Turbulence

- a crucial example in scaling and self-similarity is turbulence

Self-similarity:

- phenomenon looks the same over a range of scales if
  \( l_i < l < l_o \) in the inner to outer

\[ \psi \left( \frac{r}{R(t)} \right) = \psi \left( \frac{r}{R(t)} \right) \]

\[ R(t) = \frac{t}{T} \]

\[ t \rightarrow \infty \]

\[ r \rightarrow \infty \]

leaves a invariant

- power law dependence is symptom

\( i.e. \frac{dV}{dL} \sim L^{1/3} \)

\( \frac{d}{dx} \sim dV \sim L^{1/3} dV \)
Some examples:

1. Cascade & hierarchical fragmentation — "shattering" → 3D Fluid Turbulence

2. Aggregation ("inverse cascade")
   → colloidal aggregation, also Schmeluchowsky

3. Fractals and B-models:
   → memory of dimension
   → fractals

4. Fluid Turbulence (c.f.: Frisch)

   What is it?

   - spatio-temporal disorder

   - broad range of space-time scales

   - power transfer through broad range of scales

   - energy dissipation
- can view as consisting of sequence of basic interactions.

\[ \text{cascade} \rightarrow \text{fragmentation sequence} \rightarrow \# \text{eddys in check} \]

- aggregating/mixing cascade

- plain vanilla collision

- # particles conserved.
More characteristics:
- decay of large scales
- irreversible mixing
- can be intermittent/bursty

Key parameter:

\[ \text{Re} = \frac{\nu(L) L}{\nu} \]

Where \( \text{Re} \) is the Reynolds number, \( \nu(L) \) is the velocity at length scale \( L \), and \( \nu \) is the kinematic viscosity. For atmospheric turbulence:

\[ BL \text{ on hot day} \]

\[ \text{Re} \approx 10^8 \]

\[ \text{hour} \approx \text{few km} \]

\[ \text{lin} \approx \text{few mm} \]
Laws (Empirical)

- Recall

\[ F_d = C_D A_0 V^2 \]
\[ C_0 = C_0 (Re) \] Flat in turbulent regime

\[ \square \]

- Finite Energy Dissipation Rate

If, in experiment on turbulent flow all control parameters kept the same except viscosity, which is lowered as much as possible, energy dissipation per mass \( dE/dt \) approaches a finite limit.

Simple Terms: Energy dissipation is due to viscosity yet does not depend explicitly on \( C_0 \).

\[ \square \]

recall \( F_d \sim c_0 \rho S_A U^2 \)

\[ dE/dt \sim F_d U \sim c_0 \rho S_A U^3 \]

\[ dE/dt \sim \frac{U^3}{\eta} \]
\[ \frac{dE}{dt} = \frac{u^3}{l} = \epsilon \]

where does energy go?

\[ \Rightarrow \text{viscous dissipation} \]

\[ \text{effective large scale forcing} \]

\[ \nabla \cdot \mathbf{u} = 0 \]

\[ \dot{\langle u^3 \rangle} + \langle \nabla \cdot (\mathbf{u} \mathbf{u}) \rangle = -\nabla \langle (\mathbf{u} \mathbf{u})^2 \rangle \]

\[ -\left\langle \nabla \cdot (\mathbf{u} \mathbf{p}) \right\rangle + \left\langle F \cdot \mathbf{u} \right\rangle \]

\[ \Rightarrow \text{pressure no net effect} \]

\[ \text{forcing } \Rightarrow \nabla \langle (\mathbf{u}^2) \rangle = \left\langle F \cdot \mathbf{u} \right\rangle \]

Now necessarily \[ \left\langle F \cdot \mathbf{u} \right\rangle = \epsilon \]
\[ \bar{c} = \gamma \langle \text{\textcircled{UV}}^2 \rangle \rightarrow \text{balance} \]

\[ \text{under } \gamma \Rightarrow \text{\textcircled{UV}}_{\text{rms}} \sim \frac{1}{\sqrt{\nu}} \]

\[ \Rightarrow \text{turbulence forms singular velocity gradient} \]

\[ \Rightarrow \text{must necessarily access small scales} \]

**How?**  Cascade \rightarrow \text{hierarchical Fragmentation}

\[ \text{○ ○ ○ ○ ○} \rightarrow l_0 \]

\[ \text{○ ○ ○ ○ ○ ○ ○ ○ ○} \rightarrow l_1 \]

\[ \text{○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○} \rightarrow l_2 \]

\[ \cdots \cdots \cdots \cdots \ldots \]

\[ \sim \text{again empirical \& broad range of scales, with no gaps} \]
How described \( \rightarrow \) structure functions

\[
\sigma V(l) = \left( \frac{V(l + \ell) - V(l)}{\ell} \right)
\]

\( \ell \rightarrow \) difference in velocity across scale \( \ell \)

\( \Delta \)

\( \langle \sigma V(l)^2 \rangle \rightarrow \langle \sigma^2 V(l) \rangle \)

related energy distribution on scale

\( \rightarrow \) 2/3 law (Empirical)

\[
S_2(l) = \langle (\Delta V(e))^2 \rangle \sim l^{2/3}
\]

\( \hat{s}_2 \) fn \( S_2 \)

\( \rightarrow \) Rigorous:

\[
\langle (\sigma V(l))^3 \rangle = \frac{4}{5} \bar{e} \ell
\]

4/5 Law

energetics
What's the Story?

- K41 (Kolmogorov Phenomenology)

Ideas:

- Flux of energy in scale space from $l_0$ (input/increment scale) to $l_0$ (dissipation scale - set by $r$).
- Energy flux is same at all scales between $l_0$ and $l_0$, self-similarity.
- Energy dissipation - set as $\nu \rightarrow 0$ but not $\rightarrow 0$, breaking.
- Symmetry of stirring, etc. lost $\Rightarrow$ Symmetry restored.

Ingredients / Players

- exciter $\rightarrow$ eddy

- $l$: scale parameter, eddy scale
\[ \nu(e) \sim \left( \frac{\partial u_i \partial e_i}{2} \right) \]

Velocity increment

\[ \partial u_i \sim \left[ u_i(e+\epsilon) - u_i(e) \right] \cdot \frac{\epsilon}{\epsilon} \]

\[ \partial \vec{u} \sim \text{rms eddy fluctuation} \]

\[ \nu(e_0) \sim \nu_0 \]

\[ T(e) \sim \text{eddy transfer/lifetime} \]

\[ \to \text{characteristic scale of transfer in cascade step} \]

Now, self-similarity \( \Rightarrow \) constant flow-they rate:

\[ \epsilon = \frac{\nu(e)^2}{\gamma(e)} \]

\[ \gamma(e) \]
The \( \ell \):

- dimensionally \( \rightarrow \) lifetime of structure of scale \( \ell \)
- time to distort out of existence.

For scale \( \ell' \) which \( \ell' \) effect lifetime \( \ell' \)

\[ \ell' \gg \ell \]

Advent oddly but don't distort if

\( \Rightarrow \) irrelevant physics not change under random Galilean boost

\( \Rightarrow \) violates symmetry restoration

- scales \( \ell' \ll \ell \)

\( \Rightarrow \) irrelevant, as very little energy shears in such eddies
strongest interaction on $\ell \sim l$

Composable scalar distinct one another

cascading @ local in scale!

\[ T(\ell) \sim \ell / v(\ell) \]

\[ E \sim v(\ell)^2 \sim v(\ell)^3 \]

i.e.

\[ v_{\ell}^3 \sim v(\ell)^3 \]

\[ \frac{l_{\ell}}{l_{\ell}} \sim \ell \]

\[ v(\ell) \sim (\ell / \ell)^{\frac{1}{3}} \]
\[ V(l) = E^{2/3} l^{2/3} \]
\[ \approx v_o^2 \left( \frac{l}{l_o} \right)^{2/3} \]

- Power law
- RET monopole 2/3 law
- Dependence on \( l \), \( v_o \) only, \( \propto E \).

For \( k \) spectrum:
\[ E(k) = |\nu(k)|^2 \]

\[ E = \int dk \, |\nu(k)|^2 = \int dk \, E(k) \]
\[ \text{c.e. absorb density of states} \]

then
\[ V(l) = \int_{k_{n-1}}^{k_n} dk \, E(k) \]
$V(e)^2 \sim \epsilon^{2/3} l^{4/3} \sim \epsilon^{2/3} k^4$

$E(k) \sim \epsilon^{2/3} k^{-5/3}$

*Kolmogorov spectrum.*

\[ \frac{1}{V(e)} \sim \frac{V(e)}{l} \sim \epsilon^{4/3} l^{-2/3} \]

**N.B.**

transfer rate increases as scale decreases.

finite time to end?

"**total time**"

\[ T = \sum_{n=0}^{\infty} T_n \]

\[ = \sum_{n=0}^{\infty} \frac{\ell_0}{v_0} \left( \frac{\ell_n}{\ell_0} \right)^{2/3} \]

\[ \ln l_0 / l_0 \sim l_0^{1/2} \] for \( x < 1 \)

\[ T = \sum_{n=0}^{\infty} \frac{\ell_0}{v_0} \ell_0^{-2/3} \]
\[ T \sim \frac{\nu_l}{\nu_0} \frac{1}{1 - x^{2/3}} \]

\( T_0 \) sets cascade time.

- Cascade can go through \( \infty \) steps in finite time.
- Hence analogy with "shattering".

\( \nu \) for dissipation scale \( l_\xi \)

- Occurs when viscous diffusion kicks in and cuts off cascade.

\[ \frac{1}{T_0} \sim \frac{1}{\nu_0 l_2} \rightarrow \text{diffusive and } \nu_0 \text{ time scales cross} \]

\[ \frac{1}{l_\xi} \sim x^{2/3} \sim \nu l_2 \]

\[ \delta_l \sim \nu^{3/4} \ell_1^{1/4} \]

\( \rightarrow \) dissipation scale.
Finally

$\# \text{ of FIs} \sim \left( \frac{h_0}{l_0} \right)^3 \sim \left( \frac{h_i}{l_0} \right)^3 \sim \left( \frac{h_i}{l_i} \right) \sim (Re^{3/4})^3 \sim Re^{3/4}$

For $h_0 \sim 1 \text{ km, } l_0 \sim 1 \text{ mm} \implies N \sim \frac{18}{10}$

N.B.: What is missing?