2, \cdots, \mathbf{k}_n$. The equations for conservation of energy and momentum density (49) now reduce to

$$(\omega_1 \pm \cdots \pm \omega_n)\dot{N}_1 = \int \frac{1}{2} n_0 m v^2 \frac{\partial}{\partial \mathbf{v}}\cdot \mathsf{D}_{2n} \cdot \frac{\partial}{\partial \mathbf{v}} f \, d\mathbf{v}, \tag{63}$$

$$(\mathbf{k}_1 \pm \cdots \pm \mathbf{k}_n)\dot{N}_1 = \int n_0 m \mathbf{v} \frac{\partial}{\partial \mathbf{v}}\cdot \mathsf{D}_{2n} \cdot \frac{\partial}{\partial \mathbf{v}} f \, d\mathbf{v}, \tag{64}$$

where D_{2n} is the diffusion tensor containing the waves $\mathbf{k}_1, \cdots, \mathbf{k}_n$ given by Eq. (50). The signs of $\mathbf{k}_1, \cdots, \mathbf{k}_n$ and $\omega_1, \cdots, \omega_n$ which occur in $\mathbf{D}_{2n}$ are determined by the signs chosen in the summation $(\mathbf{k}_1 \pm \cdots \pm \mathbf{k}_n)$ as in Eq. (50).

One obvious result of Eq. (64) is that momentum given to particles in the interaction must be in the direction parallel to the vector $\mathbf{k}_1 \pm \cdots \pm \mathbf{k}_n$. This has important implications for the diffusion tensor. Since D_{2n} was shown to be a symmetric tensor, it must, therefore, be a dyad in the vector $(\mathbf{k}_1 \pm \cdots \pm \mathbf{k}_n)$. From Eq. (50) we may write $\mathbf{A} = \hat{\mathbf{K}}\,|\mathbf{A}|$ where $\mathbf{K} = \mathbf{k}_1 \pm \cdots \pm \mathbf{k}_n$. Then, Eq. (50) may be written as

$$\mathsf{D} = 2(n!)^2 \hat{\mathbf{K}}\hat{\mathbf{K}}\, |\mathbf{A}(\mathbf{k}_1 \cdots \mathbf{k}_n, \mathbf{v})|^2 \cdot \pi\, |E(k_1)|^2 \cdots |E(k_n)|^2\, \delta(Ku - \Omega), \tag{65}$$

where $\Omega = (\omega_1 \pm \cdots \pm \omega_n)$ and where u is the velocity component in the direction of $\hat{\mathbf{K}}$, i.e., $u = \hat{\mathbf{K}}\cdot\mathbf{v}$. Then, either Eq. (63 or 64) can be integrated by parts to give

$$\dot{N}_1 = \int d\mathbf{v}_\perp \frac{2\pi n_0 m\, |\mathbf{E}(\mathbf{k}_1)|^2 \cdots |\mathbf{E}(\mathbf{k}_n)|^2}{|K|^2} (n!)^2 \cdot \left|\mathbf{A}\left(\mathbf{k}_1 \cdots \mathbf{k}_n, \frac{\Omega}{K}, \mathbf{v}_\perp\right)\right|^2 \frac{\partial f}{\partial u}\bigg|_{u=\Omega/K}, \tag{66}$$

where $\mathbf{v}_\perp$ is the velocity perpendicular to u, i.e., $\mathbf{v}_\perp = \mathbf{v} - \hat{\mathbf{K}}u$. Equation (66) is valid as long as $\omega_1 \pm \cdots \pm \omega_n$ and $\mathbf{k}_1 \pm \cdots \pm \mathbf{k}_n$ do not both equal zero. If they do, these modes can exchange energy among themselves via resonant mode coupling to order $2n$–2. The growth rate for this latter process will then probably dominate the growth rate from a wave–particle interaction to order $2n$.

For instance, suppose that $\partial f/\partial u\,|_{\Omega/K} > 0$; then, the particles will give energy to the waves. Furthermore, suppose that the signs in front of $\omega_1 \cdots \omega_m$ are positive and that the signs in front of $\omega_{m+1} \cdots \omega_n$ are negative. Then, if

$$\sum_{i=1}^{m} \omega_i - \sum_{j=m+1}^{n} \omega_j > 0,$$

waves 1 through m are created and waves $m + 1$ through n are destroyed.

In order to obtain the total growth rate for a wave $\mathbf{k}_1$, one must add up all possible interactions in which this wave is involved. For instance, to fourth order

$$\dot{I}(\mathbf{k}) = \omega \dot{N}(\mathbf{k}) = |\mathbf{E}(\mathbf{k})|^2 \sum_{k'} \int \frac{d\mathbf{v}_\perp 8\pi n_0 m\omega\, |\mathbf{E}(\mathbf{k}')|^2}{|\mathbf{k} - \mathbf{k}'|^2} \cdot \left|\mathbf{A}\left(\mathbf{k}_1 - \mathbf{k}', \frac{\omega - \omega'}{|\mathbf{k} - \mathbf{k}'|}, \mathbf{v}_\perp\right)\right|^2 \cdot \frac{\partial f}{\partial u}\bigg|_{(\omega-\omega')/(\mathbf{k}-\mathbf{k}')} + (\mathbf{k}' \to -\mathbf{k}'). \tag{67}$$

Once $I(\mathbf{k})$ is expressed in terms of $|E(\mathbf{k})|^2$, Eq. (67) is an expression for the growth rate to fourth order in terms of the appropriate diffusion tensor. That is,

$$\frac{1}{4\pi} \frac{\partial}{\partial \omega} [\omega\, \epsilon\, (\omega, \mathbf{k})] \frac{\partial\, |\mathbf{E}(\mathbf{k})|^2}{\partial t} = \dot{I}(\mathbf{k}).$$

By using the transformations given in Sec. IV and the fact that D_{2n} is independent of the reference frame, it is not difficult to verify that the growth rate γ, defined by $\partial\, |\mathbf{E}(\mathbf{k})|^2/\partial t = 2\gamma\, |E(\mathbf{k})|^2$, is also frame independent.

VII. NONRESONANT DIFFUSION

The theory of diffusion to any order in perturbation theory has been developed under the assumption that the wave frequencies are real. However, in the last few sections we have seen that to order $2n$ the waves do indeed grow or damp in order that total energy of waves and resonant particles be conserved. To order $2n$, the growth rate γ is proportional to the electric field strength to the $2n-2$ power.

If the waves are growing, it is well known[21,22] that the resonant diffusion theory must be modified by adding to the diffusion constant the so-called nonresonant diffusion constant given by

$$\mathsf{D}_{nr} = \sum_{\mathbf{k},p} \frac{e^2}{m^2} \frac{|\mathbf{E}(\mathbf{k})|^2 \gamma}{(\omega - \mathbf{k}\cdot\mathbf{v})^2} \frac{\mathbf{kk}}{|\mathbf{k}|^2}. \tag{68}$$

Equation (68) is valid for particle velocities such that $(\omega - \mathbf{k}\cdot\mathbf{v})^2 \gg \gamma^2$. If the growth rate proportional to $|\mathbf{E}(\mathbf{k})|^{2n-2}$ is inserted, Eq. (68) will be a correction to the $2n$th-order resonant diffusion constant which must be included in the theory.

VIII. APPLICATIONS TO NONLINEAR LANDAU DAMPING

To illustrate the procedure, we now calculate the diffusion constant and damping rates of electron plasma oscillations to fourth order. The fourth-order diffusion tensor is given by

$$\mathsf{D}_4 = \int_{-\infty}^{\infty} \frac{\langle \mathbf{F}_2(t)\mathbf{F}_2(t+\tau)\rangle}{2}\, d\tau, \tag{69}$$

where $\mathbf{F}_2(t)$ is the particle acceleration to second order in the electric field and at frequency $(\omega - \omega') - (\mathbf{k} - \mathbf{k}')\cdot\mathbf{v}$. There are two contributions to $\mathbf{F}_2(t)$; the second iteration of the first term of Eq. (25) which has been calculated in Sec. III, and the first iteration of the second term in Eq. (25). Hence, we must calculate $\mathbf{E}(\mathbf{k}, \mathbf{k}')$ as defined in Eq. (25). Using a well-known iteration procedure,[1] one can obtain

$$\frac{e}{m}\mathbf{E}(\mathbf{k},\mathbf{k}') = \frac{-(e^2/m^2)\omega_p^2(\mathbf{k}-\mathbf{k}')}{\epsilon(\mathbf{k}-\mathbf{k}', \omega-\omega')\,|\mathbf{k}-\mathbf{k}'|^2} \cdot \int \frac{d\mathbf{v}}{(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v} - (\omega-\omega') - i\delta}\,\mathbf{E}(\mathbf{k}) \cdot \frac{\partial}{\partial\mathbf{v}} \frac{1}{\mathbf{k}'\cdot\mathbf{v} - \omega' - i\delta}\,\mathbf{E}(-\mathbf{k}')\cdot\frac{\partial f}{\partial\mathbf{v}}, \tag{70}$$

where ϵ is the dielectric constant of the plasma, i.e.,

$$\epsilon(\mathbf{k},\omega) = 1 + \frac{\omega_p^2}{\mathbf{k}^2}\int \frac{\mathbf{k}\cdot(\partial f/\partial\mathbf{v})}{\mathbf{k}\cdot\mathbf{v} - \omega - i\delta}\, d\mathbf{v}. \tag{71}$$

Symmetrizing Eq. (70) with respect to exchanging $\mathbf{k}$ and $-\mathbf{k}'$ and combining with Eq. (26) yields

$$\mathsf{D}_4 = \sum_{\mathbf{k},\mathbf{k}'} 16\frac{\pi e^4}{m^4} |\mathbf{E}(\mathbf{k})|^2\, |\mathbf{E}(\mathbf{k}')|^2 \cdot \frac{(\mathbf{k}-\mathbf{k}')(\mathbf{k}-\mathbf{k}')(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')^2(\mathbf{k}_\perp\cdot\mathbf{v})^2}{\omega_p^6} \cdot \delta[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v} - (\omega-\omega')] \tag{72}$$

after some algebra. To obtain Eq. (72), one must expand denominators such as $(\omega - \mathbf{k}\cdot\mathbf{v})^{-1}$ and keep only the lowest nonvanishing terms. Also, $\hat{k}$ is a unit vector in the direction of $\mathbf{k}$ and the vector $\mathbf{k}_\perp$ is the component of $\mathbf{k}$ which is perpendicular to $(\mathbf{k} - \mathbf{k}')$. Note that the contribution to D_4 from waves at $\mathbf{k}$ and $\mathbf{k}'$ is zero if $\mathbf{k}$ and $\mathbf{k}'$ are either parallel or perpendicular. In a one-dimensional problem in which all wave vectors are parallel, it has been shown[14] that D_4 is smaller than Eq. (72) roughly by a factor of $(kv_T/\omega_p)^2$.

From Eq. (72) one can immediately write the damping rate from Eq. (67). The result is

$$\dot{I}_\mathbf{k} = \frac{\partial}{2\pi\,\partial t}|\mathbf{E}(\mathbf{k})|^2 = |\mathbf{E}(\mathbf{k})|^2 \sum_{k'} \frac{32\pi e^4 n_0}{m^3}\,\omega\, |\mathbf{E}(\mathbf{k}')|^2 \cdot \frac{(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')^2}{\omega^6} |\mathbf{k}'_\perp|^2 v_T^2 \frac{\partial f(u)}{\partial u}\bigg|_{(\omega-\omega')/|\mathbf{k}-\mathbf{k}'|}, \tag{73}$$

where Eq. (67) has been integrated over perpendicular velocities assuming $f(v)$ is isotropic. Note that if $\partial f(u)/\partial u\,|_{(\omega-\omega')/|\mathbf{k}-\mathbf{k}'|}$ is positive, net energy is gained instead of lost by the wave spectrum. That is, the plasmon which is created has more energy than the plasmon which is destroyed. Thus, the system will be nonlinearly unstable. However, the net energy gain will be small since the total number of plasmons is conserved. Thus, fourth-order diffusion provides a means by which the plasma can tap free energy[6] which is available in the form of a positive slope around

$$u = (\omega - \omega')/|\mathbf{k} - \mathbf{k}'|.$$

This energy is not necessarily available in the linear theory. In fact, diffusion to order $2n$ can provide a means by which the plasma can tap free energy in the form of a positive slope anywhere on the distribution function. Thus, it would appear that the only distribution function which cannot couple energy to the waves is a monotonically decreasing function of energy.

Let us now consider a two-dimensional plasma and look for those regions of velocity space for which $D_4 \neq 0$ if the wave spectrum is well columnated around wave vector $\mathbf{k}_0$ on the axis, or around phase velocity

---

[21] R. E. Aamodt and W. E. Drummond, Phys. Fluids 9, 1816 (1964).

[22] B. B. Kadomtsev, *Plasma Turbulence* (Academic Press Inc., New York, 1964), p. 15.

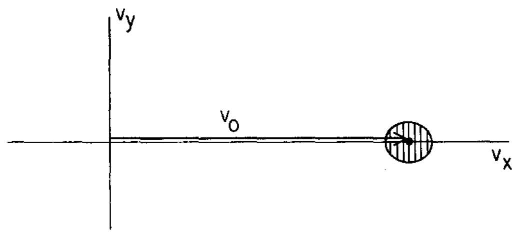

FIG. 2. Phase velocities of waves in spectrum.

$$\mathbf{v}_0 = \frac{\omega_p^2 + \frac{3}{2}|\mathbf{k}_0|^2 v_T^2}{|\mathbf{k}_0|^2 \omega_p} \mathbf{k}_0.$$

The regions of nonzero wave spectrum are shown as the shaded regions of phase velocity space in Fig. 2, or as the shaded region of $\mathbf{k}$ space in Fig. 3. Let us now look for the values which the beat phase velocities

$$\frac{\omega - \omega'}{|\mathbf{k} - \mathbf{k}'|^2}(\mathbf{k} - \mathbf{k}') = \tfrac{3}{2}v_T^2 \frac{|\mathbf{k}|^2 - |\mathbf{k}'|^2}{|\mathbf{k} - \mathbf{k}'|^2 \omega_p}(\mathbf{k} - \mathbf{k}')$$

can assume. If $\mathbf{k}$ and $\mathbf{k}'$ are both along the x axis, then the beat velocity ranges from $3v_T(k_{\min} v_T/\omega_p)$ to $3v_T(k_{\max} v_T/\omega_p)$. The beat phase velocity can also be perpendicular to the x axis if $\mathbf{k} - \mathbf{k}'$ is perpendicular to $\mathbf{k}_0$ and if $\mathbf{k} \neq \mathbf{k}'$. It is also easy to see that if $\mathbf{k} - \mathbf{k}'$ is perpendicular to $\mathbf{k}_0$, the magnitudes of $\mathbf{k}$ and $\mathbf{k}'$ can be made to approach each other, so the beat velocity can approach zero in the y direction. The beat velocity can also point in the negative x direction if $k_x < k_x'$ while $|\mathbf{k}|^2 > |\mathbf{k}'|^2$. Therefore, the beat phase velocity can take on values in the shaded region of Fig. 4. Since this region is seen to enclose the origin in velocity space, there can be fourth-order diffusion anywhere in velocity space even though the wave spectrum is well columnated along $\mathbf{k}_0$.

IX. APPLICATION TO NONLINEAR INSTABILITIES

In the previous section we have seen how a plasma can be nonlinearly unstable if

$$\partial f/\partial u\, |_{(\omega-\omega')/(k-k')} > 0.$$

However, the growth rate is very small since, for

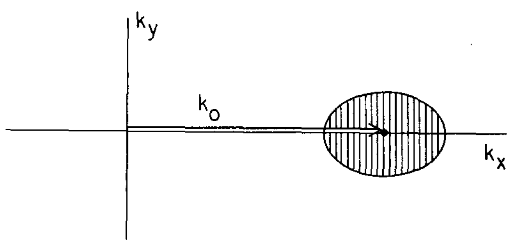

FIG. 3. Wavenumbers of waves in spectrum.

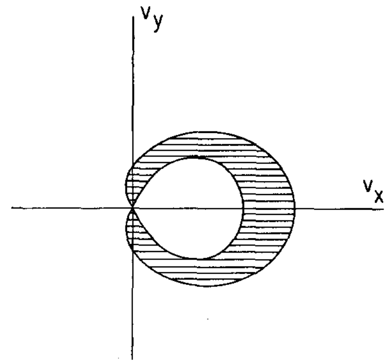

FIG. 4. Beat phase velocities.

every plasmon which is created, one is also destroyed. If the dynamics of the coupling is such that both plasmons are either created or destroyed, the instability may proceed much more quickly. In one simple case we will show that if f were to remain constant, the total wave energy may become infinite in a finite time. Such nonlinear instabilities have been studied where the coupling mechanism is the resonant interaction between three waves of negative and positive energy.[23,24] Recently, it has been shown that two waves can interact with particles to give such a nonlinear instability.[25] One way in which the coupling to particle differs from the coupling to waves is that the energies of the coupling waves can all be of the same sign.

We will demonstrate these concepts by looking at a very simple nonlinear instability. Consider a one-dimensional plasma with three components: a neutralizing positive background, a cold plasma with density n_0 and zero streaming velocity, and a group of energetic particles with thermal velocity v_T about an average streaming velocity v_b and with density μn_0 where $\mu \ll 1$. This distribution function is shown in Fig. 5. The density of energetic particles will be assumed so low that the dispersion relation of cold plasma waves is unchanged; that is, all waves have frequency ω_p and positive energy $I_k = |E(k)|^2/2\pi$.

The energetic particles, of course, contribute a "bump on tail," making waves with phase velocity $v_p \approx v_b$ linearly unstable. However, a wave with $k \approx 2\omega_p/v_b$ can couple nonlinearly with a wave with $k' \approx 0$ and particles of velocity

$$v = (\omega + \omega')/|k + k'| \approx v_b.$$

[23] M. L. Sloan and R. E. Aamodt, Phys. Rev. Letters 19, 1127 (1967).
[24] B. Coppi, Princeton Matt. Report 545 (1967).
[25] M. Rosenbluth, B. Coppi, and R. N. Sudan, Bull. Am. Phys. Soc. 13, 283 (1968).

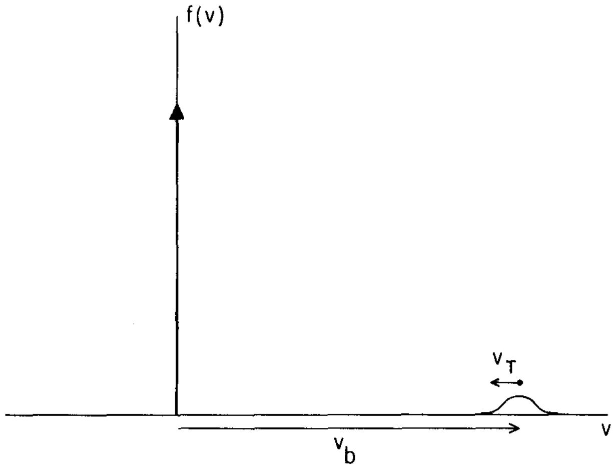

FIG. 5. The velocity distribution function.

For such an interaction, it was seen in Sec. V that $\dot{N}_k = \dot{N}_{k'}$, so that each wave gains or loses energy.

To find the growth rate, we must first find the velocity space diffusion constant from the amplitude of the driving force at frequency $(\omega + \omega') - (k + k')v$. There are two contributions to this amplitude, the second iteration of the equation of motion given by

$$\frac{1}{2} \sum_{k,k',p} \left(\frac{e}{m}\right)^2 E(k)E(k') \frac{k + k'}{(k'v - \omega')(kv - \omega')} \tag{74}$$

and the driven field given by

$$\frac{e}{m} E(k + k', \omega + \omega') = -\frac{1}{4} \sum_{k,k',p} \frac{4}{3}\left(\frac{e}{m}\right)^2 \frac{\omega_p^2(k + k')}{(k + k')^2} E(k)E(k') \cdot \int \frac{dv'}{(k + k')v' - (\omega + \omega') - i\delta} \cdot \left(\frac{\partial}{\partial v'} \frac{1}{k'v' - \omega' - i\delta} \frac{\partial f}{\partial v'} + \frac{\partial}{\partial v'} \frac{1}{kv' - \omega - i\delta} \frac{\partial f}{\partial v'}\right). \tag{75}$$

In Eq. (75) we have made use of the fact that $\epsilon(k + k', \omega + \omega') \approx 0.75$. Then, assuming $\omega = \omega_p$ and $f(v) = \delta(v)$, Eq. (75) can be integrated and added to Eq. (74), yielding

$$D_4 = \frac{2\pi}{4} \sum_{k,k',p} \left(\frac{e}{m}\right)^4 |E(k)|^2 \, |E(k')|^2 \, (k + k')^2 \cdot \left(\frac{1}{(\omega_p - kv)(\omega_p - k'v)} - \frac{1}{\omega_p^2}\right)^2 \cdot \delta[2\omega_p - (k + k')v]. \tag{76}$$

Using Eq. (66), we can find the growth rate from the interaction of two waves at $k \approx 2\omega_p/v_p$ and $k' \approx 0$ with resonant particles. The result is

$$\frac{d\,|E(k)|^2}{dt} = \frac{d\,|E(k')|^2}{dt} = 2\pi^2 m\omega_p n_0 \left(\frac{e}{m}\right)^4 \cdot \left(\frac{1}{(\omega_p - kv)(\omega_p - k'v)} - \frac{1}{\omega_p^2}\right)^2 \cdot \frac{\partial f}{\partial u}\bigg|_{u=2\omega_p/(k+k')} |E(k)|^2 \, |E(k')|^2, \tag{77}$$

where we have used the fact that $I_k = |E(k)|^2/2\pi$ for waves in a cold plasma. Equation (77) may be solved by noting that $d\,|E(k)|^2/d\,|E(k')|^2 = 1$ for all time. Thus,

$$|E(k')|^2 = |E(k)|^2 + C, \tag{78}$$

where C is determined by the condition at time $t = 0$. Then, Eq. (77) can be rewritten in the form

$$\frac{d\,|E(k)|^2}{dt} = \alpha C\,|E(k)|^2 + \alpha\,|E(k)|^4, \tag{79}$$

where α is the factor multiplying $|E(k)|^2\,|E(k')|^2$ in Eq. (77). It is positive as long as $\partial f/\partial u\,|_{2\omega_p(k+k')}$ is positive. The change in f will be described by a diffusion equation. However, to make matters simple (albeit not physical), let us assume that f and hence α are constant in time. If $|E(k)|^2$ at time $t = 0$ is denoted by E_0, and $|E(k')|^2$ at $t = 0$ by E_0', the solution of Eq. (79) is

$$|E(k)|^2 = \frac{[(E_0' - E_0)/E_0']e^{(E_0'-E_0)\alpha t}}{1 - (E_0/E_0')e^{(E_0'-E_0)\alpha t}}. \tag{80}$$

Thus, $|E(k)|^2 = \infty$ for

$$\frac{E_0}{E_0'} e^{(E_0'-E_0)\alpha t} = 1.$$

Clearly, the above can only be satisfied for positive α.

Since the total wave energy goes to infinity in a finite time, these instabilities are sometimes called explosive instabilities. If there are more than two waves, the situation is less clear. The total growth rate of a wave at k is given by

$$\frac{d\,|E(k)|^2}{dt} = 2\pi^2\,|E(k)|^2\, m\omega \sum_{k'} n_0 \left(\frac{e}{m}\right)^4 \cdot \left(\frac{1}{(\omega_p - kv)(\omega_p - k'v)} - \frac{1}{\omega_p^2}\right)^2 \cdot |E(k')|^2 \frac{\partial f}{\partial u}\bigg|_{2\omega_p/(k+k')}. \tag{81}$$

In Eq. (81) some k' in the sum will be such that $\partial f/\partial u\,|_{2\omega_p/(k+k')}$ is positive, while for other k', it will be negative. To find out if the net effect is toward stability or instability, one must add the contributions from all waves.

X. CONCLUSIONS

It has been shown that to any order in perturbation theory the dominant equation for the ensemble average distribution function is a diffusion equation. A method of calculating diffusion constants has been outlined as well as a method of calculating growth rates in terms of the resonant diffusion constant. This method was then applied to problems in nonlinear Landau damping and nonlinear instabilities.

ACKNOWLEDGMENTS

One of us (W.M.M.) would like to thank Professor K. U. Ingard for many very valuable discussions on these matters.

This work was supported principally by the National Science Foundation (Grant GK-2581).